The nilpotent cone in rank one and minimal surfaces

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The two moduli spaces

Joint work with 李琼玲 (Qiongling Li) and Andrew Sanders.

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$\mathcal{N}_X(G)$: The nilpotent cone in the moduli space of (semi-stable) G-Higgs bundles on X.

$\mathcal{B}_\Sigma(\mathbb{H}^3)$: The moduli space of equivariant branched minimal immersions from $\widetilde{\Sigma}$ to $\mathbb{H}^3$.

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The **Hitchin fibration**:

$$H : \mathcal{M}_X(G) \ni (E, \ldots, \varphi) \longrightarrow \mathrm{tr}(\varphi^2) \in H^0(X, K^2)\,.$$

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The **Hitchin fiber**: for $q_2 \in H^0(X, K^2)$,

$$H^{-1}(q_2) \subset \mathcal{M}_X(G)$$

a half-dimensional Lagrangian subvariety.

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The **nilpotent cone**, when $q_2 = 0$,

$$\mathcal{N}_X(G) := H^{-1}(0) .$$

This is the most singular fiber.

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Mirror symmetry:
- $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$ are **Langlands dual groups**.
- $\mathcal{M}_X(SL(2, \mathbb{C}))$ and $\mathcal{M}_X(PSL(2, \mathbb{C}))$ are mirror dual spaces.

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Topology of the moduli space:

$$\mathcal{N}_X(G) \hookrightarrow \mathcal{M}_X(G)$$

this inclusion is a homotopy equivalence

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Minimal surfaces

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$$f : \widetilde{\Sigma} \longrightarrow \mathbb{H}^3 \,,$$

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$\mathcal{B}_\Sigma(\mathbb{H}^3)$: The moduli space of such pairs.

Minimal surfaces and nilpotent cones

The pull back of the hyperbolic metric of $\mathbb{H}^3$ induces a conformal structure on Σ.
This gives a map to the Teichmüller space $\mathcal{T}(\Sigma)$:

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Denote the fiber over $X \in \mathcal{T}(\Sigma)$ by $\mathcal{B}_X(\mathbb{H}^3)$.

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$\mathcal{B}_X(\mathbb{H}^3)$ is “more or less” the nilpotent cone $\mathcal{N}_X(SO(3,\mathbb{C}))$.

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$\mathcal{B}^d_X(\mathbb{H}^3)$ the subset of the (f, ρ) with Euler number d.
A **stratum** of $\mathcal{B}_X$.

Strata

We can understand the strata: the map

$$\mathcal{B}^d_X(\mathbb{H}^3) \ni (f,\rho) \longrightarrow D \in \mathrm{Symm}^{2g-2-d}(X)\,.$$

is a vector bundle of rank $g-1+d$.

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Minimal surfaces inside quasi-Fuchsian hyperbolic manifolds are here.

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We explicitly describe the topology of $\mathcal{B}_X(\mathbb{H}^3)$.

E.g. we prove that $\mathcal{B}_X(\mathbb{H}^3)$ has two connected components (even *d* and odd *d*).

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$\mathcal{B}_X^d(\mathbb{H}^3)$ is a vector bundle over $\mathcal{B}_X^d(\mathbb{H}^2)$.

$SO(3,\mathbb{C})$-Higgs bundles

Assume from now on that $G = SO(3,\mathbb{C})$.
Describe the elements of $\mathcal{M}_X(SO(3,\mathbb{C}))$:

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- (Semi-stability) For all B-isotropic φ-invariant line sub-bundle $L \subset E$, $\deg L \leq 0$.

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When is (E,ω,B,φ) in $\mathcal{N}_X(SO(3,\mathbb{C}))$?

2 cases:

- When $\varphi = 0$. Here,
  $(E,\omega,B,\varphi) = (E,\omega,B,0) \in \mathcal{M}_X(SO(3))$.
  We define the **Euler number** $d := 0$.

Nilpotent $SO(3,\mathbb{C})$-Higgs bundles

For $SO(3,\mathbb{C})$-Higgs bundles, $\det(\varphi) = 0$.
φ is either 0 or generically of rank 2.

If $\varphi \neq 0$, denote by $\ker \varphi \subset E$ the unique line sub-bundle s.t.

$$\varphi|_{\ker \varphi} = 0 .$$

When is (E, ω, B, φ) in $\mathcal{N}_X(SO(3,\mathbb{C}))$?

2 cases:

- When $\varphi = 0$. Here, $(E, \omega, B, \varphi) = (E, \omega, B, 0) \in \mathcal{M}_X(SO(3))$. We define the **Euler number** $d := 0$.
- When $\varphi \neq 0$ and $\ker \varphi$ is B-isotropic. We define the **Euler number** $d := -\deg(\ker \varphi)$, with

$$1 \leq d \leq 2g - 2 .$$

The strata

Denote by $\mathcal{N}_X^d(G)$ the subset of Higgs bundles with Euler number d.

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For $1 \leq d \leq 2g-2$, $\mathcal{N}_X^d(G)$ is not closed.

Explicit description of the non-closed strata

Fix $1 \leq d \leq 2g - 2$.
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The zero section ($\beta = 0$) is the sub-space of $SO(2,1)$-Higgs bundles:

$$\mathcal{N}_X^d(SO(2,1)) \simeq \mathrm{Symm}^{2g-2-d}(X)\,.$$

This is also the sub-space of the variations of Hodge structure.

Minimal immersions and nilpotent cone

Using our parameters, we can write the Hitchin's equations and understand their solutions.

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$\mathcal{N}_X^0(SO(3,\mathbb{C}))$ would correspond to pairs (f,ρ), where f is constant and ρ goes to $SO(3)$.

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This will tell us the shape of $\mathcal{N}_X(G)$.

Theorem

Let $1 \leq d', d \leq 2g - 2$.

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$$\partial \mathcal{N}_X^d(G) \cap \mathcal{N}_X^{d'}(G) \neq \emptyset$$

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$$d > d' \qquad \textit{AND} \qquad d - d' \textit{ is even} \ .$$

In other words, when the parameter β goes to ∞ in $\mathcal{N}_X^d(G)$, you can only converge to a stratum with smaller Euler number, and only with an even difference.

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We define

$$S_{d,d'} = \left\{ 2T + D \mid T \in \mathrm{Symm}^q(X), D \in \mathrm{Symm}^{2g-2-d}(X) \right\} ,$$

with $S_{d,d'} \subset \mathrm{Symm}^{2g-2-d'}(X)$. *Then*

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$$\partial \mathcal{N}_X^d(G) \cap \mathcal{N}_X^{d'}(G) = p^{-1}(S_{d,d'}),$$

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In other words, when converging, the new branching points come with even multiplicity, and all candidate limits are achieved.

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Suppose $(f_n, \rho_n) \in \mathcal{B}_X(\mathbb{H}^3)$ *is a sequence with fixed branch type* $(n_1, \ldots, n_k)$. *Then, up to extracting a subsequence, one of the following occurs.*

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1. (f_n, ρ_n) *converges to a pair* (f, ρ) *of branch type* $(n_1 + 2m_1, \ldots, n_k + 2m_k, 2m_{k+1}, \ldots, 2m_{k+s})$*, with* $m_i \geq 0$*.*

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Moreover, every branched minimal immersion of the kind described in point 1 can arise as a limit.

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As a slogan, new branching points are created with even multiplicity, and every candidate limit is reachable.

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As a slogan, every even branching can be perturbed away.

A special case: the $\mathbb{C}^*$-flow

The $\mathbb{C}^*$-flow is an action of $\mathbb{C}^*$ on $\mathcal{M}_X(G)$:

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This is the “generic case”.

$\mathbb{C}^*$-flow – Unstable case

If E is unstable, it contains a φ-invariant B-isotropic subbundle M, with $\deg(M) = d' > 0$.

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Simpson showed that $d' < d$, and $\exists a \in H^0(X, KM^{-1})$ such that the limit Higgs bundle is parametrized by $(M, a, 0) \in \mathcal{N}_X^{d'}(SO(2,1))$.

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$$E_0 = M \oplus \mathcal{O} \oplus M^{-1}, \qquad \overline{\partial}_{E_0} = \overline{\partial},$$

$$\omega_0 = 1 \in H^0(X, \mathcal{O}), \quad B_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varphi_0 = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & -a & 0 \end{pmatrix}.$$

$\mathbb{C}^*$-flow – Our results

We computed that $a = cb^2$, for some holomorphic section b. In particular, the new branching has even order, and $d - d'$ is even.

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Moreover, all possible candidate limits are realized: Given $(M, a, 0) \in \mathcal{N}_X^{d'}(SO(2,1))$, we can describe all the $\mathbb{C}^*$-orbits that converge to $(M, a, 0)$ (the **unstable manifold** of $(M, a, 0)$).

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The unstable manifold of $(M, a, 0)$ is usually not irreducible, it has an irreducible component U_b for every divisor b such that $b^2 < a$.

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Then U_b *is the union of a family of* $\mathbb{C}^*$*-orbits, where* $\lim_{t \to 0}$ *is* $(M, a, 0)$ *and* $\lim_{t \to \infty}$ *is* $(L, c, 0)$. *This family is parametrized by*

$$\prod_{i=1}^{k} (\mathbb{C}^* \times \mathbb{C}^{n_i - 1}) .$$