Probing Hawking radiation through capacity of entanglement

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Overview

1. Introduction: entropy of Hawking radiation
2. Review: capacity of entanglement
3. Capacity formula in 2d dilaton gravity
4. Capacity of Hawking radiation in toy model

Outline

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Hawking radiation from an evaporating BH

- Suppose the initial state of matter is pure

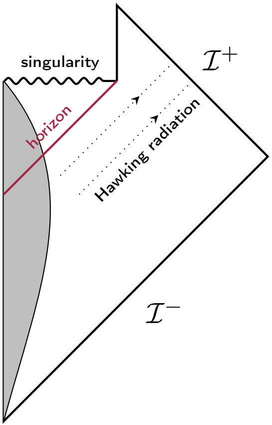
$$\rho_{\text{pure}} = |\Psi\rangle\langle\Psi|$$

but after gravitational collapse a black hole is formed

- BH starts to evaporate due to Hawking radiation
- After the evaporation of BH, the system is in a mixed state of thermal radiation:

$$\rho_{\text{pure}} \xrightarrow[\text{BH evaporation}]{} \rho_{\text{mixed}}$$

which appears to contradict with unitarity [Hawking 76]



Page curve for the radiation

To model an evaporating BH with radiation, suppose

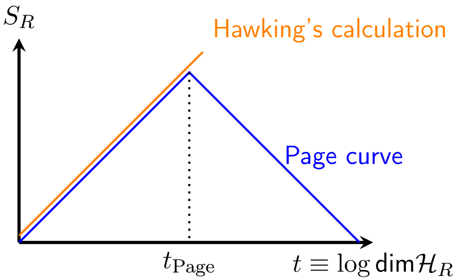
$$|\Psi\rangle \in \mathcal{H}_{\text{BH}} \otimes \mathcal{H}_R \,, \qquad \mathcal{H}_{\text{BH}} : \text{BH system} \,, \qquad \mathcal{H}_R : \text{radiation system}$$

- For a pure state $|\Psi\rangle$ Page showed [Page 93] when $\dim \mathcal{H}_R \ll \dim \mathcal{H}_{\text{BH}}$ the radiation system is almost maximally entangled :

$$S_R \approx \log \dim \mathcal{H}_R$$

- In the opposite limit, $\dim \mathcal{H}_R \gg \dim \mathcal{H}_{\text{BH}}$, from unitarity:

$$S_R \approx \log(\dim \mathcal{H}_{\text{tot}} - \dim \mathcal{H}_R)$$



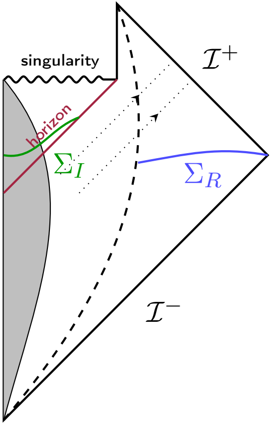
Island formula for the radiation entropy

- To reconcile with the Page curve, the entropy of radiation should be calculated by the island formula [Penington 19, Almheiri-Engelhardt-Marolf-Maxfield 19, Almheiri-Mahajan-Maldacena-Zhao 19]:

$$S_R = \min_{\Sigma_I} \left\{ \operatorname{ext}_{\Sigma_I} \left[\frac{\text{Area}(\partial \Sigma_I)}{4G_N} + S_{\text{mat}}(\Sigma_R \cup \Sigma_I) \right] \right\}$$

  - Σ_R: radiation region R
  - Σ_I: island region I

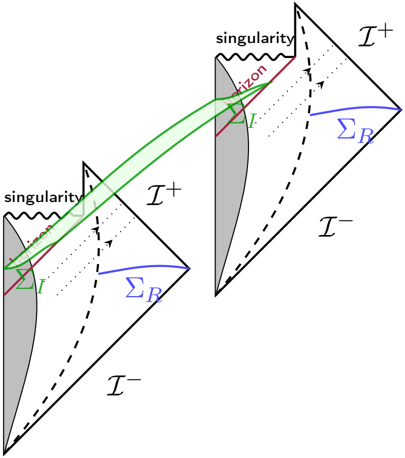
- No island → linear growth at early time
- With island → saturation or decay at late time



Replica wormholes

- The island formula is a generalization of the Ryu-Takayanagi formula for entanglement entropy [Ryu-Takayanagi 06], which has a gravitational path integral derivation [Lewkowycz-Maldacena 13, ···]

- The island regions are accounted for by replica wormholes [Almheiri-Maldacena-Hartman-Shaghoulian-Tajdini 19, Penington-Shenker-Stanford-Yang 19]



Goal of this talk

- We will examine if Hawking radiation (or replica wormholes) can be captured by capacity of entanglement, a quantum information measure other than entanglement entropy
- Derive a formula for the capacity in 2d dilaton gravity
- Calculate the capacity for a toy model of radiating black holes
- The capacity has a peak or discontinuity at the Page time, showing a good probe of the radiation

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Entanglement entropy

Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$

Entanglement entropy

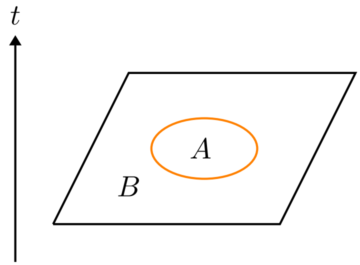
$$S_A = -\text{Tr}_A \left[\rho_A \log \rho_A\right]$$

- The reduced density matrix

$$\rho_A \equiv \text{Tr}_B[\rho_{\text{tot}}]$$

- For a pure ground state $|\Psi\rangle$

$$\rho_{\text{tot}} = |\Psi\rangle \langle\Psi|$$



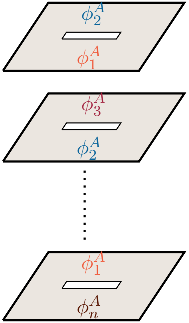
Replica trick and Rényi entropy

Entanglement entropy

$$S_A = \lim_{n\to 1} S_n$$

n^{th} Rényi entropy

$$S_n \equiv \frac{1}{1-n}\log \mathrm{Tr}_A[\rho_A^n] = \frac{1}{1-n}\log Z(n)$$



$Z(n)$: partition function on the n-fold cover branched over A

Analogy to statistical mechanics

We regard $Z(n) \equiv \mathrm{Tr}_A[\rho_A^n]$ as a thermal partition function at an inverse temperature $\beta \equiv n$:

Statistical mechanics		Rényi analogue	
Inverse temperature:	β	Replica parameter:	n
Hamiltonian:	H	Modular Hamiltonian:	$H_A = -\log \rho_A$
Partition function:	$Z(\beta) = \mathrm{Tr}\left[e^{-\beta\, H}\right]$	Replica partition function:	$Z(n) = \mathrm{Tr}_A\left[e^{-n\, H_A}\right]$
Free energy:	$F(\beta) = -\beta^{-1} \log Z(\beta)$	Replica free energy:	$F(n) = -n^{-1} \log Z(n)$
Energy:	$E(\beta) = -\partial_\beta \log Z(\beta)$	Replica energy:	$E(n) = -\partial_n \log Z(n)$
Thermal entropy:	$S(\beta) = \beta^2\, \partial_\beta F(\beta)$	Refined Rényi entropy:	$\tilde{S}^{(n)} = n^2\, \partial_n F(n)$
Heat capacity:	$C(\beta) = -\beta\, \partial_\beta S(\beta)$	Capacity of entanglement:	$C^{(n)} = -n\, \partial_n \tilde{S}^{(n)}$

Capacity of entanglement

- The "thermal" entropy is *not* the Rényi entropy

$$S_n = -\frac{1}{n-1}\log Z(\beta) = \frac{n}{n-1}\,F(\beta) \neq \beta^2\,\partial_\beta F(\beta)$$

but a refined one (improved Rényi/modular entropy [Dong 16, Nakaguchi-TN 16]):

$$\tilde{S}^{(n)} \equiv \beta^2\,\partial_\beta F(\beta) = n^2\,\partial_n\left(\frac{n-1}{n}\,S_n\right)$$

- The capacity of entanglement [Yao-Qi 10] is non-negative for a unitary theory:

$$C^{(n)} \equiv -n\,\partial_n \tilde{S}^{(n)} = n^2 \langle\, (H_A - \langle\, H_A\,\rangle_n)^2\,\rangle_n \geq 0$$

where $\langle\, X\,\rangle_n \equiv \mathrm{Tr}_A\left[X\,e^{-n\,H_A}\right]/Z(\beta)$

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2d dilaton gravity + large-c CFT

- A general dilaton gravity in two dimensions ($4G_N = 1$):

$$I = I_{\text{grav}} - \log Z_{\text{CFT}} \ , \qquad I_{\text{grav}} = I_{\text{EH}} + I_{\text{dil}}$$

  - Einstein term (topological):

$$I_{\text{EH}} = -\frac{S_0}{4\pi}\int_{\Sigma_2}\mathcal{R} - \frac{S_0}{2\pi}\int_{\partial\Sigma_2}\mathcal{K} = -S_0\,\chi[\Sigma_2]$$

  - Dilaton term (constraint):

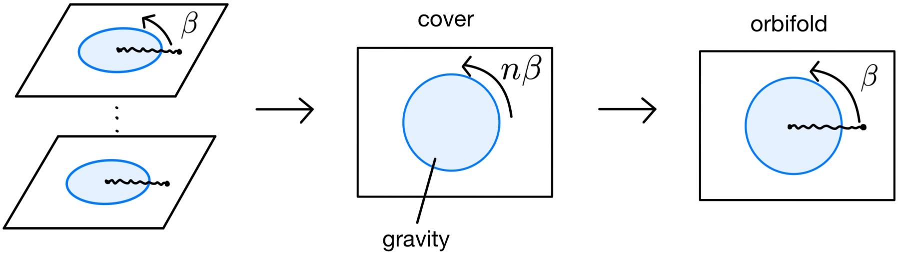
$$I_{\text{dil}} = -\frac{1}{4\pi}\int_{\Sigma_2}\Phi\,\left[\mathcal{R} + U(\Phi)(\nabla\Phi)^2 + V(\Phi)\right] - \frac{\Phi_b}{2\pi}\int_{\partial\Sigma_2}\mathcal{K}$$

- For JT gravity on AdS_2:

$$\Phi = \phi \ , \quad U = 0 \ , \quad V = 2$$

Replica calculation in gravitational path integral

- Two descriptions of the replica geometry [Lewkowycz-Maldacena 13]:
  - n-fold cover $\widetilde{\mathcal{M}}_n$: no singularity in gravity region
  - orbifold $\mathcal{M}_n \equiv \widetilde{\mathcal{M}}_n / \mathbb{Z}_n$: conical singularities at $\mathbb{Z}_n$ fixed points in the gravity region



Replica partition function

- The on-shell actions related by [Dong 16, Nakaguchi-TN 16]

$$\frac{1}{n} I_{\text{grav}}[\widetilde{\mathcal{M}}_n] = I_{\text{grav}}\,[\mathcal{M}_n] + \left(1 - \frac{1}{n}\right) \underbrace{\mathcal{A}^{(n)}}_{\text{localized on singularities}}$$

  - The "area" term:

$$\mathcal{A}^{(n)} = \sum_i \left[S_0 + \Phi^{(n)}(w_i)\right] \qquad w_i : \text{conical singularities in gravity region}$$

- In the semiclassical limit, $1 \ll c \ll 1/G_N$:

$$\begin{aligned} -\frac{1}{n} \log \operatorname{Tr} \rho^n &= \frac{1}{n} I_{\text{grav}}[\widetilde{\mathcal{M}}_n] - \frac{1}{n} \log Z_{\text{CFT}}[\widetilde{\mathcal{M}}_n] \\ &= I_{\text{grav}}[\mathcal{M}_n] + \left(1 - \frac{1}{n}\right) \mathcal{A}^{(n)} - \frac{1}{n} \log Z_{\text{CFT}}[\widetilde{\mathcal{M}}_n] \end{aligned}$$

Island and capacity formulas in 2d dilaton gravity

- The island formula [Almheiri-Maldacena-Hartman-Shaghoulian-Tajdini 19, Penington-Shenker-Stanford-Yang 19]:

$$S \equiv \frac{1}{1-n} \log \operatorname{Tr} \rho^n \Big|_{n=1} = \sum_{i \in \partial \Sigma_I} [S_0 + \Phi(w_i)] + S_{\mathsf{CFT}}(w_i)$$

- The capacity formula [Kawabata-TN-Okuyama-Watanabe 21]:

$$C \equiv C^{(1)} = -2\partial_n \left(\frac{1}{1-n} \log \operatorname{Tr} \rho^n \right) \Big|_{n=1} = - \sum_{i \in \partial \Sigma_I} \partial_n \Phi^{(n)}(w_i) \big|_{n=1} + C_{\mathsf{CFT}}(w_i)$$

Implication of the formula: discontinuity at Page time

- When there are two competing saddle solutions (with and without island)
  - The island formula picks up the dominant saddle with least entropy:

$$S = \min_{\Sigma_I} \operatorname{ext}_{\Sigma_I} \left[\sum_{i \in \partial\Sigma_I} [S_0 + \Phi(w_i)] + S_{\mathsf{CFT}}(w_i) \right]$$

  - Thus the entropy is continuous at the Page time: $\Delta S \equiv S_{\text{no-island}} - S_{\text{island}}\big|_{\text{Page}} = 0$
  - The capacity measures the fluctuation around a given saddle:

$$C = - \sum_{i \in \partial\Sigma_I} \partial_n \Phi^{(n)}(w_i)\Big|_{n=1} + C_{\mathsf{CFT}}(w_i) \qquad \text{for } \Sigma_I \text{ fixed by the island formula}$$

  - The capacity is not necessarily continuous and is expected to show a discontinuity:

$$\Delta C \equiv C_{\text{no-island}} - C_{\text{island}}\big|_{\text{Page}} \neq 0$$

Outline

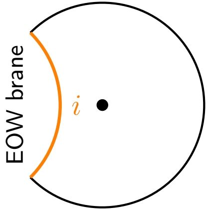
EOW brane model [Penington-Shenker-Stanford-Yang 19]

- A quantum mechanical model of a radiating black hole:

$$|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle_B \, |i\rangle_R \,, \qquad |\psi_i\rangle_B : \text{BH microstates} \,, \quad |i\rangle_R : \text{Hawking radiation}$$

- EOW brane model: $I = I_{\text{JT}} + \mu \int_{\text{brane}} \mathrm{d}s$ Branes carry internal DOF i

$$|\psi_i\rangle_B \, |i\rangle_R \iff$$



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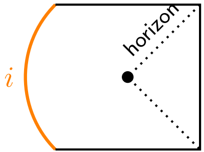


Euclidean

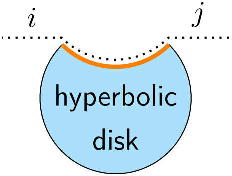
Lorentzian

Replica wormhole

- The reduced density matrix of R:

$$\rho_R = \frac{1}{k} \sum_{i,j=1}^{k} \langle \psi_i | \psi_j \rangle_B \, |i\rangle\langle j|_R$$

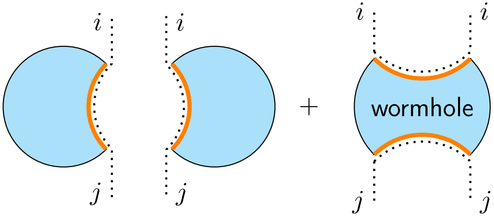
$\langle \psi_i | \psi_j \rangle_B =$



$= Z_1 \, \delta_{ij}$

- $^{\exists}$replica wormhole: $\langle \psi_i | \psi_j \rangle_B = \delta_{ij} + e^{-S_0/2} \, R_{ij}$ (R_{ij}: random variable)

$\left| \langle \psi_i | \psi_j \rangle_B \right|^2 =$



$= Z_1^2 \, \delta_{ij} + Z_2$

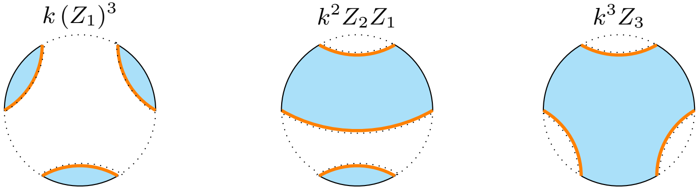
Planar approximation

- In the planar limit, $e^{S_0} \gg 1$ with $k\, e^{-S_0}$ fixed

$$\mathrm{Tr}_R[\rho_R^n] \approx \frac{1}{k^{n-1}} \left[1 + \binom{n}{2} \cdot \frac{k\, Z_2}{(Z_1)^2} + \cdots + \frac{k^{n-1} Z_n}{(Z_1)^n} \right]$$

$Z_n (\propto e^{S_0})$: replica wormhole partition function of disk topology with n boundaries

Example $(n = 3)$:



fully disconnected   partly connected   fully connected

Entanglement entropy at early and late times

- $\dim \mathcal{H}_R = k \iff \#$ of radiation particles $\approx \log k$
- $\log k$: time of BH evaporation
  - **Early time** $(\log k \ll S_0)$: fully disconnected solution dominates

$$\mathrm{Tr}_R[\rho_R^n] \approx \frac{1}{k^{n-1}} \qquad \Rightarrow \qquad S_R \approx \log k$$

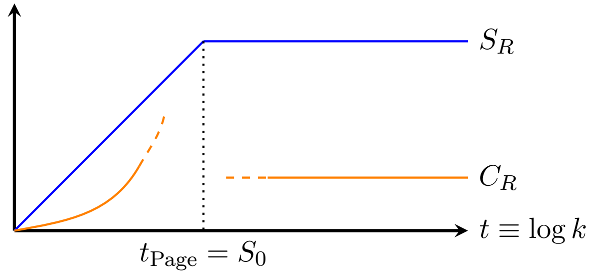
  - **Late time** $(\log k \gg S_0)$: fully connected solution dominates

$$\mathrm{Tr}_R[\rho_R^n] \approx \frac{Z_n}{(Z_1)^n} \qquad \Rightarrow \qquad S_R \approx \lim_{n \to 1} (1 - \partial_n) \log Z_n$$

Capacity and Page curve

- The asymptotic behavior of the capacity:

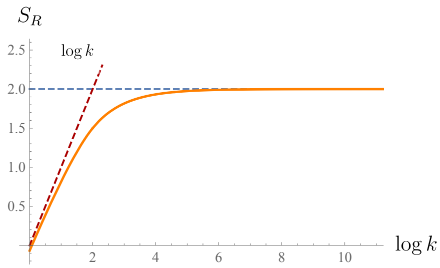
$$C_R \approx \begin{cases} k\,\frac{Z_2}{(Z_1)^2} \propto e^{\log k} & \textbf{(early time)} \\ \lim_{n\to 1} \partial_n^2 \log Z_n & \textbf{(late time)} \end{cases}$$

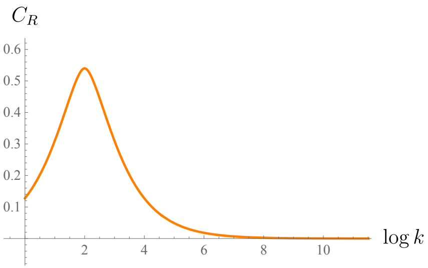


What happens for the capacity around the Page time?

Microcanonical ensemble

- Replica partition functions Z_n can be solved analytically in the microcanonical ensemble by fixing the energy of BH (in planar limit):
  - Entanglement entropy reproduces the Page curve for an eternal BH
  - The capacity shows a peak around the Page time and decays to zero at late time

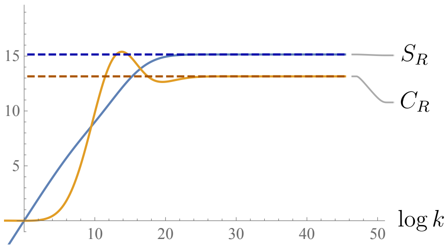




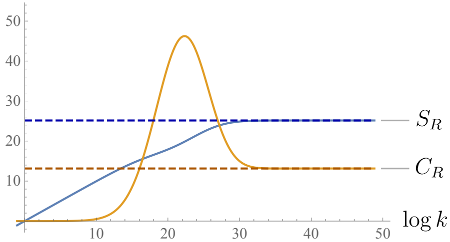
Canonical ensemble

- Numerically calculate Z_n in the canonical ensemble by fixing the inverse temperature β of BH (in planar limit):
  - Entanglement entropy reproduces the similar Page curve as in the microcanonical ensemble
  - The capacity shows a peak around the Page time and approaches to a constant at late time

$\beta = 3, \mu = 5, S_0 = 5$



$\beta = 3, \mu = 5, S_0 = 15$



Summary and future direction

- The capacity of entanglement can be a good probe of Hawking radiation
  - The capacity formula implies a discontinuity at the Page time
  - EOW model: sensitive to the dominant replica wormhole saddle, dependent on the choice of ensembles

- Future problems
  - Capacity formula in higher dimensions
  - Application of the formula to realistic black holes