Conformal Bootstrap:
Non-perturbative QFT's under siege

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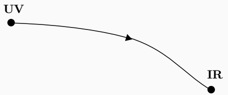
Iberian Strings 2017

Ubiquitous CFT's

- CFT's are the building blocks of quantum field theories: they are signposts in the space of quantum theories



- Promising way to lean more about quantum gravity via AdS/CFT correspondence
- Many real physical systems where scale invariance shows up

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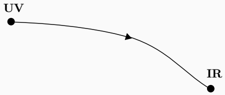
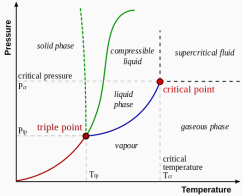


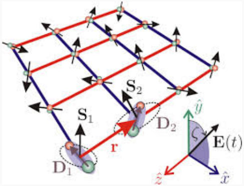
- Promising way to lean more about quantum gravity via AdS/CFT correspondence
- Many real physical systems where scale invariance shows up

• Condensed matter:

critical point of many fluids,



Superfluid ^{4}He $\to$ $O(2)$-model



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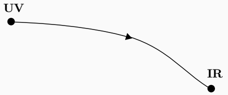

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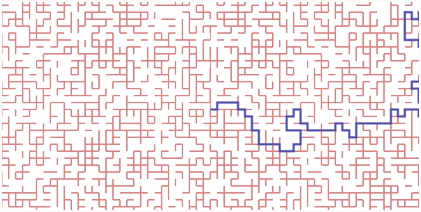
- Percolation$\rightarrow$ $Q = 1$ Potts model



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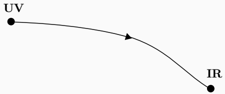

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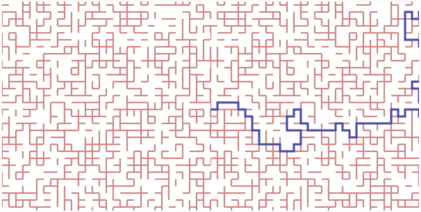
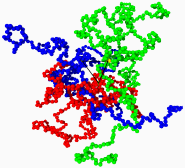


- Polymers → Self-avoiding random walk (O($N = 0$) vector model)



Nature is perverse...

- In 2D, great results have been achieved thanks to a richer structure (infinite dimensional symmetry)
- In D>2, general believe that no quantitative result could be obtained...
  - perturbative expansion (in couplings, dimensions, number of fields,...)
  - supersymmetry
  - strong/weak dualities (AdS/CFT, ...)

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- In 2D, great results have been achieved thanks to a richer structure (infinite dimensional symmetry)
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Many real physical systems do not fall in the regime of validity of any of the above assumptions.

Ex: $O(N)$ vector model:

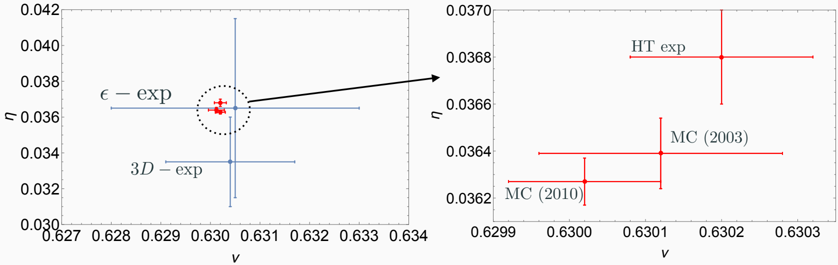
- $N = 1$ (Ising model): Ferromagnetism, Vapor to liquid transition,...
- $N = 2$ (XY model): superfluid ^{4}He,
- $N = 3$ (Heisenberg model): magnetism

Genuine, non-supersymmetric, small-N, non perturbative systems.

Tools at disposal

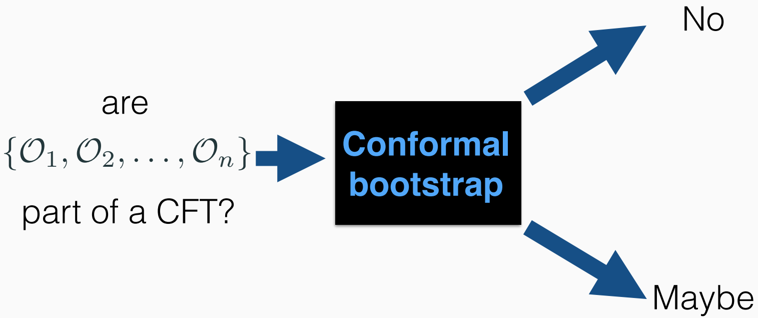
Alternative approaches to non-perturbative theories: how to compute critical exponents (η,ν,...)?

- More exotic field theory approaches
- MonteCarlo simulations (spin systems)



- QFT on the lattice
- Truncated Conformal Space Approach
- .....

The conformal bootstrap aims to develop a systematic and rigorous method to study the properties of conformal invariant fixed points in any dimension, complementary to existing techniques.



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Conformal bootstrap main ideas

What is a CFT?

Theories invariant under the *conformal algebra* $SO(D|2)$ which includes:

- translations
- Lorentz transformations
- dilatations
- "inversion"

They are described by three ingredients:

1) Operator content: representations totally characterized by scaling dimension Δ and spin ℓ of the primary (lowest dimension operator). All other are called descendants.

2) Interactions between operators: encoded in the Operator Product Expansion (OPE)

$$\mathcal{O}_{\Delta_1} \times \mathcal{O}_{\Delta_2} \sim \sum_{\mathcal{O}} C_{12\mathcal{O}} \underbrace{\left(\mathcal{O}_{\Delta,\ell} + \text{descendants}\right)}_{\text{fixed by conformal symmetry}}$$

$C_{12\mathcal{O}}$ are called OPE coefficients

3) Crossing symmetry constraints: see next slides...

The power of conformal invariance

In CFT we are interested in computing correlations functions $\langle \mathcal{O}_i(x_1)...\mathcal{O}_j(x_n)\rangle$:

Two point functions of primaries: completely fixed

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{x_{12}^{2\Delta_i}} \qquad x_{12} \equiv |x_1 - x_2| \qquad \Delta_i = [\mathcal{O}_i]$$

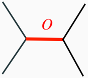
Three point functions of primaries: fixed modulo a constant.
Use OPE to reduce higher point functions to smaller ones

$$\langle \underbracket{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_3\rangle \propto \begin{cases} C_{123} \underbrace{(\langle \mathcal{O}_3\mathcal{O}_3\rangle + \text{descendants} \ )}_{\text{fixed by conformal symmetry}} & \text{if } \mathcal{O}_3 \in \mathcal{O}_1 \times \mathcal{O}_2 \\ 0 & \text{otherwise} \end{cases}$$

Four point functions

Use OPE to reduce higher point functions to smaller ones:

$$\langle \underbrace{\mathcal{O}(x_1)\mathcal{O}(x_2)}\underbrace{\mathcal{O}(x_3)\mathcal{O}(x_4)}\rangle \sim \sum_{\mathcal{O}}$$



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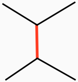

Crossing symmetry is the statement that the two expansions must give the same result!

Definition of a CFT:

A Conformal Field Theory is an infinite set of primary operators $\mathcal{O}_{\Delta,\ell}$ and OPE coefficients C_{ijk} that satisfy crossing symmetry for all set of four-point functions.

Four point functions (more in details)

Recalling the OPE

$$\mathcal{O}(x_1) \times \mathcal{O}(x_2) = \sum_{\mathcal{O}'} \frac{C_{\mathcal{O}'}}{x_{12}^{2d-\Delta}} (\mathcal{O}'_{\Delta,\ell} + \text{descendants}) \qquad\qquad d = [\mathcal{O}]$$

Four point functions (more in details)

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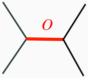
$$\langle \underbrace{\mathcal{O}(x_1)\mathcal{O}(x_2)} \underbrace{\mathcal{O}(x_3)\mathcal{O}(x_4)} \rangle \sim \sum_{\mathcal{O}}$$



$$= \frac{u^{-d}}{(x_{13}^{2d} x_{24}^{2d})} \sum_{O'_{\Delta,l}} C^2_{\mathcal{O}'} \underbrace{\left(\langle O'_{\Delta,\ell} O'_{\Delta,\ell} \rangle + \text{descendants} \right)}_{\text{function of } u,\, v \text{ only by conformal symmetry}}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Four point functions (more in details)

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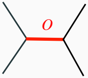

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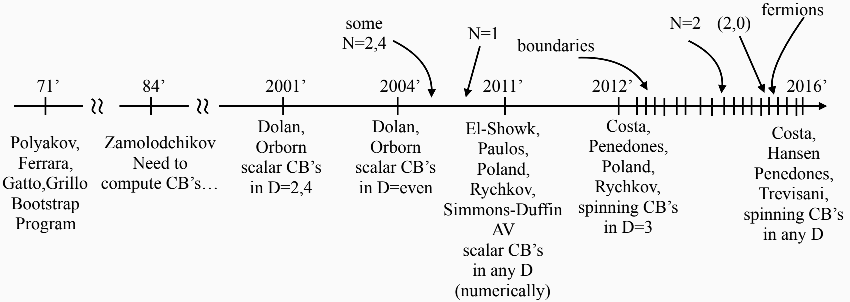
Conformal Blocks:

$$g_{\Delta,l}(u,v) \equiv \langle O'_{\Delta,\ell} O'_{\Delta,\ell} \rangle + \text{descendants}$$

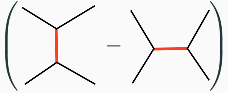
They sum up the contribution of an entire representation

Conformal blocks (CB's) gold rush



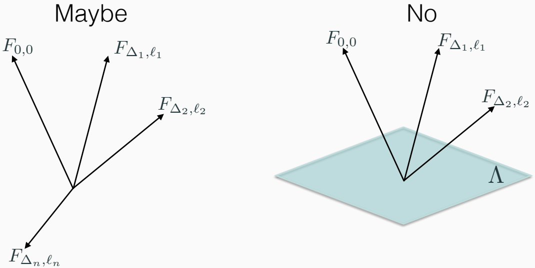
The Bootstrap program

- Crossing equation for $\langle O(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$:



$$\sum_{\Delta,\ell} C^2_{\Delta,\ell} \underbrace{\left(\;\cdots\; - \;\cdots\; \right)}_{\text{Known functions } F_{\Delta,\ell}} \equiv \sum_{\Delta,l} C^2_{\Delta,l} \underbrace{\left(u^{-d} g_{\Delta,\ell}(u,v) - v^{-d} g_{\Delta,\ell}(v,u) \right)}_{F_{d,\Delta,\ell}} = 0$$

- Unitarity: $C^2_{\Delta,\ell} \geq 0$



Existence of Λ can be recast into a *linear (or semi-definite) programming problem* and checked numerically. [Rattazzi,Rychkov,Tonni, AV] 2008

A toy model:

Ising Model:

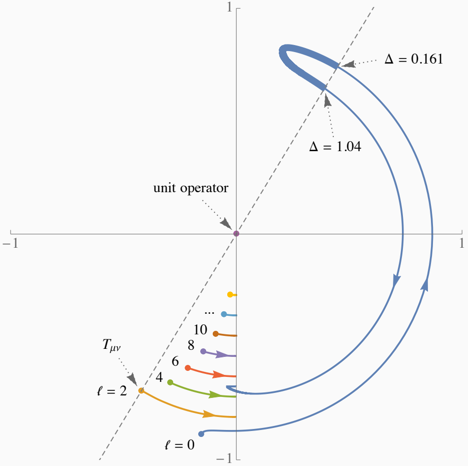
$$\sigma \times \sigma \sim 1 + \sum_{\mathcal{O}_{\Delta,\ell}} C_{\Delta,\ell} \mathcal{O}_{\Delta,\ell} + ...$$

Crossing symmetry of $\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$ implies:

$$\sum_{\Delta,\ell} C^2_{\Delta,\ell} F_{\Delta,\ell}(u,v) = 0$$

with:

- $\Delta_\sigma = 1/8$
- $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{24}^2}{x_{13}^2 x_{24}^2}$
- Project crossing constraint on *ad hoc* plane (linear combination of crossing constraint evaluated at 3 different points)



The Bootstrap program

Rules of the game:

- Choose one or more operators $\mathcal{O}_1, \mathcal{O}_2, ...$
- Consider all four point functions containing those operators $< \mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1 >, < \mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2 >, ...$
- Make assumptions on the operators (and coefficients) appearing in the OPE's $\mathcal{O}_i \times \mathcal{O}_j$
- Check numerically if assumptions made are consistent with crossing symmetry
- If not consistent: there is no CFT with that operator content (crossing symmetry is a necessary condition)

A few applications

Comparison with 2D results

Minimal models: family of 2D CFT's completely solved:

$$\sigma \times \sigma \sim 1 + \epsilon + ...$$

... contains:

- Other Virasoro primaries
- Virasoro Descendants
- Conformal descendants

Consider the plane Δ_σ, Δ_ϵ:

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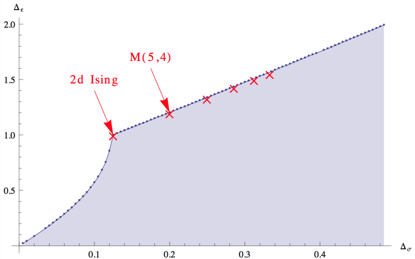


Bound on maximal value of Δ_ϵ

A kink signals the presence of the Ising Model

[S.Rychkov, AV 2009]

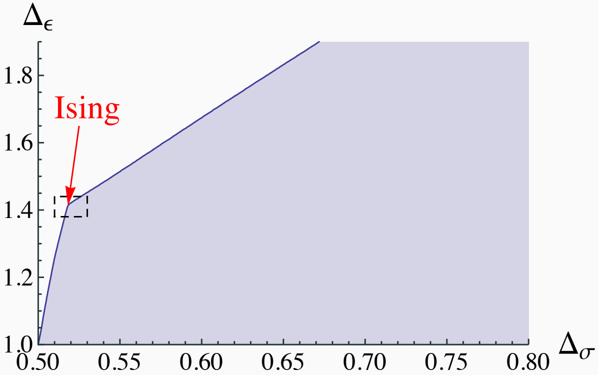
Important

No use of Virasoro algebra. Extend the method to 3D right away

A surprise in 3D

Choose:
σ = spin field with dimension Δ_σ ($\to \eta$)
ϵ = energy density with dimension Δ_ϵ ($\to \nu$)

$$\begin{array}{ccll} \sigma \times \sigma & \sim & 1 + \epsilon + \epsilon' + \ldots. & \mathbb{Z}_2 - \text{even} \\ \sigma \times \epsilon & \sim & \sigma + \sigma' + \ldots. & \mathbb{Z}_2 - \text{odd} \\ \epsilon \times \epsilon & \sim & 1 + \epsilon + \epsilon' + \ldots. & \mathbb{Z}_2 - \text{even} \end{array}$$



[El-Showk,Paulos,Poland,Rychkov,Simmons-Duffin, AV] 2011

Let us input a few assumptions

Choose:
σ = spin field with dimension Δ_σ $(\to \eta)$
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$$
\begin{aligned}
\sigma \times \sigma &\sim 1 + \epsilon + \epsilon' + \ldots. \quad \mathbb{Z}_2 - \text{even} \\
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\epsilon \times \epsilon &\sim 1 + \epsilon + \epsilon' + \ldots. \quad \mathbb{Z}_2 - \text{even}
\end{aligned}
$$

So far we have assumed anything about the CFT besides unitarity.

Additional assumptions:

- In the Ising model one needs to tune only the temperature in order to flow to the IR fixed point:
  only one relevant $\mathbb{Z}_2$ − even deformation: ϵ
- equation of motions predicts $\Box\sigma \sim \sigma^3$:
  second magnetic perturbation $\sim \sigma^5 \Rightarrow$ irrelevant

3D Ising Model: the triumph of conformal bootstrap

Choose:
$\sigma =$ spin field with dimension Δ_σ $(\to \eta)$
$\epsilon =$ energy density with dimension Δ_ϵ $(\to \nu)$

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Use $< \sigma\sigma\sigma\sigma >, < \sigma\sigma\epsilon\epsilon >, < \epsilon\epsilon\epsilon\epsilon >$

Assume only σ and ϵ have dimension smaller than 3: allowed values for Δ_σ, Δ_ϵ?

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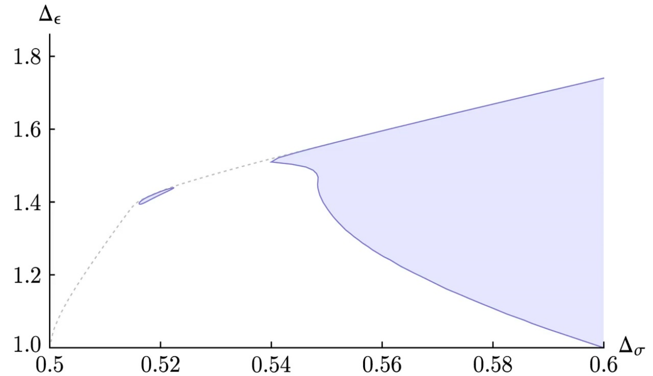

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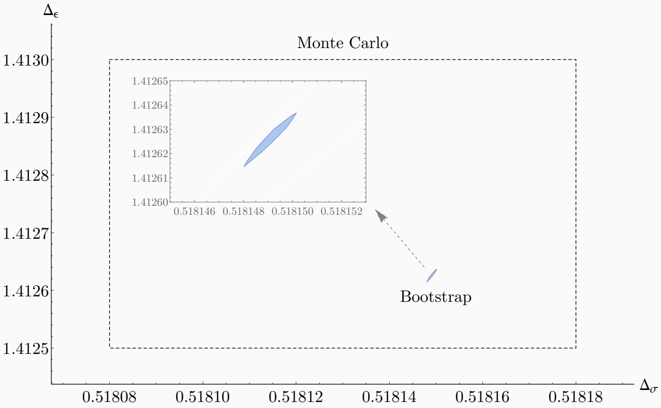
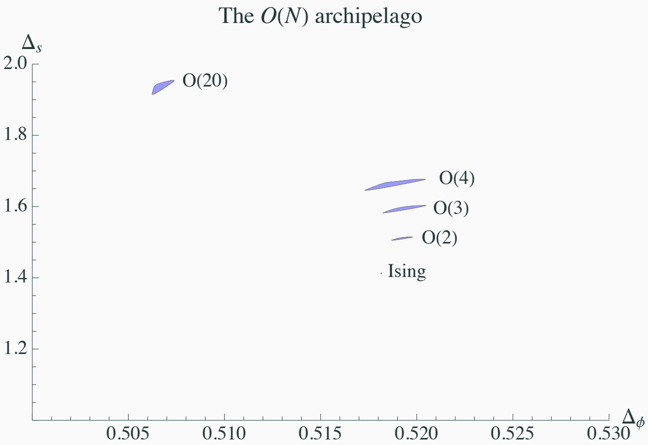


[Poland,Simmons-Duffin,AV, '11]
[El-Showk,Paulos,Poland,Rychkov, Simmons-Duffin,AV, '12 & '14]
[Poland,Simmons-Duffin, Kos '14]
[Simmons-Duffin, '15]
[Poland,Simmons-Duffin,Kos,AV, '16]

An island for any N



[Kos,Poland,Simmons-Duffin,AV] 2015

Going Beyond

Conserved currents

Universal handle on any CFT: need to study universal operators in the theory.

- Conserved global symmetry currents J^{μ}
- Energy momentum tensor $T^{\mu\nu}$

Two possible strategies:

1. Use supersymmetry to relate correlations functions of conserved currents to those of the scalar belonging to the same super-multiplet
2. Study correlation functions of spinning operators

Super Conformal Field Theories

- $\mathcal{N} = 1, D = 4$
  real multiplet $\mathcal{J}^A \subset J^A, J^A_\mu$
  [Stergiu, in progress]
- $\mathcal{N} = 2, D = 4$
  semi-short multiplet $\hat{\mathcal{C}}_{0,(0,0)} \supset \phi, J^{(R)}_\mu, J^{(r)}_\mu, T_{\mu\nu}$
  [Beem, Lemos, Liendo, Rastelli, van Rees]
- $\mathcal{N} = 3, D = 4$
  semi-short multiplet $\hat{\mathcal{B}}_{[R,0]} \supset \phi, T_{\mu\nu}$
  [Lemos, Liendo, Meneghelli, Mitev]
- $\mathcal{N} = 4, D = 4$
  1/2 BPS multiplet $\mathcal{B}_{[0,2,0]} \supset \phi^{IJ}, J^{(R)}_\mu, J^{(r)}_\mu, T_{\mu\nu}$
  [Beem, Rastelli, van Rees]
- $(2,0), D = 6$
  1/2 BPS multiplet $\mathcal{D}[2,0] \supset \phi^{ij}, T_{\mu\nu}$
  [Beem, Lemos, Rastelli, van Rees]

See Madalena's talk for a detailed application.

A window to all CFTs (with a global symmetry)

Goal: study bootstrap equations in 3D for $\langle J_\mu J_\nu J_\rho J_\sigma \rangle$, with J_μ a conserved current.

Size of the problem:

	Generic		Equal & conserved	
	+	-	+	-
3pf	5	4	2(ℓ even)	1 ($\ell \neq 1$)
4pf	41		17	

- Parametrization

$$< J^\mu(x_1) J^\nu(x_2) J^\rho(x_3) J^\sigma(x_4) > = \sum_{k=1}^{41} \frac{f_i(u,v) \mathcal{Q}_i^{\mu\nu\rho\sigma}(x)}{(x_{12})^2 (x_{34})^2}$$

- Crossing symmetry has the form

$$f_i(u,v) = \sum_{k=1}^{17} M(u,v)_{ij} f_j(v,u)$$

Minimal set of crossing conditions

- Conservation relates the 17 a priori independent structures with 14 differential equations of the form

$$\sum_j \left(K^0_{ij} + K^u_{ij}\partial_u + K^v_{ij}\partial_v \right) f_j(u,v) = 0, \qquad i = 1, ..., 14 \qquad j = 1, ..., 17$$

- By inspecting the Kernel of the matrices K one can show that the minimal and necessary set of information needed to integrate the conservation equations is
  1. 5 functions $\widetilde{f}_i$ of both conformal ratios satisfying $\widetilde{f}_i(u,v) = \pm\widetilde{f}_i(v,u)$
  2. 10 functions $\widetilde{g}_j$ defined at $u = v$
  3. the values of 2 functions $\widetilde{h}_k$ a at $u = v = 1/4$.
- Moreover, the form of conservation equation is such that if the initial conditions satisfies crossing symmetry, the same would be for the integrated solution.

Conformal collider bounds

Gedankenexperiment:

- Switch on a local perturbation $\mathcal{O}_{\Delta,\ell}$ in a CFT. This create a perturbation growing in time and space
- Measure the angular distribution of the flux $\mathcal{E}(\theta)$ at some distance.
- It integral measures the total energy deposited in a "calorimeter"

Formally we want to compute the expectation value of the energy density $\mathcal{E} = T_{--}$ in the vacuum created by the local operator $\mathcal{O}_{\Delta,\ell}|0>$:

$$\langle \mathcal{E}(\theta) \rangle = \frac{\langle O^\dagger_{\Delta,\ell} \mathcal{E}(\theta) O_{\Delta,\ell} \rangle}{\langle O^\dagger_{\Delta,\ell} O_{\Delta,\ell} \rangle}$$

Conformal collider bounds

If we choose $\langle O_{d-1,1} = J_\mu$ the above flux can be computed in terms of the universal CFT data entering the three point function $\langle J_\mu J_\nu T_{\rho\sigma}\rangle$

$$\langle \mathcal{E}(\theta)\rangle \propto \left[1 - 4d(d-1)\gamma \left(\cos^2\theta - \frac{1}{d-1}\right)\right] .$$

Positivity of this energy correlator implies the bounds

$$-\frac{1}{4d} \leq \gamma \leq \frac{1}{4d(d-2)}$$

which are saturated by free scalars and free fermions, respectively.

The parameter γ can be defined in terms of the AdS bulk action:

$$S_{AdS} = C_J \int d^{d+1}x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \gamma R_{AdS}^2 W^{\mu\nu\tau\rho}F_{\mu\nu}F_{\tau\rho}\right]$$

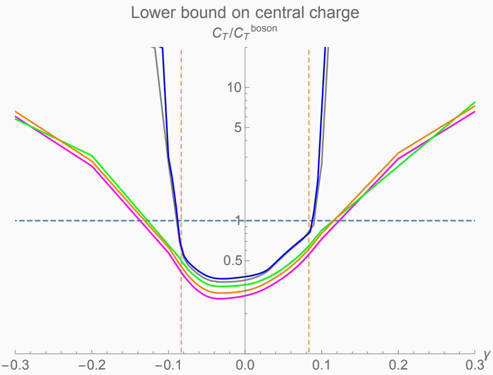
(W is the Weyl tensor)

Conformal collider bounds

Can we find numerical evidences in $d = 3$ of the bound $-\frac{1}{12} \leq \gamma \leq \frac{1}{12}$?

The argument assumes the CFT posses a local Energy momentum tensor
$\Leftrightarrow$ central charge $c_T < \infty$.

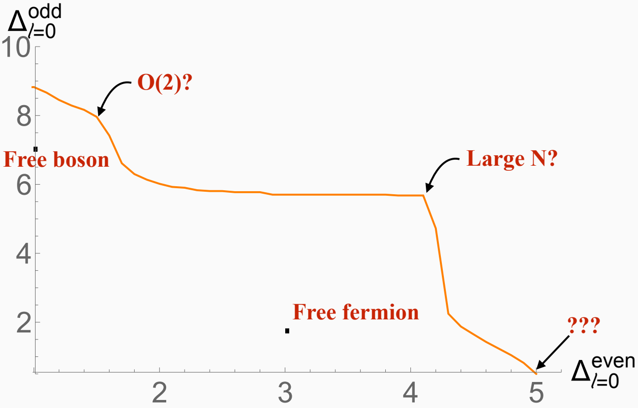
Let us compute numerically a lower bound on the central charge as a function of γ



Colors correspond to increasing computing power. [work in progress, w/ Dymarsky, Penedones, Trevisani]

Scalar operators

The OPE $J_\mu \times J_\nu$ contains both even and odd scalar under Parity. How high can their dimension be in a generic CFT?



Conclusions

Summary of results

- The Conformal bootstrap is carving out the space of conformal field theories in $D \geq 2$.
- Three dimensional CFT's (neglected for 40 years) are now under siege: critical exponents of condensed matter systems can be computed with precision competitive with MC
- The study of conserved currents will open a windows on all unitary CFTs, even allowing to discover theories we know nothing about.