# Quantum Error Correction and Holographic Information from Bilocal Holography

(arXiv:2106.00349 work with E. Gandote, N.H. Tahiridimbisoa and H.J.R. Van Zyl)


Robert de Mello Koch

School of Science
Huzhou University
.
and
.
Mandelstam Institute for Theoretical Physics
University of the Witwatersrand


September 20, 2021

# References


S. R. Das and A. Jevicki, "Large N collective fields and holography," Phys. Rev. D **68**, 044011 (2003) [arXiv:hep-th/0304093 [hep-th]].

R. dMK, A. Jevicki, K. Jin and J. P. Rodrigues, "$AdS_4/CFT_3$ Construction from Collective Fields," Phys. Rev. D **83**, 025006 (2011) [arXiv:1008.0633 [hep-th]].

# Vector model / Higher spin duality

Free $O(N)$ vector model in $2+1$ dimensions

$$S = \int d^3x \sum_{a=1}^{N} \left( \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a \right)$$

Single trace primaries: $O_{\Delta=1}(t,\vec{x}) = \sum_{a=1}^{N} \phi^a(t,\vec{x})\phi^a(t,\vec{x})$

$$J_{\mu_1\mu_2\cdots\mu_{2s}}(t,\vec{x})\alpha^{\mu_1}\alpha^{\mu_2}\cdots\alpha^{\mu_{2s}} = \sum_{a=1}^{N}\sum_{k=0}^{2s} \frac{(-1)^k \ :(\alpha\cdot\partial)^{2s-k}\phi^a \ (\alpha\cdot\partial)^k\phi^a:}{k!(2s-k)!\Gamma(k+\frac{1}{2})\Gamma(2s-k+\frac{1}{2})}$$

The gravity dual is higher spin gravity in $AdS_4$, with a bulk scalar and a gauge field $A_{\mu_1\cdots\mu_{2s}}$ for each conserved current.

# Large $N$ Higher spin equations of motion

Metsaev, arXiv:hep-th/9906217

Work in Poincare patch of $AdS_4$

$$X^+ \equiv X^2 + X^0, \quad X^- \equiv X^2 - X^0, \quad X \equiv X^1, \quad Z \tag{1}$$

Work in lightcone gauge

$$A_{+\mu_2\cdots\mu_{2s}} = 0 \tag{2}$$

All components $A_{-\mu_2\cdots\mu_{2s}}$ are determined by constraints. Dynamical fields are $X, Z$ polarizations.

Free equation of motion is

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-} + \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right) A_{XXZ\cdots ZX} = 0 \tag{3}$$

# Bilocal Holography

Basic claim: Holography is accomplished by a change to gauge invariant (bilocal) field variables in the CFT. A change of coordinates is needed to give the bulk interpretation of the bilocal theory.

The loop expansion parameter of the original CFT is $\hbar$. After changing to invariant (bilocal) field variables the loop expansion parameter is $1/N$ matching the loop expansion parameter of the dual gravity.

The bilocal transforms in a tensor product. The complete collection of higher spin fields transform in a direct sum. The natural change of basis

$$V_{[\frac{1}{2},0]} \otimes V_{[\frac{1}{2},0]} \longrightarrow V_{[1,0]} \oplus \bigoplus_{s=2,4,\cdots} V_{[s+1,s]} \tag{4}$$

determines a map between CFT and bulk coordinates.

(Think of addition of angular momentum: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.)

# Change of Field Variables

To solve QFT "all" we need to do is evaluate a complicated integral.

$$\int d\phi^a e^{-S(\phi^a)} \qquad a = 1, \cdots, N \tag{5}$$

Hard to do when $N \to \infty$, but things simplify when the theory has an $O(N)$ symmetry, so the action is an $O(N)$ invariant.

Suppose the $\phi^a$ are in vector rep of $O(N)$. Then $S$ is a function of $\sigma = \phi^a\phi^a$, the unique invariant. One integration variable and not $N$ - much simpler!

$$\int d\sigma e^{-N\tilde{S}(\sigma)} \tag{6}$$

$N$ appears because we had a total of $N$ variables. Saddle point approximation produces an expansion with $1/N$ the loop counting parameter.

2+1 Minkowski in light cone coordinates $x^+ \equiv x^0 + x^2$, $x^- \equiv x^2 - x^0$, $x \equiv x^1$. Invariant variables

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \sum_{a=1}^{N} \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2) \tag{7}$$

# Change of Field Variables

2+1 Minkowski in light cone coordinates $x^+ \equiv x^0 + x^2$, $x^- \equiv x^2 - x^0$, $x \equiv x^1$. Invariant variables

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \sum_{a=1}^{N} \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2) \tag{8}$$

The field $\sigma(x^+, x_1^-, x_1, x_2^-, x_2)$ develops a large $N$ expectation value. It is the fluctuation $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ about this large $N$ background that maps to bulk AdS fields.

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \sigma_0(x^+, x_1^-, x_1, x_2^-, x_2) + \frac{1}{\sqrt{N}}\eta(x^+, x_1^-, x_1, x_2^-, x_2)$$

$\sigma_0(x^+, x_1^-, x_1, x_2^-, x_2)$ is the large $N$ two point function.

# Change of Co-ordinates

An infinite number of spinning fields in $\mathrm{AdS}_4$ = a single field on $\mathrm{AdS}_4 \times \mathrm{S}^1$

Co-ordinates: $X^+ \equiv X^2 + X^0 \qquad X^- \equiv X^2 - X^0, \qquad X \equiv X^1 \qquad Z$

$$ds^2 = \frac{dX^+ dX^- + dX^2 + dZ^2}{Z^2}$$

Fields: $A_{\mu_1 \cdots \mu_{2s}}(X^+, X^-, X, Z)$, $\Phi(X^+, X^-, X, Z)$

Co-ordinates: $X^+ \equiv X^2 + X^0 \qquad X^- \equiv X^2 - X^0, \qquad X \equiv X^1 \qquad Z \qquad \theta$

$$ds^2 = \frac{dX^+ dX^- + dX^2 + dZ^2}{Z^2}$$

Fields: $\Phi(X^+, X^-, X, Z, \theta)$

$$\Phi(X^+, X^-, X, Z, \theta) = \sum_{s=0}^{\infty} \cos(2s\theta)\, \Phi_{2s}(X^+, X^-, X, Z) \tag{9}$$

$$A_{- \, - \cdots -} = \frac{\partial^{2s}}{\partial X^{-\,2s}} \Phi_{2s}(X^+, X^-, X, Z) \tag{10}$$

# Change of Coordinates

The bilocal transforms in $V_{\frac{1}{2},0} \otimes V_{\frac{1}{2},0}$ $\qquad$ ($L^A \in$so(2,3))

$$L^A_{\otimes}\sigma \quad = \quad \left(L^A\phi^a(x^+,x_1^-,x_1)\phi^a(x^+,x_2^-,x_2)+\phi^a(x^+,x_1^-,x_1)L^A\phi^a(x^+,x_2^-,x_2)\right)$$

$$V_{\frac{1}{2},0} \otimes V_{\frac{1}{2},0} = V_{1,0} \oplus \bigoplus_{s=2,4,6,\cdots} V_{s+1,s} \tag{11}$$

The complete collection of higher spin fields fill out the reducible representation $V_{1,0} \oplus \bigoplus_{s=2,4,6,\cdots} V_{s+1,s}$

$$L^A_{\oplus}\Phi(X^+,X^-,X,Z,\theta) = \sum_{s=0}^{\infty} \cos(2s\theta)\; L^A_{2s}\Phi_{2s}(X^+,X^-,X,Z)$$

We want to change from the natural representation ($L^A_{\otimes}$) of the CFT to the representation that is natural for the bulk gravity ($L^A_{\oplus}$).

# Change of Coordinates

Symmetry: $X^- \to X^- + a$ in gravity; $x^- \to x^- + b$ in CFT.

$$x_1 = X + Z\tan\left(\frac{\theta}{2}\right) \qquad x_2 = X - Z\cot\left(\frac{\theta}{2}\right) \qquad x^+ = X^+$$

$$p_1^+ = P^+\cos^2\left(\frac{\theta}{2}\right) \qquad p_2^+ = P^+\sin^2\left(\frac{\theta}{2}\right)$$

$$X = \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+} \qquad Z = \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+}$$

$$P^+ = p_1^+ + p_2^+ \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

$$L_\oplus^A \Phi = 2\pi P^+ \sin\theta \; L_\otimes^A \eta$$

# Summary: Bilocal Holography

$$\begin{aligned}\sigma(x^+,x_1^-,x_1,x_2^-,x_2) &= \sum_{a=1}^{N}\phi^a(x^+,x_1^-,x_1)\phi^a(x^+,x_2^-,x_2)\\ &= \sigma_0(x^+,x_1^-,x_1,x_2^-,x_2)+\frac{1}{\sqrt{N}}\eta(x^+,x_1^-,x_1,x_2^-,x_2)\end{aligned}$$

$$X = \frac{p_1^+x_1+p_2^+x_2}{p_1^++p_2^+} \qquad Z = \frac{\sqrt{p_1^+p_2^+}(x_1-x_2)}{p_1^++p_2^+} \qquad X^+ = x^+$$

$$P^+ = p_1^++p_2^+ \qquad \theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$$

$$\begin{aligned}\Phi(X^+,P^+,X,Z,\theta) &= 2\pi P^+\sin\theta\,\eta(X^+,P^+\cos^2\frac{\theta}{2},X+Z\tan\frac{\theta}{2},P^+\sin^2\frac{\theta}{2},X-Z\cot\frac{\theta}{2})\\ &= \sum_{s=0}^{\infty}\cos(2s\theta)\,\Phi_{2s}(X^+,X^-,X,Z)\end{aligned}$$

# Bulk Reconstruction

Does the proposed bulk field $\Phi(X^+, P^+, X, Z, \theta)$ obey the correct bulk equation of motion with the correct boundary condition?

CFT equation of motion:

$$\left(\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} + \frac{\partial^2}{\partial x^2}\right)\phi^a(x^+, x^-, x) = 0$$

implies

$$\left(\frac{\partial}{\partial X^+}\frac{\partial}{\partial X^-} + \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Z^2}\right)\Phi\left(X^+, X^-, X, Z, \theta\right) = 0$$

$$(p_1^+ + p_2^+)^s \cos\left(2s \arctan\sqrt{\frac{p_2^+}{p_1^+}}\right) = \mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k (p_1^+)^{s-k}(p_2^+)^k}{\Gamma\left(s-k+\frac{1}{2}\right)\left(\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!\right)}$$

implies that $(\mathcal{N} = \Gamma\left(\frac{1}{2}\right) s! \Gamma\left(s+\frac{1}{2}\right))$

$$\frac{\partial^s}{\partial X^{-s}}\Phi_s(X^+; X^-, X, 0) = 16\pi\mathcal{N}\sum_{k=0}^{s}\frac{(-1)^k \partial_-^{s-k}\phi^a(X^+, X^-, X)\partial_-^k\phi^a(X^+, X^-, X)}{\Gamma\left(s-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)k!(s-k)!}$$

# Subregion Duality

Which subregion of the CFT (if any) is dual to a given subregion of the bulk?

Consider localized CFT excitations, at time $x^+$, with the first at $(x_1, p_1^+)$ and the at $(x_2, p_2^+)$, described as wavepackets, tightly peaked at $x_1$ and $x_2$ along direction $x$ transverse to the light cone, and smeared along $x^-$.

$$\left(X - \frac{x_1 + x_2}{2}\right)^2 + Z^2 = \left(\frac{x_1 - x_2}{2}\right)^2 \tag{12}$$

The bulk excitation is on a semicircle in the $X, Z$ plane, with radius $(x_1 - x_2)/2$ and center $X = (x_1 + x_2)/2$ and $Z = 0$. To locate the excitation specify angle $\theta$

$$\tan\theta = \frac{Z}{X - \frac{x_1+x_2}{2}} = \frac{2\sqrt{p_1^+ p_2^+}}{p_1^+ - p_2^+} \tag{13}$$

This angle $\theta$ is the angle $\theta$ appearing in the map.

# Subregion Duality

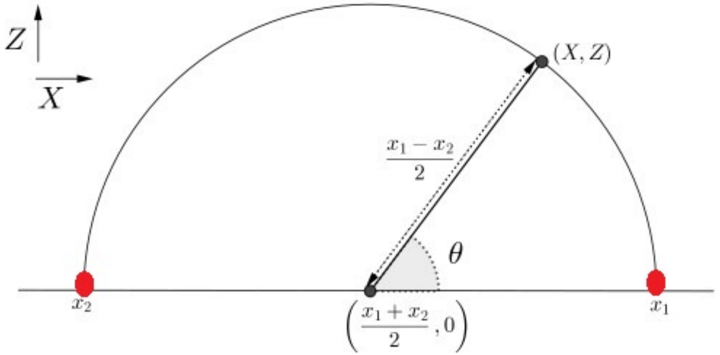



Figure: The bilocal describing excitations localized at $(x_1, p_1^+)$ and $(x_2, p_2^+)$ corresponds to a bulk excitation localized at $(X, Z)$ as shown. This figure lives on a constant $x^+ = X^+$ slice. The angle $\theta$ is $\theta = 2\tan^{-1}\left(\sqrt{\frac{p_2^+}{p_1^+}}\right)$.

# Bulk Reconstruction

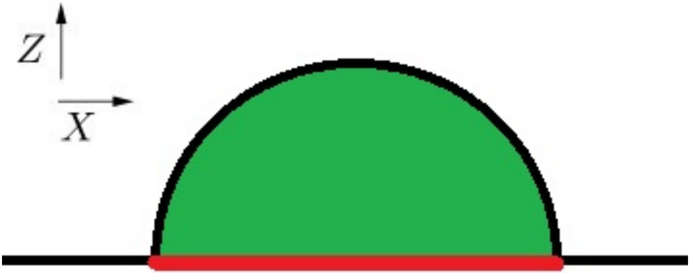



Figure: Using biloclas restricted to the red region of the CFT we are able to reconstruct bulk field living in the area shaded in green. This figure is for fixed time $x^+ = X^+$.

# Subregion Duality

CFT subregion: $x^+ = 0$, $-\infty \leq x^- \leq \infty$, $-L \leq x \leq L$.
Bulk excitations: $X^+ = 0$ and

$$X^2 + Z^2 \leq L^2 \qquad -\infty \leq X^- \leq \infty \qquad 0 \leq \theta \leq \pi \tag{14}$$

The metric of $\mathrm{AdS}_4$ on a constant $X^+$ slice is

$$ds^2 = \frac{dX^2 + dZ^2}{Z^2} \tag{15}$$

Consider region $\mathcal{E}_\mathcal{R}$ of the $(X, Z)$ plane bounded by $Z = 0$ and the curve $Z = Z(X)$. The length of the boundary of this region is given by

$$\ell = \int dX \frac{\sqrt{Z'^2 + 1}}{Z} \qquad Z' = \frac{dZ}{dX} \tag{16}$$

Minimizing we find

$$\frac{dZ}{dX} = \frac{\sqrt{R^2 - Z^2}}{Z} \qquad \Rightarrow \qquad X^2 + Z^2 = R^2 \tag{17}$$

$\mathcal{E}_\mathcal{R}$ is reminiscent of the entanglement wedge.

# Bulk Reconstruction

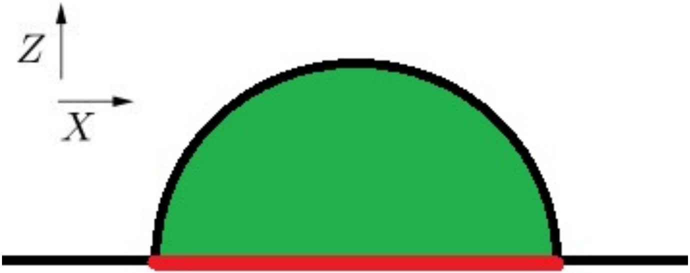



Figure: Using biloclas restricted to the red region of the CFT we are able to reconstruct bulk field living in the area shaded in green. This figure is for fixed time $x^+ = X^+$.

# Code Subspace

An observer can't access the whole Hilbert space. In particular, an observer

1. can't access infinite energy states.
2. can only access a subset of all fields.

These limitations are formalized in the code subspace.

We will consider a code subspace obtained by

a. excited a finite number of spinning fields.
b. the occupation numbers are all bounded.

# Bulk Reconstruction

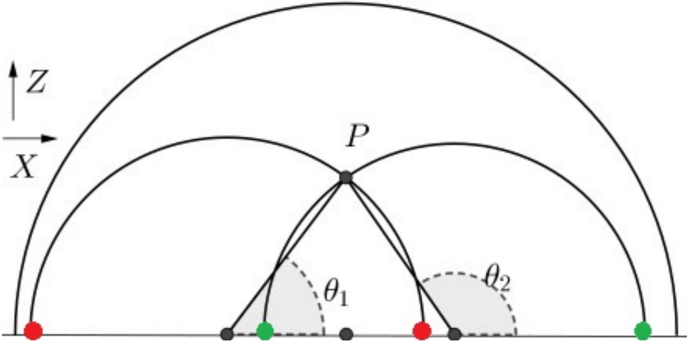



Figure: It is possible to choose distinct bilocals from the subregion $\mathcal{R}$ that correspond to different semicircles passing through point $P$. This allows us to obtain the value of the bulk field $\Phi(X^+, X^-, X, Z, \theta)$ at $P$ for a number of distinct $\theta$.

$$\Phi(X^+, p^+, X, Z, \theta) = \sum_{s=0}^{k} \cos(2s\theta)\, \Phi_{2s}(X^+, X^-, X, Z)$$

# Quantum Error Correction

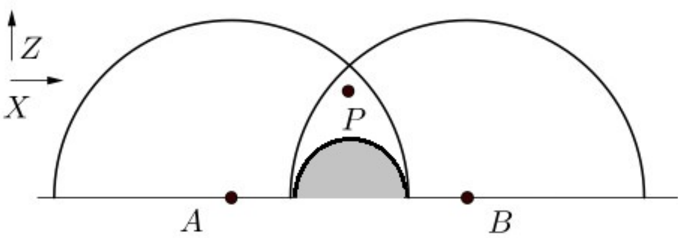



Figure: The fields at point $P$ can be reconstructed using bilocals taken from region $A$ or region $B$. It is however not possible to reconstruct the bulk fields at point $P$ using bilocals from the intersection $A \cap B$.

# Quantum Error Correction

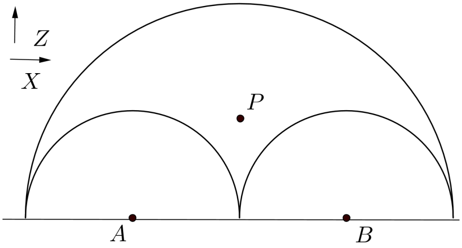
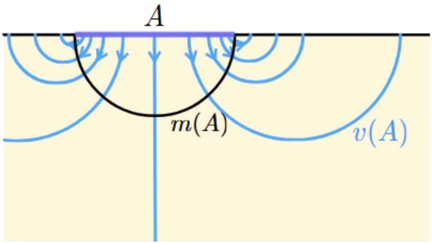



Figure: It is not possible to reconstruct the bulk operator at $P$ if we restrict to either subregion $A$ or $B$. The reconstruction is however possible if bilocals from $A \cup B$ are used.

# Bit Threads

By Ryu-Takyanagi formula:

$$S(A) = \frac{1}{4G_N}\mathrm{area}(m(A)) \tag{18}$$




Examples of flows $v^\mu$ for empty AdS is a family of semi-circles [Agon, de Boer, Pedraza, arXiv:1811.08879]

# Entanglement Wedge Reconstruction

$$
\begin{aligned}
\eta(x^+, p_1^+, x_1, p_2^+, x_2) \quad &= \quad \int_{-\infty}^{\infty} dx^- \, \frac{e^{i(p_1^+ + p_2^+)x^-}}{2\sqrt{p_1^+ p_2^+}} \\
&\times \quad \Phi\left( x^+, x^-, \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+}, \frac{\sqrt{p_1^+ p_2^+}(x_1 - x_2)}{p_1^+ + p_2^+}, 2\tan^{-1}\sqrt{\frac{p_2^+}{p_1^+}} \right)
\end{aligned}
$$

By convention take $x_1 > x_2$. Set

$$
\alpha = \frac{p_1^+}{p_1^+ + p_2^+} \qquad \beta = \frac{p_2^+}{p_1^+ + p_2^+} \qquad \alpha + \beta = 1 \qquad 0 \le \alpha, \beta \le 1
$$

$Z = \sqrt{\alpha\beta}(x_1 - x_2)$ is zero when $\alpha = 0$ or $\beta = 0$, and is maximum value of $\frac{1}{2}(x_1 - x_2)$ at $\alpha = \frac{1}{2} = \beta$.

$X = \alpha x_1 + \beta x_2$ ranges from $x_2$ when $\beta = 1$, $\alpha = 0$, to $x_1$ when $\alpha = 0$, $\beta = 1$.

$\Rightarrow$ a bilocal with points $x_1, x_2$ inside $A$ is reconstructed using bulk fields within $\mathcal{E}_A$.

# Holography of Information

Finite $N$ constraints: For $N=1$ we have

$$\sigma(x^+,p^+,x_1,p^+,y_1)\sigma(x^+,p^+,x_2,p^+,y_2)$$
$$=\phi(x^+,p^+,x_1)\phi(x^+,p^+,y_1)\phi(x^+,p^+,x_2)\phi(x^+,x_2,p^+,y_2)$$
$$=\phi(x^+,p^+,x_1)\phi(x^+,p^+,y_2)\phi(x^+,p^+,x_2)\phi(x^+,x_2,p^+,y_1)$$
$$=\sigma(x^+,p^+,x_1,p^+,y_2)\sigma(x^+,p^+,x_2,p^+,y_1)$$

This constraint between the bilocal degrees of freedom can be written as

$$\det M = 0 \qquad M=\begin{bmatrix}\sigma(x^+,p^+,x_1,p^+,y_1) & \sigma(x^+,p^+,x_1,p^+,y_2)\\ \sigma(x^+,p^+,x_2,p^+,y_1) & \sigma(x^+,p^+,x_2,p^+,y_2)\end{bmatrix}$$

which simply expresses the fact that the $2\times 2$ matrix $M$ has rank 1.

# Holography of Information

$p_1^+ = p_2^+ = p_3^+$

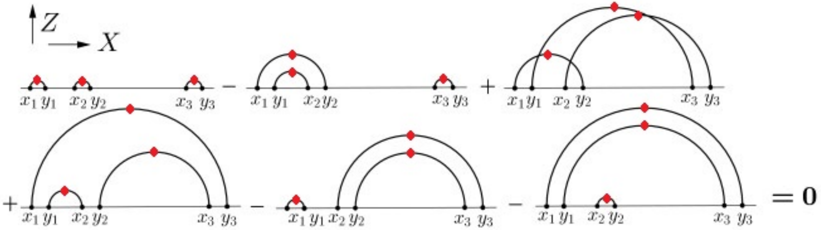



Figure: Constraints that arise when $N = 2$.

# Future Directions

Can we construct a similar geometric interpretation for the case that we perform a canonical quantization of the CFT?

Would be nice to study something like the thermofield double state, where we expect horizons in the bulk.

The idea that changing to invariant variables implements holography is probably more general than just vector models. For multi-matrices we can't easily construct a collective field theory. The space of gauge invariants is enormous and its difficult to formulate the change of variables.

For one matrix models the space of gauge invariants is the space of eigenvalues. This simplicity is what makes progress possible.

What is a useful set of variables for the multi-matrix problem? It is on this very first step that we get stuck.

Thanks for your attention!