Conformal mass in Lovelock AdS gravity

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- Conclusions

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i) *Variational principle* must be satisfied for suitable boundary conditions.

This is achieved by supplementing the bulk action by appropriate boundary terms such that the on-shell action is stationary:

$$I_{\text{var}}[g] = \int\limits_{M} \mathcal{L}(g) + \int\limits_{\partial M} B$$

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$$I_{\mathrm{var}}[g] = \int_M \mathcal{L}(g) + \int_{\partial M} B$$

ii) *Removal of IR divergences* present in I, δI by adding boundary terms (counterterms) in a way consistent with the previous point:

$$I[g] = I_{\mathrm{var}}[g] + \int_{\partial M} B_{\mathrm{ct}}$$

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- **There are also other methods** that we will not discuss here, such as obtaining the Quasilocal Brown-York Stress Tensor (1993), Holographic Stress Tensor by Henningson and Skenderis (1998), also background-dependent methods – Abbott-Deser procedure (1982) and Hamiltonian method by Henneaux-Teitelboim (1985), etc.

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Basic advantages / disadvantages of these two methods:

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- Comparison of both methods has been done and they match in EH AdS
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We shall start with a summary of both methods.

Ashtekar-Magnon-Das method

- Asymptotic symmetries in AdS space $\leftrightarrow$ Conformal isometries
- Conserved charges are expected to be expressed in terms of the Weyl tensor, $W_{\mu\nu\alpha\beta}$, and the Killing fields, ξ^i, on the boundary

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- [Ashtekar, Magnon, Das '99] showed that any conserved charge in AAdS gravity can be expressed as a surface integral of a quantity involving only **the electric part of the asymptotic Weyl tensor and the asymptotic Killing field**
- [Hollands, Ishibashi, Marolf '05] gave a formal proof

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- [Hollands, Ishibashi, Marolf '05] gave a formal proof
- **Penrose's conformal completion of asymptotically AdS spacetime** brings the boundary at ∞ of AAdS space to a finite distance by performing a conformal transformation $\boxed{\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}}$

$(M, g_{\mu\nu})$ *physical AAdS spacetime,* $g_{\mu\nu}$ *obeys the Einstein eqs.*
$(\tilde{M}, \tilde{g}_{\mu\nu})$ *unphysical spacetime, mathematically convenient*

- The boundary with topology $\mathbb{R} \times S^{D-2}$ is attached to the spacetime

Ashtekar-Magnon-Das method

- **Conformal factor satisfies** $\Omega = 0$, $\nabla_\mu \Omega \neq 0$ on the boundary $\partial\tilde{M}$;

  $\Omega \sim \frac{1}{r}$ is a good radial coordinate for AAdS spaces in the vicinity of $\partial\tilde{M}$

- **Fall-off of the Weyl tensor**

  $\Omega^{4-D}\tilde{W}_{\mu\nu\alpha\beta}$ is smooth on $\tilde{M}$ and vanishes on $\partial\tilde{M}$

  (Global AdS $W^{\mu\nu}_{\alpha\beta} = 0$; Non-vacuum state $W^{\mu\nu}_{\alpha\beta} \propto \frac{GM}{r^{D-1}}$ by dim. analysis)

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- **Electric part of the Weyl tensor =** Weyl projected to the boundary

  $$\tilde{E}^j_i = \frac{1}{D-3}\,\tilde{W}^{j\nu}_{i\mu}\,\tilde{n}^\mu\,\tilde{n}_\nu \quad \text{(trace-free and symmetric)}$$

  x^i = local coordinates at the boundary

  $\tilde{n}_\mu$ = outward-pointing normal to the boundary $(\tilde{n}^2 = -1)$

- **Now we have all elements to write out the AMD formula.**

Ashtekar-Magnon-Das method

- **Conformal mass**

$$\mathcal{H}[\xi] = -\frac{\ell}{8\pi G}\int\limits_{\tilde{\Sigma}_\infty} d^{D-2}x\sqrt{\tilde{\sigma}}\,\tilde{E}_i^j\,\xi^i\,\tilde{u}_j$$

$\tilde{u}_j$	outward-pointing unit normal to the boundary at constant t
$\tilde{\Sigma}_\infty; \tilde{\sigma}_{mn}$	asymptotic boundary of spatial section at constant t; its metric
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- **Conformal mass in terms of the physical metric**

$$\begin{aligned}\mathcal{H}[\xi] &= -\frac{\ell}{8\pi G}\int_{\tilde{\Sigma}_\infty} d^{D-2}x\sqrt{\sigma}\,E_i^j\,\xi^i\,u_j\\ E_i^j &= \frac{1}{D-3}\,W_{i\mu}^{j\nu}\,n^\mu n_\nu\end{aligned}$$

It is a very simple formula in EH AdS space!

We only need E_i^j and ξ^i.

Kounterterm charges

- **Boundary quantities on AAdS boundary placed at $\rho = \text{const}$**

h_{ij}	*boundary metric*
$K_{ij} \sim \partial_\rho h_{ij}$	*extrinsic curvature*
$\mathcal{R}^i{}_{jkl}(h)$	*intrinsic curvature*

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- **Kounterterms regularization method in AdS gravity in $D = d + 1$**

- Consists in **adding a unique surface term to the action** that regularizes Lovelock AdS gravity action and depends explicitly on the **extrinsic curvature**

$$I = I_{\text{bulk}} + c_d \int\limits_{\partial M} d^d x \, B_d \left(h, K, \mathcal{R} \right)$$

- It is covariant and background-independent.
- δI vanishes on-shell thanks to the fall-off of the Weyl tensor (and fixed c_d)
- The (Euclidean) action is finite, conserved charges are finite and their values are correct when evaluated on a solution

Kounterterm charges

- **Equivalent to standard Holographic Renormalization method** (when expanded asymptotically)

$$c_d \int\limits_{\partial M} d^d x \, B_d = \int\limits_{\partial M} B_{\mathrm{GGH}}(h, K) + \int\limits_{\partial M} d^{D-1} x \, \sqrt{-h} \, \mathcal{L}_{\mathrm{ct}} \, (h, \mathcal{R}, \nabla \mathcal{R}) + \cdots$$

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1. Generalized Gibbons-Hawking boundary term ($B_{\mathrm{GGH}} = \sqrt{-h}\, K + \cdots$)
   $\Rightarrow$ Dirichlet boundary condition on h_{ij}

2. Dirichlet counterterms $\mathcal{L}_{\mathrm{ct}}$ do not depend on K_{ij}, obtained by holographic renormalization [Henningson, Skenderis '98]

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- **Charges include the vacuum energy for the global AdS**
  (= Casimir energy for boundary CFT) [Balasubramanian, Kraus '99]

$$E_0 = (-k)^n \, \frac{\text{Vol}(\Sigma_{D-2})}{8\pi G} \, \frac{\ell^{2n-2}(2n-1)!!^2}{(2n)!} \, \delta_{D,2n+1}$$

Kounterterm charges

Boundary term in even dimensions $D = 2n$ [Kofinas, Olea '07]

- Euler theorem

$$\int_M d^{2n}x\,\mathcal{E}_{2n}(R) = \chi(M) + \int_{\partial M} d^{2n-1}x\,B_{2n-1}$$

- B_{2n-1} is a given polynomial in h_{ij}, K_{ij} and $\mathcal{R}_{ij}^{kl}(h)$
- The coupling c_{2n-1} is fixed from the variational problem
- Boundary conditions $\delta K_j^i = 0$ on ∂M
- They match with $\delta h_{ij} = 0$ in ALAdS spacetimes because $K_j^i = \frac{1}{\ell}\,\delta_j^i + \cdots$

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Boundary term in odd dimensions $D = 2n+1$ [Olea '07, Kofinas, Olea '07]

- Transgression form

$$\int_M d^{2n+1}x\, \mathcal{T}_{2n+1}(R) = I_{\mathrm{CS}}[A] - I_{\mathrm{CS}}[\bar{A}] - \int_{\partial M} d^{2n}x\, B_{2n}$$

- I_{CS} is the Chern-Simons action for AdS (A) or Lorentz $(\bar{A})$ group

Comparison of charges in AdS gravity

Conformal mass vs. Kounterterm mass in Einstein-Hilbert AdS gravity

They are in exact agreement up to a definitional difference

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- The AMD charge by construction has vanishing AdS background, hence it does not contain the vacuum energy
- One has to assume an asymptotic fall-off for the Weyl tensor as considered by AMD [Jatkar, Kofinas, Miskovic, Olea '14]
- Holographic renormalization also gives the same vacuum energy, E_0
- Holographic methods contain additional finite terms, scheme-dependent counterterms that do not contribute to the charge
- This comparison has been done only up to five dimensions because of technical difficulties [Miskovic, Olea '06, '07]

Lovelock AdS gravity

- **Lovelock action**

$$I=\frac{1}{16\pi G}\int d^Dx\sqrt{-g}\sum_{p=0}^{N}\alpha_p\mathcal{L}_p(R)$$

- $\mathcal{L}_p$ = polynomial of order p in the Riemann curvature

$$\mathcal{L}_p(R)=\frac{1}{2^p}\,\delta^{\mu_1\cdots\mu_{2p}}_{\nu_1\cdots\nu_{2p}}\,R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}$$

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- $N =$ maximal degree of the curvature, $N = [(D-1)/2]$
- $\alpha_p =$ coupling constants, *e.g.*

$$\alpha_0 = -2\Lambda = (D-1)(D-2)/\ell^2$$
$$\alpha_1 = 1 \qquad \text{(Einstein-Hilbert)}$$
$$\alpha_2 = \alpha \qquad \text{(Gauss-Bonnet)}$$

- In general, α_p are arbitrary, but some restrictions may apply; If GB is coming from String Theory, then $\alpha > 0$; In the context of AdS/CFT, $\{\alpha_p\}$ are restricted by causality in CFT [Camanho, Edelstein '10], etc.

- **Equations of motion**

$$0 = -\frac{\alpha_0}{2}\,\delta_\nu^\mu + \alpha_1\left(R_\nu^\mu - \frac{1}{2}R\,\delta_\nu^\mu\right) + L_\nu^\mu$$

$$L_\nu^\mu = \sum_{p=2}^{N} \frac{\alpha_p}{2^{p+1}}\,\delta_{\nu\mu_1\cdots\mu_{2p}}^{\mu\nu_1\cdots\nu_{2p}}\,R_{\nu_1\nu_2}^{\mu_1\mu_2}\cdots R_{\nu_{2p-1}\nu_{2p}}^{\mu_{2n-1}\mu_{2p}}$$

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- **AdS vacuum is determined by the effective AdS radius ℓ_{eff}**

$$\lambda = -1/\ell^2_{\text{eff}}\,, \qquad R^{\mu\nu}_{\alpha\beta} = -\frac{1}{\ell^2_{\text{eff}}}\,\delta^{\mu\nu}_{\alpha\beta}$$

E.o.m $\Rightarrow$ $$\Delta(\lambda) = \sum_{p=0}^{N} \frac{\alpha_p\,\lambda^p}{(D-2p-1)!} = 0 \quad \Rightarrow \quad \ell_{\text{eff}} = \ell_{\text{eff}}(\alpha_p)$$

- **Degeneracy of the vacuum**

Simple vacuum	$\Delta'(\lambda)$	$\neq 0$
Degenerate vacuum of order k	$\Delta^{(p)}(\lambda)$	$= 0\,, \quad p = 1,\ldots,k-1$
	$\Delta^{(k)}(\lambda)$	$\neq 0$

Lovelock AdS gravity

- **General spherically symmetric (black hole) solution**

  with the metric function $f(r) = -g_{tt} = 1/g_{rr}$

$$\sum_{p=0}^{N} \frac{\alpha_p}{(D-2p-1)!} \left(\frac{k - f(r)}{r^2} \right)^p = \frac{\mu}{r^{D-1}} + \text{matter}$$

- **Asymptotic behavior of around a k-degenerate vacuum**

$$f(r) = k + \frac{r^2}{\ell_{\text{eff}}^2} + a \left(\frac{\mu}{r^{D-2k-1}} \right)^{1/k} + b \left(\frac{\mu^2}{r^{2D-2k-2}} \right)^{1/k} + \cdots$$

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- **Example – Einstein-Gauss-Bonnet AdS gravity**, $\alpha_0, \alpha_1, \alpha_2 \neq 0$

  Equations of motion $\delta^{\mu\mu_1\mu_2\mu_3\mu_4}_{\nu\nu_1\nu_2\nu_3\nu_4}\left(R^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell_+^2}\,\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\right)\left(R^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell_-^2}\,\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\right) = 0$

  If α is such that $\ell_+ = \ell_-$, the vacuum is $k = 2$ degenerate. In $D = 5$, this α corresponds to Chern-Simons AdS gravity.

Kounterterm charges in Lovelock AdS gravity

- **Even dimensions**, $D = 2n$ [Kofinas, Olea '06; Miskovic, Olea '11]

$$B_{2n-1} = 2n\sqrt{-h}\int\limits_0^1 du\,\delta^{[2n-1]}\,K\left(\frac{1}{2}\,\mathcal{R} - u^2K^2\right)^{n-1}$$

$$c_{2n-1} = -\frac{1}{16\pi nG}\sum_{p=1}^{n-1}\frac{p\alpha_p}{(2n-2p)!}\left(-\ell_{\mathrm{eff}}^2\right)^{n-p}$$

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- **Odd dimensions**, $D = 2n+1$ [Kofinas, Olea '06; Miskovic, Olea '11]

$$B_{2n} = 2n\sqrt{-h}\int_0^1 du\int_0^t ds\,\delta^{[2n]}\,K\delta\left(\frac{1}{2}\,\mathcal{R} - u^2K^2 + \frac{s^2}{\ell_{\text{eff}}^2}\,\delta^2\right)^{n-1}$$

$$c_{2n} = \frac{1}{16\pi nG}\left[\int_0^1 du\,\left(u^2-1\right)^{n-1}\right]^{-1}\sum_{p=1}^{n}\frac{(-1)^p\,p\alpha_p\,\ell_{\text{eff}}^{2n-2p}}{(2n-2p+1)!}$$

Kounterterm charges in Lovelock AdS gravity

Noether charge

$$Q[\xi] = \int_{\Sigma_\infty} d^{D-2}y \, \sqrt{\sigma} \, u_j \, \xi^i \left(q_i^j + q_{(0)i}^j \right)$$

$q_i^j \quad \longrightarrow$ black hole mass

$q_{(0)i}^j \quad \longrightarrow$ vacuum energy

In global AdS space $q_i^j = 0$

In even dimensions $q_{(0)i}^j = 0$

Kounterterm charges in Lovelock AdS gravity

Expression for the charge density tensor $Q_i^j = q_i^j + q_{(0)i}^j$

- **Even dimensions** $D = 2n$, $\qquad q_{(0)i}^j = 0$

$$q_i^j = \frac{1}{16\pi G 2^{n-2}} \, \delta^{[2n-1]j} \sum_{p=1}^{n-1} \frac{p\alpha_p}{(2n-2p)!} \, K_i R^{p-1} \left[\delta^{[2]n-p} - \left(-\ell_{\text{eff}}^2 R \right)^{n-p} \right]$$

$$R_{kl}^{ij} = \mathcal{R}(h)_{kl}^{ij} - K_k^i K_l^j + K_l^i K_k^j \text{ (Gauss-Codazzi relation)}$$

Kounterterm charges in Lovelock AdS gravity

Expression for the charge density tensor $Q_i^j = q_i^j + q_{(0)i}^j$

- **Even dimensions** $D = 2n$, $\quad q_{(0)i}^j = 0$

$$q_i^j = \frac{1}{16\pi G 2^{n-2}} \, \delta^{[2n-1]j} \sum_{p=1}^{n-1} \frac{p\alpha_p}{(2n-2p)!} K_i R^{p-1} \left[\delta^{[2]n-p} - \left(-\ell_{\text{eff}}^2 R \right)^{n-p} \right]$$

$$R_{kl}^{ij} = \mathcal{R}(h)_{kl}^{ij} - K_k^i K_l^j + K_l^i K_k^j \text{ (Gauss-Codazzi relation)}$$

- **Odd dimensions** $D = 2n+1$, $\quad q_{(0)i}^j \neq 0$

$$q_i^j = \frac{1}{16\pi G \, (2n-1)! 2^{n-2}} \, \delta_{i_1 \cdots i_{2n}}^{j j_2 \cdots j_{2n}} K_i^{i_1} \delta_{j_2}^{i_2}$$
$$\times \left[\sum_{p=1}^{n} \frac{p\alpha_p \, (2n-1)!}{(2n-2p+1)!} R_{j_3 j_4}^{i_3 i_4} \cdots R_{j_{2p-1} j_{2p}}^{i_{2p-1} i_{2p}} \delta_{j_{2p+1} j_{2p+2}}^{i_{2p+1} i_{2p+2}} \cdots \delta_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} \right.$$
$$\left. + \, 16\pi G \, (2n-1)! n c_{2n} \int_0^1 du \left(R_{j_3 j_4}^{i_3 i_4} + \frac{u^2}{\ell_{\text{eff}}^2} \delta_{j_3 j_4}^{i_3 i_4} \right) \cdots \left(R_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} + \frac{u^2}{\ell_{\text{eff}}^2} \delta_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} \right) \right]$$

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Goal: to find when $q_i^j \propto E_i^j$ **because then** $Q[\partial_t] = \mathcal{H}[\partial_t]$

Conformal charges in Lovelock AdS gravity

Important step – factorization of the charge by the AdS tensor

- **AdS curvature** $\boxed{F_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} + \frac{1}{\ell_{\text{eff}}^2}\,\delta_{\mu\nu}^{\alpha\beta}}$ (vanishes for global AdS)
- **Relation between the Weyl tensor and AdS curvature**

  Einstein AdS gravity: $W_{\mu\nu}^{\alpha\beta} = F_{\mu\nu}^{\alpha\beta}$ $\quad \ell_{\text{eff}} = \ell$

  Lovelock AdS gravity : $W_{\mu\nu}^{\alpha\beta} \neq F_{\mu\nu}^{\alpha\beta}$ $\quad \ell_{\text{eff}} \neq \ell$

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- Since $q^j_i = 0$ in the vacuum, it suggests that q^j_i is factorizable by F^{ij}_{kl}

$$q^j_i = a_n\, \delta^{jj_2\cdots j_{2n-1}}_{i_1\cdots i_{2n-1}}\, K^{i_1}_i\, F^{i_2 i_3}_{j_2 j_3}\, \mathcal{P}^{i_4\cdots i_{2n-1}}_{j_4\cdots j_{2n-1}}(F)$$

Here: $n = [D/2]$ Integer related to the dimension

a_n Constant that depends on c_{D-1} and ℓ_{eff}

$\mathcal{P}^{i_4\cdots i_{2n-1}}_{j_4\cdots j_{2n-1}}$ Dimension-dependent polynomial of F

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- $\mathcal{P}_{j_4\cdots j_{2n-1}}^{i_4\cdots i_{2n-1}}$ is obtained using the factorization formula

$$a^{n-1} - (-b)^{n-1} = (n-1)(a+b)\int_0^1 du\,[u(a+b)-b]^{n-2}$$

- **Another important information: Fall-off of the Weyl tensor**

  Weyl tensor ≠ AdS curvature

$$W_{\alpha\beta}^{\mu\nu} = R_{\alpha\beta}^{\mu\nu} + \frac{1}{\ell_{\text{eff}}^2}\,\delta_{\alpha\beta}^{\mu\nu} + X_{\alpha\beta}^{\mu\nu}$$

$$\boxed{X_{\alpha\beta}^{\mu\nu} = \frac{1}{D-2}\,\delta_{[\alpha}^{[\mu} H_{\beta]}^{\nu]} - \left(\frac{2H_{\lambda}^{\lambda}}{(D-1)(D-2)} + \frac{\alpha^*}{\ell_{\text{eff}}^4}\right)\delta_{\alpha\beta}^{\mu\nu} \neq 0}$$

- In order to find an asymptotic behavior of the tensor $F_{\alpha\beta}^{\mu\nu} = R_{\alpha\beta}^{\mu\nu} + \frac{1}{\ell_{\text{eff}}^2}\,\delta_{\alpha\beta}^{\mu\nu}$ in AAdS spaces, first we repeat the original argument of Ashtekar and Das about the fall off of the Weyl tensor in EH gravity, and after that we analyse the AdS curvature in EGB gravity.

Conformal charges in Lovelock AdS gravity

- **Asymptotic behavior of W in EH AdS gravity**

- Global AdS space: $W|_{\mathrm{AdS}} = 0$
- Non-vacuum state with total mass M:
  $W - W|_{\mathrm{AdS}} \sim \frac{GM}{r^a}$ leads to $\boxed{W^{\mu\nu}_{\alpha\beta} \sim \frac{GM}{r^{D-1}}}$
- The power factor a is determined by dimensional analysis, from $[G] = L^{D-2}$, $[M] = 1/L$ and $[W] = 1/L^2$

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- **Asymptotic behavior of F in EGB and Lovelock AdS gravity**

- EH gravity: $X|_{\text{EHAdS}} = (W - F)|_{\text{EHAdS}} = 0$
- Thus, a difference between W and F in EGB gravity is due to existence of a GB term and the quadratic mass corrections of a non-vacuum state:
$\boxed{X^{\mu\nu}_{\alpha\beta} \sim \alpha \, \frac{(GM)^2}{r^{2D-2}}}$
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- **We confirm the validity of the above argument in the space of solutions.**

Conformal charges in Lovelock AdS gravity

From Noether to conformal charge in even dimensions

- The proof is similar in odd dimensions as well
- We showed a factorization of the charge $\boxed{q_i^j = a_n \, \delta^{j[2n-1]} K_i \, F \, \mathcal{J}(F)}$
- The polynomial $\mathcal{P}(R)$ is given in integral representation by

$$\mathcal{J}(F) = \delta^{[2n-1]j} \sum_{p=1}^{n-1} \frac{p(n-p)\alpha_p}{(2n-2p)!} \left(F - \frac{1}{\ell_{\text{eff}}^2} \delta^{[2]} \right)^{p-1} \times$$

$$\times \int_0^1 du \left(\frac{1}{\ell_{\text{eff}}^2} \delta^{[2]} + (u-1) \; F \right)^{n-p-1}$$

- **Now it is explicit that q_i^j vanishes for the global AdS ($F=0$)**

Conformal charges in Lovelock AdS gravity

- **Asymptotically AdS behavior of the metric**

$$u_j \sim \mathcal{O}(r), \qquad \sqrt{\sigma} \sim \mathcal{O}(r^{D-2})$$

$\Rightarrow$ The only contribution to the charge comes from $q_i^j \sim \mathcal{O}(r^{D-1})$

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- **Asymptotic behavior of other tensors**

$$\begin{aligned} F_{kl}^{ij} &= W_{kl}^{ij} + \mathcal{O}(1/r^{2D-2}) \\ K_j^i &= -\frac{1}{\ell_{\text{eff}}}\,\delta_j^i + \mathcal{O}\left(1/r^2\right) \end{aligned}$$

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- **Charge density tensor evaluated:** $q_i^j = a_n\,\delta_{[i\cdots]}^{[j\cdots]}\,W\mathcal{J}(0) + \cdots$

- The integral $\mathcal{J}(0)$ becomes trivial

- We have to use the fact that the Weyl tensor is traceless $(W_{\mu\beta}^{\mu\alpha} = W_{r\beta}^{r\alpha} + W_{k\beta}^{k\alpha} = 0) \quad \Rightarrow \quad W_{rj}^{ri}$ is E_j^i

Conformal charges in Lovelock AdS gravity

- Gathering all those properties together, we get

$$\boxed{q_i^j = -\frac{(2n-4)!\ell_{\text{eff}}}{32\pi G}\,\Delta'(\ell_{\text{eff}})\,\delta_i^{[3]j}\,W}$$

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$$q_i^j = -\frac{(2n-4)!\ell_{\text{eff}}}{32\pi G}\,\Delta'(\ell_{\text{eff}})\,\delta_i^{[3]j}\,W$$

- The electric part of the Weyl tensor is $E_i^j = \frac{1}{D-3}\,W_{i\mu}^{j\nu}\,n^\mu n_\nu$ so that

$$Q[\xi] = -\frac{(D-3)!(D-3)\,\ell_{\text{eff}}}{8\pi G}\,\Delta'(\ell_{\text{eff}})\int\limits_{\Sigma_\infty} d\Sigma\,E_i^j\,\xi^i\,u_j$$

- In EGB gravity, the result is in agreement with [Pang '11] (up to $\ell \to \ell_{\text{eff}}$), where the conformal techniques were used, and with the conformal mass obtained in [Miskovic, Kofinas, Olea '15]
- **The result is valid only for simple vacua where $\Delta'(\ell_{\text{eff}}) \neq 0$**
- **This means that Chern-Simons gravity, or Lovelock Unique Vacuum gravity, does not have well-defined conformal mass**

Conclusions

- Kounterterm charges $Q[\xi]$ and the AMD conformal mass $\mathcal{H}[\xi]$ always match in Einstein AdS gravity
- In Lovelock AdS gravity, conformal mass is well-defined only for non-degenerate vacua
- The only difference with respect to holographic charges appears in the odd-dimensional case, where there is a piece that gives rise to the vacuum energy.

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- Better understanding of the holographic properties of the Weyl tensor and AdS curvature is necessary
  [*in progress*, Anastasiou, Olea, Papadimitriou '17].
- Case of boundary topology different than $\mathbb{R} \times S^{D-2}$ has not been covered
- In $D = 4$, the charge adcquires the magnetic mass when the Pontryagin invariant is added to the action [Aranada, Aros, Miskovic, Olea '16].

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T H A N K Y O U !