PHYSIK

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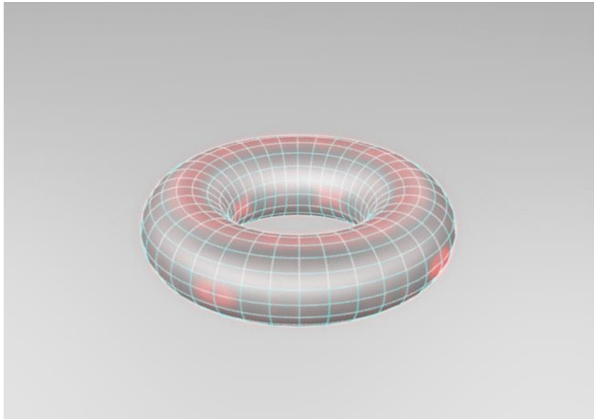
Topology of mixed states

Michael Fleischhauer

University of Kaiserslautern



SFB/TR 185

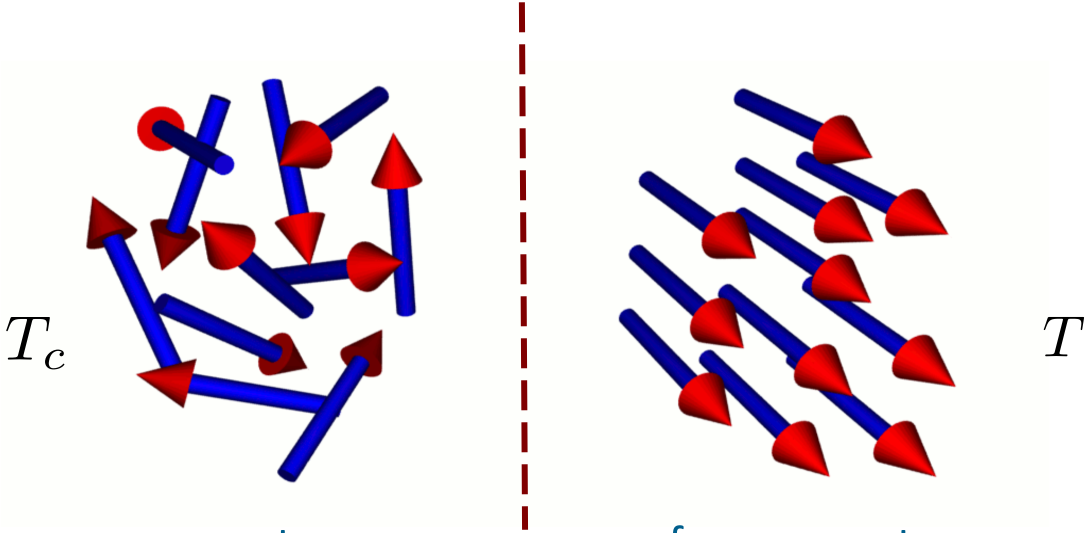
QM^3 - Quantum Matter meets Math, Lissabon, 07.06.2021

Phases of matter

- **Landau-Ginzburg**

phases of matter differ by the way they break symmetries

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$



$T > T_c$

$T < T_c$

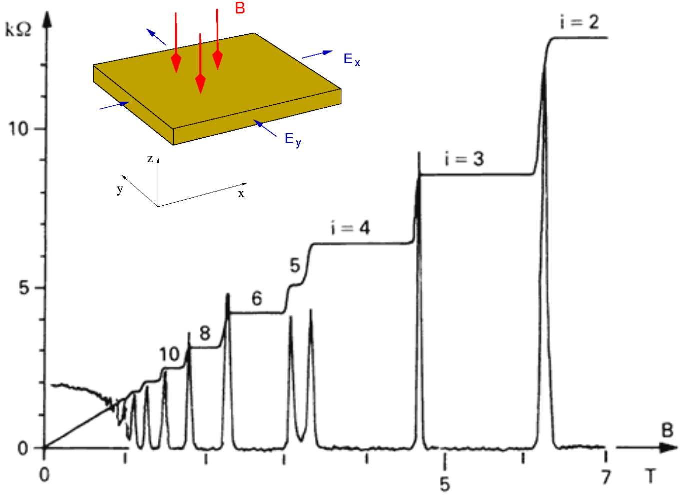
paramagnet

ferromagnet

Phases of matter

- **Quantum Hall effect**

no broken symmetries !



v. Klitzing, PRL 1980

Thouless, Kohmoto, Nightingale, denNijs, (TKNN) PRL 1982:

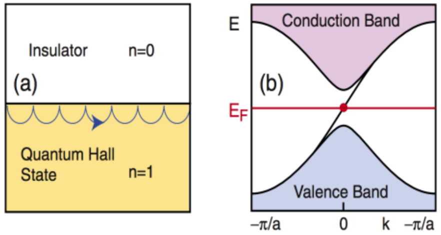
topology of the wavefunction, characterized by integer quantum numbers!

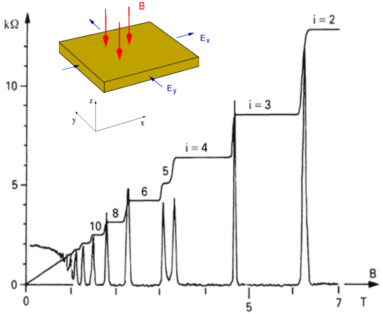
Topological quantum systems

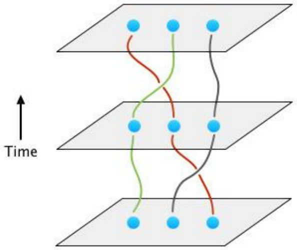
topological protection

quantized bulk transport

exotic quantum states







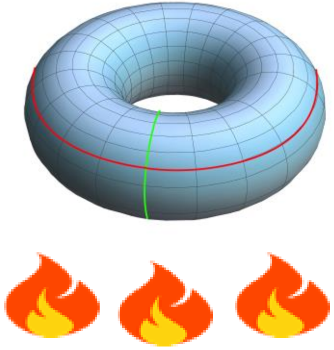
protected edge states & edge transport

Hall conductivity & resistance normal

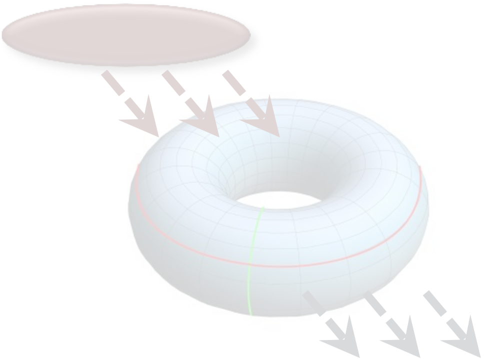
Abelian & non-Abelian anyons

mixed states ?

- topology at finite T:
  what is left ??



- topology in non-equilibrium
  driven, open systems??



mixed states ?

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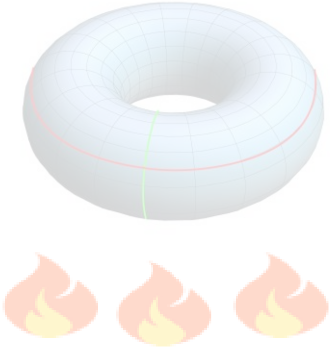

- topology in non-equilibrium
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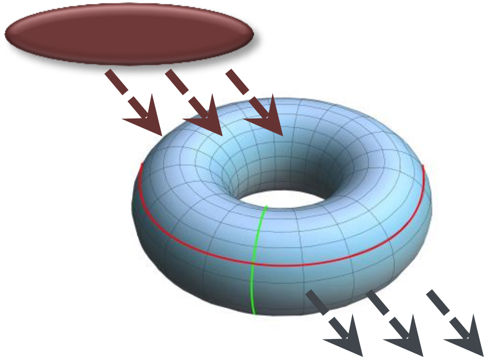
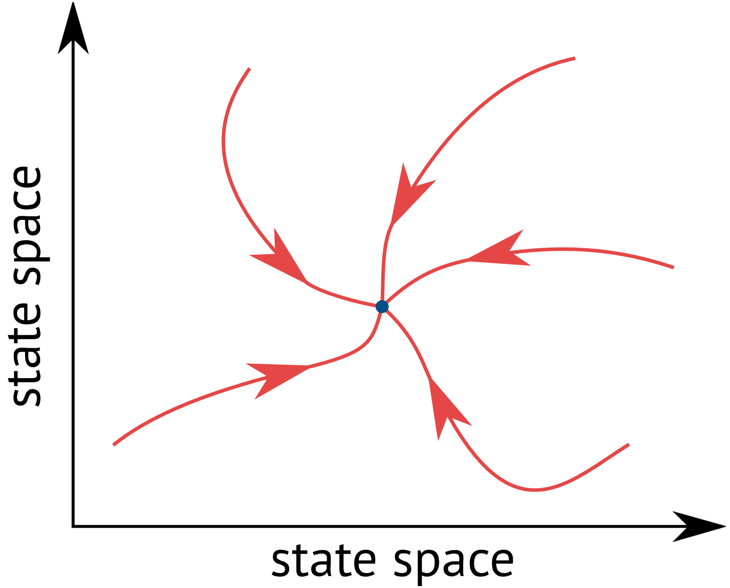


mixed states ?

- steady state of open systems is an **attractor** of the dynamics:



Outline

Topological classification by symmetries of Hamiltonians

Gaussian mixed states and fictitious Hamiltonian

Topological invariants: Geometric Phases

Broken time-reversal symmetry: Z index

Time-reversal symmetric systems: Z_2 index

Interactions

Measurable consequences

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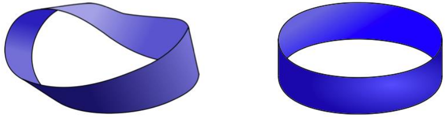
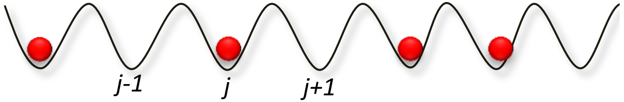
Classification of topological systems



Symmetry operations in Fock space

Fermions on a lattice



Unitary	$\hat{U}$	$\hat{U}\,\hat{a}_i\,\hat{U}^{-1} = u_{ij}\hat{a}_j$	$\hat{U}\,i\hat{U}^{-1} = i$
Time-reversal	$\hat{T}$	$\hat{T}\,\hat{a}_i\,\hat{T}^{-1} = u^t_{ij}\hat{a}_j$	$\hat{T}\,i\,\hat{T}^{-1} = -i$
Charge conjugation	$\hat{C}$	$\hat{C}\,\hat{a}_i\,\hat{C}^{-1} = u^c_{ij}\hat{a}^\dagger_j$	$\hat{C}\,i\,\hat{C}^{-1} = i$
Chiral transform	$\hat{S} = \hat{T} \circ \hat{C}$	$\hat{S}\,\hat{a}_i\,\hat{S}^{-1} = u^s_{ij}\hat{a}^\dagger_j$	$\hat{S}\,i\,\hat{S}^{-1} = -i$

this is an exhaustive list !!!

Ten-fold way

There are only 10 different classes under Fock-space transformations !

- not invariant under trafo: “0”
- invariant and $\hat{G}^2 = +1$ “+1”
- Invariant and $\hat{G}^2 = -1$ “-1”

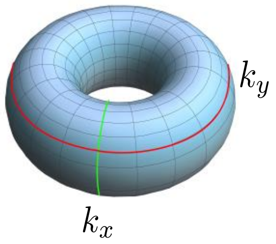
$$\hat{G} = \hat{T}, \hat{C}, \hat{S}$$

T	0	+1	-1
C	0	+1	-1
S	0	+1	-

} 3 x 3 = 9 cases

only nontrivial if T = 0 and C = 0
1 additional case

Non-interacting fermions



Class	T	C	S	1	2	3	4
A	0	0	0	0	$\boxed{\mathbb{Z}}$	0	$\boxed{\mathbb{Z}}$
AIII	0	0	+	$\boxed{\mathbb{Z}}$	0	$\boxed{\mathbb{Z}}$	0
AI	+	0	0	0	0	0	$\boxed{\mathbb{Z}}$
BDI	+	+	+	$\boxed{\mathbb{Z}}$	0	0	0
D	0	+	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}}$	0	0
DIII	−	+	+	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}}$	0
AII	−	0	0	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}}$
CII	−	−	+	$\boxed{\mathbb{Z}}$	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}_2}$
C	0	−	0	0	$\boxed{\mathbb{Z}}$	0	$\boxed{\mathbb{Z}_2}$
CI	+	−	+	0	0	$\boxed{\mathbb{Z}}$	0

← Intrinsic topology

symmetry-protected

Gaussian open systems

Gaussian mixed states

determined by single-particle correlations: fictitious Hamiltonian

$$\hat{\rho} \sim \exp\left\{-\frac{1}{2}\underline{\hat{c}}^{\dagger\top}\underline{\underline{G}}\,\underline{\hat{c}}\right\} \qquad \langle \hat{c}_i^\dagger \hat{c}_j \rangle = \frac{1}{2}\left[1-\tanh\left(\frac{G}{2}\right)\right]_{ij}$$

- Gaussian states in **thermal equilibrium** require quadratic Hamiltonian

$$\underline{\underline{G}} = \beta\left(\underline{\underline{H}} - \mu\right) \qquad \beta = 1/k_B T$$

- Gaussian **non-equilibrium steady state** (NESS) require linear Lindblad generators

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i\left[\hat{H},\rho\right] + \frac{1}{2}\sum_{\mu}\left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\}\right) = 0 \qquad L_j \sim \alpha\,\hat{c}_j^\dagger + \beta\,\hat{c}_j$$

Topological invariants:
Geometric phases

Topological invariants

Chern insulators

TR broken

Quantum Hall (1980)

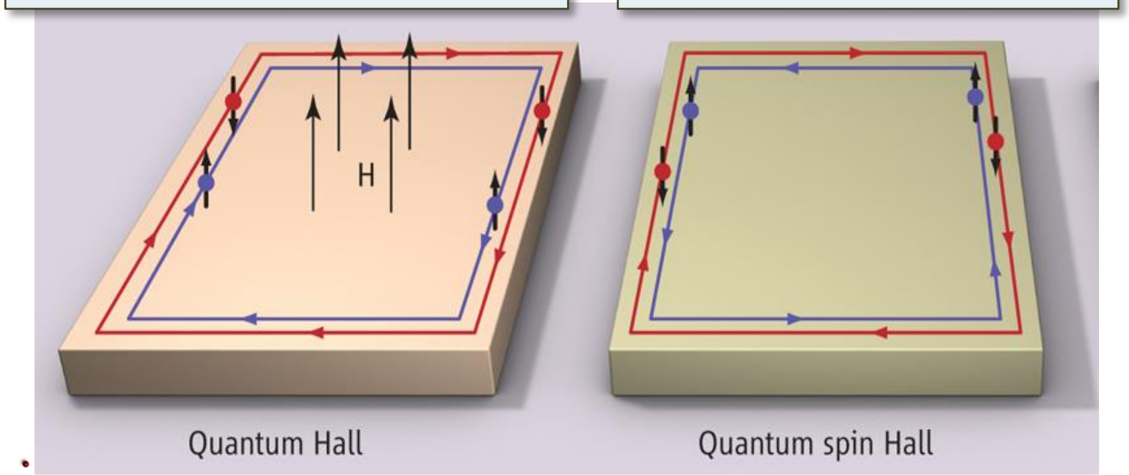
Z

Topol. insulators

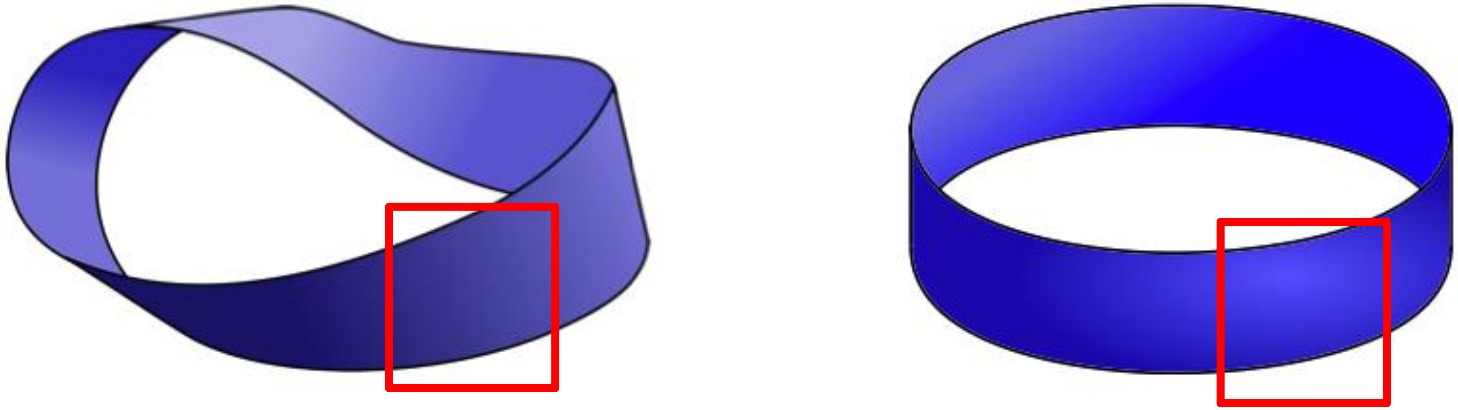
TR symmetric

Quantum Spin Hall (2007)

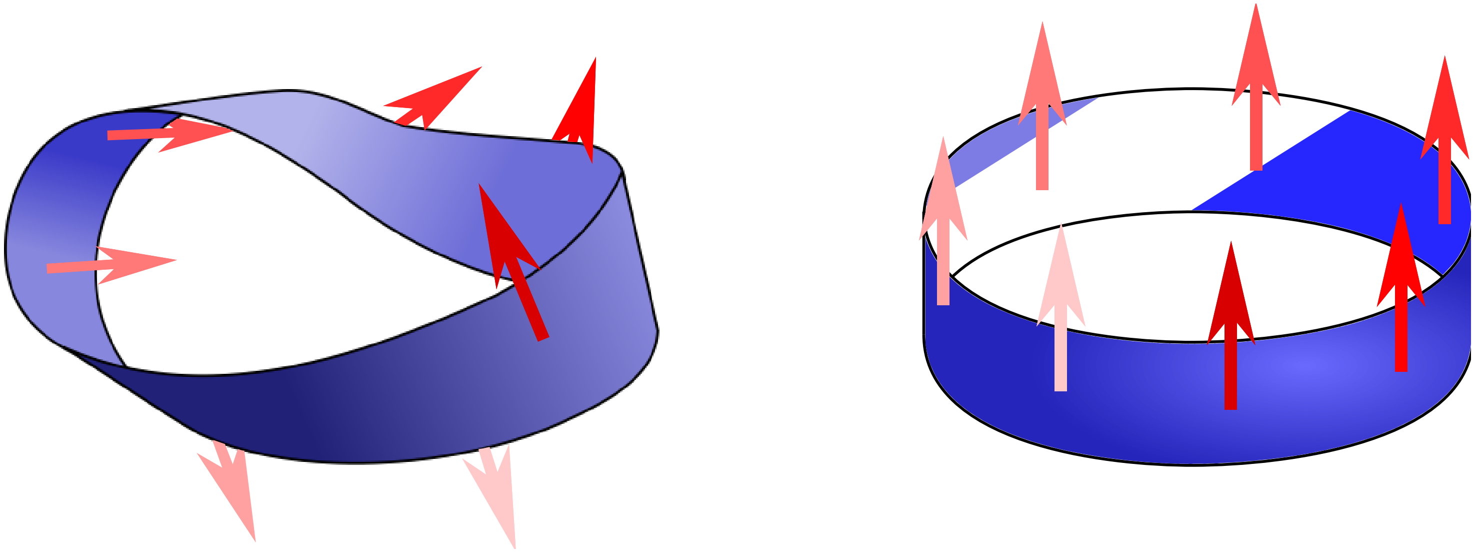
Z_2



Science **340**, 153 (2013)



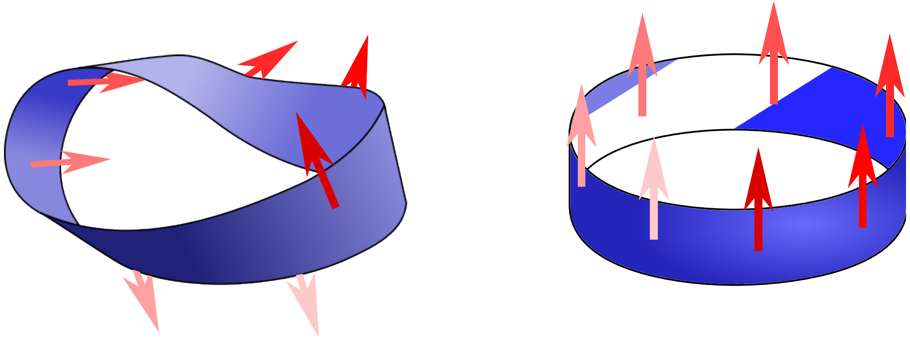
locally indistinguishable



differ by global properties !

Geometric phase

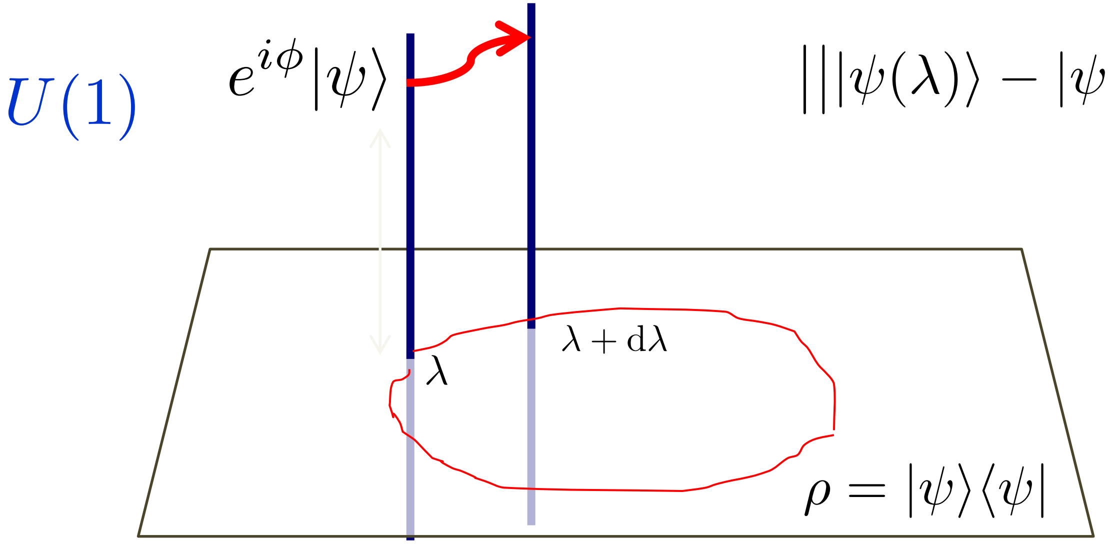
- Zak (Berry) phase



$$\phi_{\mathrm{Zak}} = \int_{-\pi/a}^{\pi/a} dk\, \langle u_k | i\partial_k | u_k \rangle$$

→ density matrix ?

Berry parallel transport

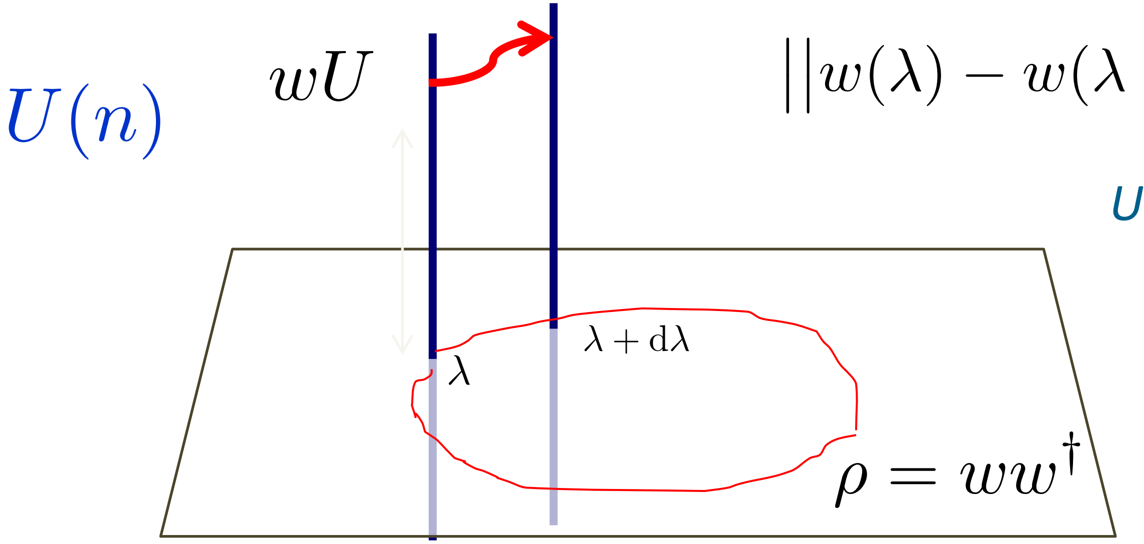


$$|| |\psi(\lambda)\rangle - |\psi(\lambda + \mathrm{d}\lambda)\rangle || = \min$$

Berry (Zak) phase: picked up at parallel transport cycle

$$\phi_{\mathrm{Zak}} = \int_{-\pi/a}^{\pi/a} dk \, \langle u_k | i\partial_k | u_k \rangle$$

Uhlmann parallel transport



$$||w(\lambda) - w(\lambda + \mathrm{d}\lambda)|| = \min$$

U(n) gauge freedom

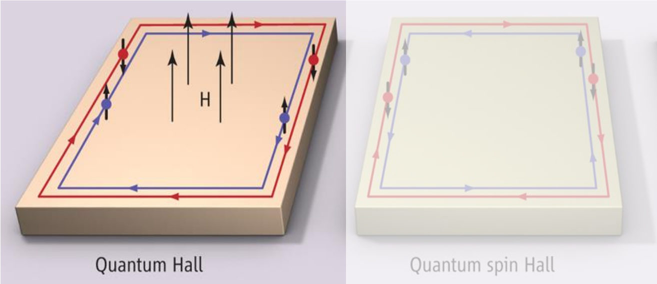
$$w \to w\,U$$
$$w^\dagger \to U^\dagger\,w^\dagger$$

U(1) Uhlmann phase

$$e^{i\phi} = \oint \mathrm{d}\lambda\;\mathrm{Tr}\left[w\partial_\lambda w^\dagger\right]$$

Chern number

TR-broken



1+1 and 2-dimensional systems

- 1D: winding number

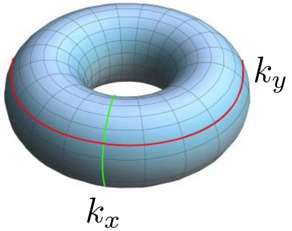
$$\hat{H} = \hat{H}(\lambda)$$

$$\nu = \frac{1}{2\pi} \oint d\lambda \, \frac{\partial \phi_{\text{Zak}}}{\partial \lambda}$$

- 2D: Chern number

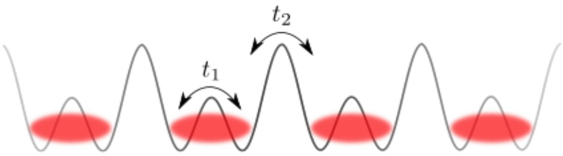
$$C = \frac{i}{2\pi} \iint_{\text{BZ}} \mathrm{d}^2 k \sum_{\alpha} \left\{ \langle \partial_{k_y} u_k^\alpha | \partial_{k_x} u_k^\alpha \rangle - \langle \partial_{k_x} u_k^\alpha | \partial_{k_y} u_k^\alpha \rangle \right\}$$

$$= \frac{1}{2\pi} \int_{\text{BZ}} dk_y \frac{\partial \phi_x^{\text{Zak}}}{\partial k_y} = -\frac{1}{2\pi} \int_{\text{BZ}} dk_x \frac{\partial \phi_y^{\text{Zak}}}{\partial k_x}$$



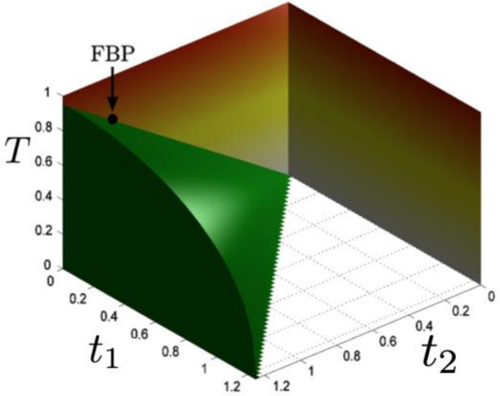
Failure of the Uhlmann phase

- SSH model at finite T (1D)   (class BDI)



Viyuela, Rivas, Martin-Delgado PRL (2014)

Huang, Arovas PRL (2014)



- asymmetric Qi-Wu-Zhang model at finite T (2D)   (class A)

$$H(k) = \sum_j d^j(k)\,\hat{\sigma}_j \qquad d^1 = \sin(k_x) \quad d^2 = 3\sin(k_y) \quad d^3 = 1 - \cos(k_x) - \cos(k_y)$$

$$C = \frac{1}{2\pi}\int dk_y \left(\frac{\partial\phi(k_y)}{\partial k_y}\right) \neq C' = \frac{1}{2\pi}\int dk_x \left(\frac{\partial\phi(k_x)}{\partial k_x}\right)$$

Budich, Diehl Phys.Rev. B (2015)

Invariants & quantized bulk transport

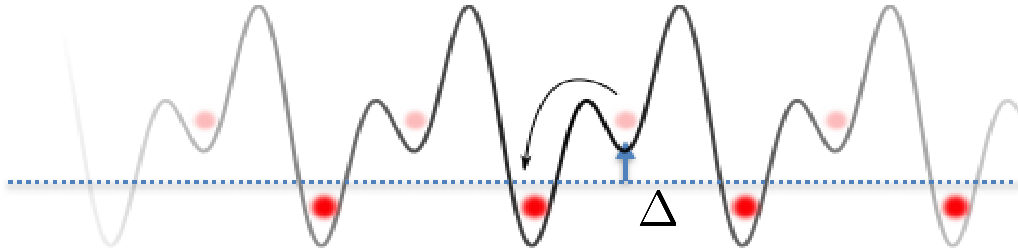
Thouless, Kohmoto, Nightingale, den Nijs (TKNN) PRL (1982)

$$\boxed{\Delta n = \frac{1}{2\pi} \oint d\lambda \, \frac{\partial \phi_{\mathrm{Zak}}}{\partial \lambda}}$$

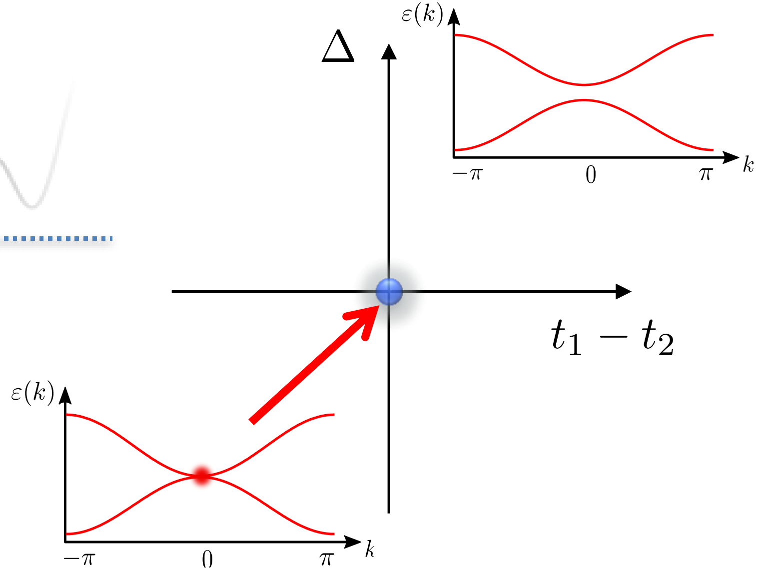
Charge pumps (Thouless)

- Rice-Mele model (1+1 D)

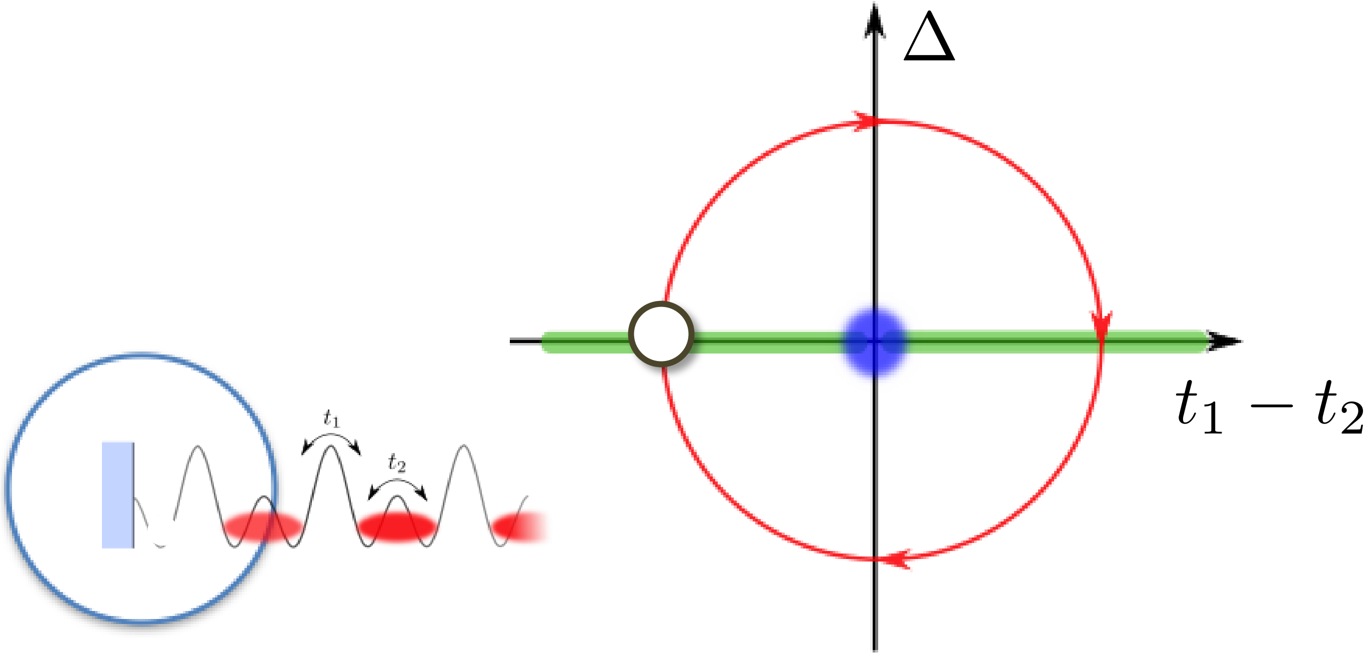
$$\hat{\mathcal{H}}_{\mathrm{RM}} = -t_1 \sum_{j:\mathrm{even}} c^\dagger_{j+1} c_j - t_2 \sum_{j:\mathrm{odd}} c^\dagger_{j+1} c_j + h.a. + \Delta \sum_j (-1)^j \, c^\dagger_j c_j$$



Rice & Mele, Phys. Rev. Lett. (1982)



Charge pumps (Thouless)



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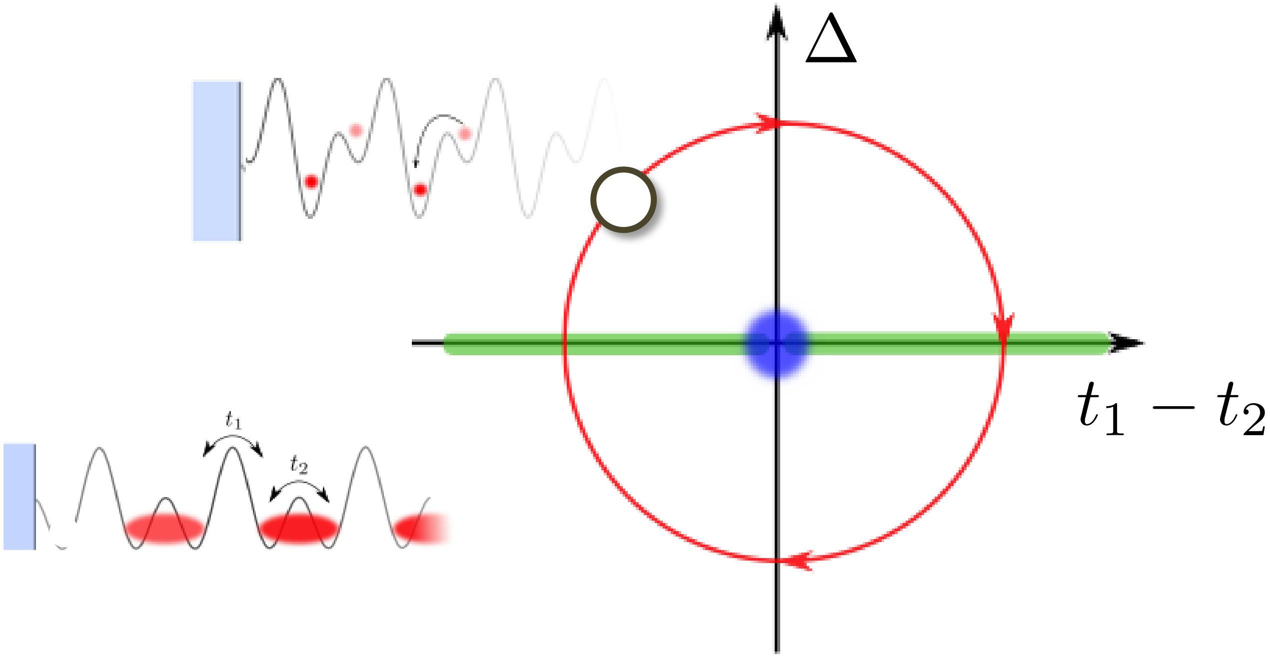

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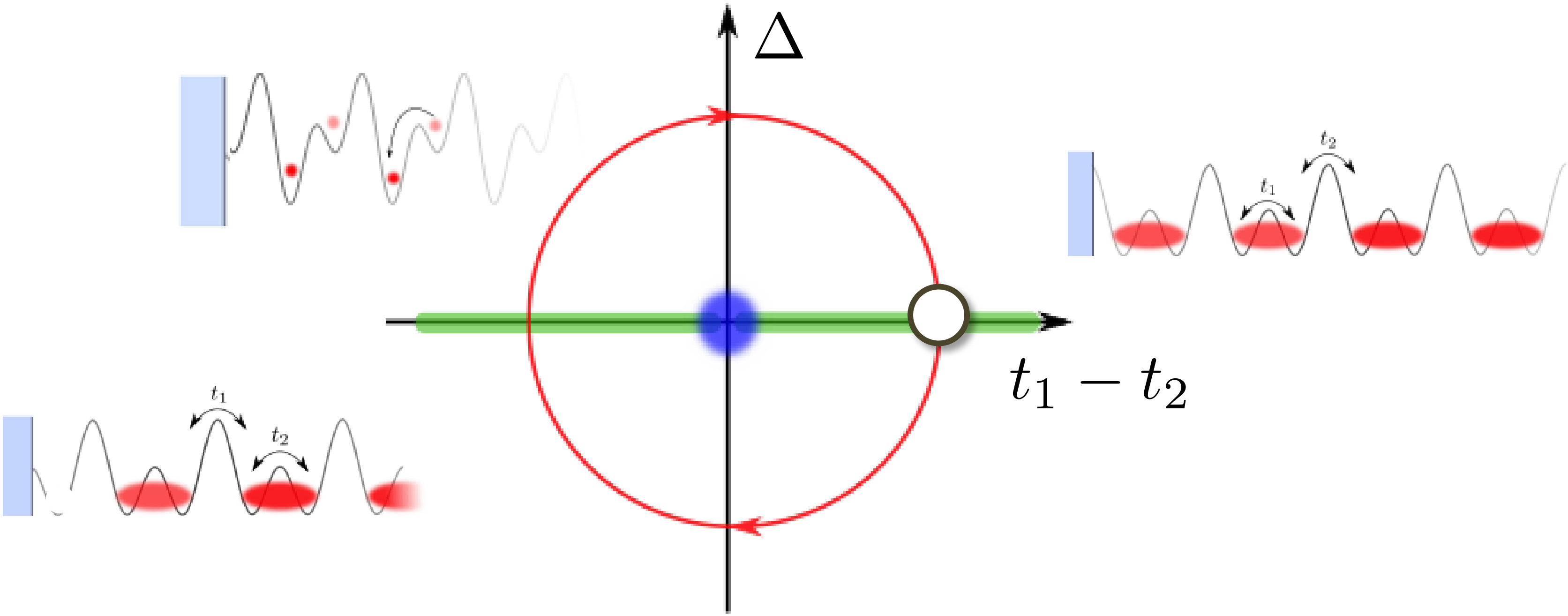

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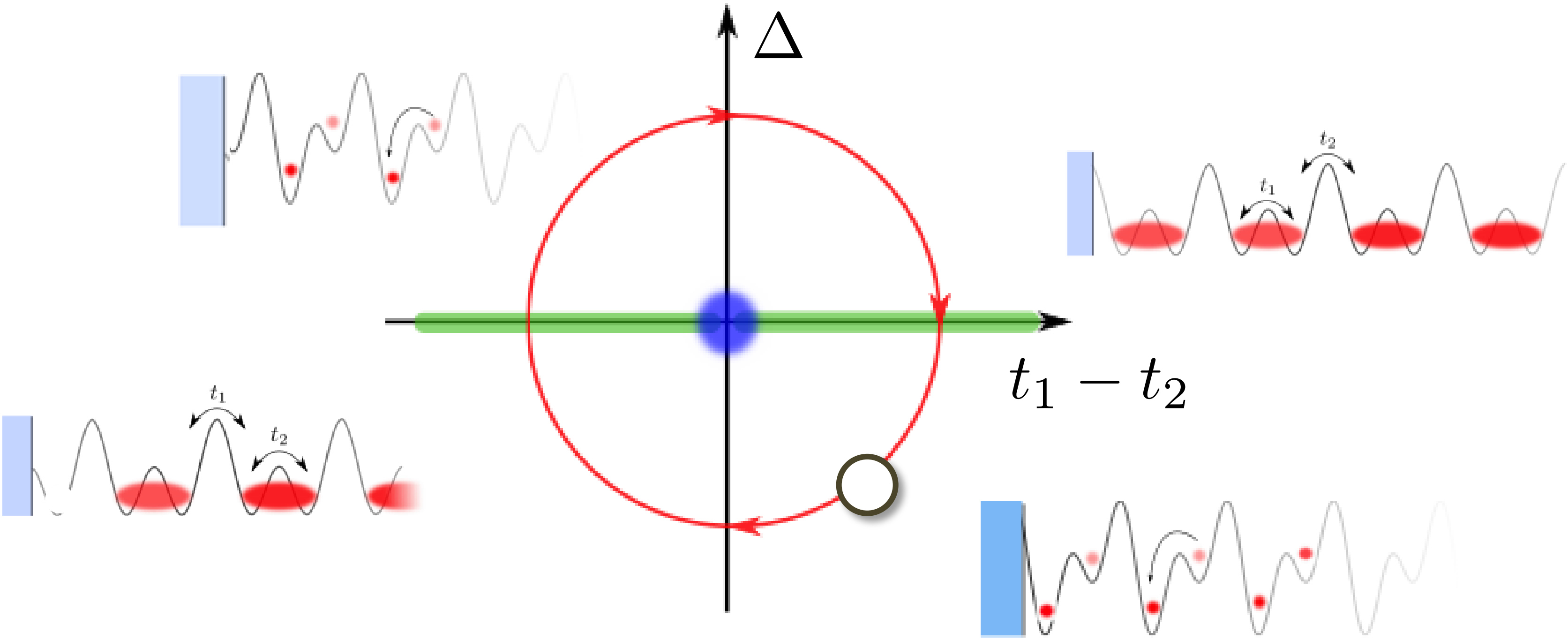
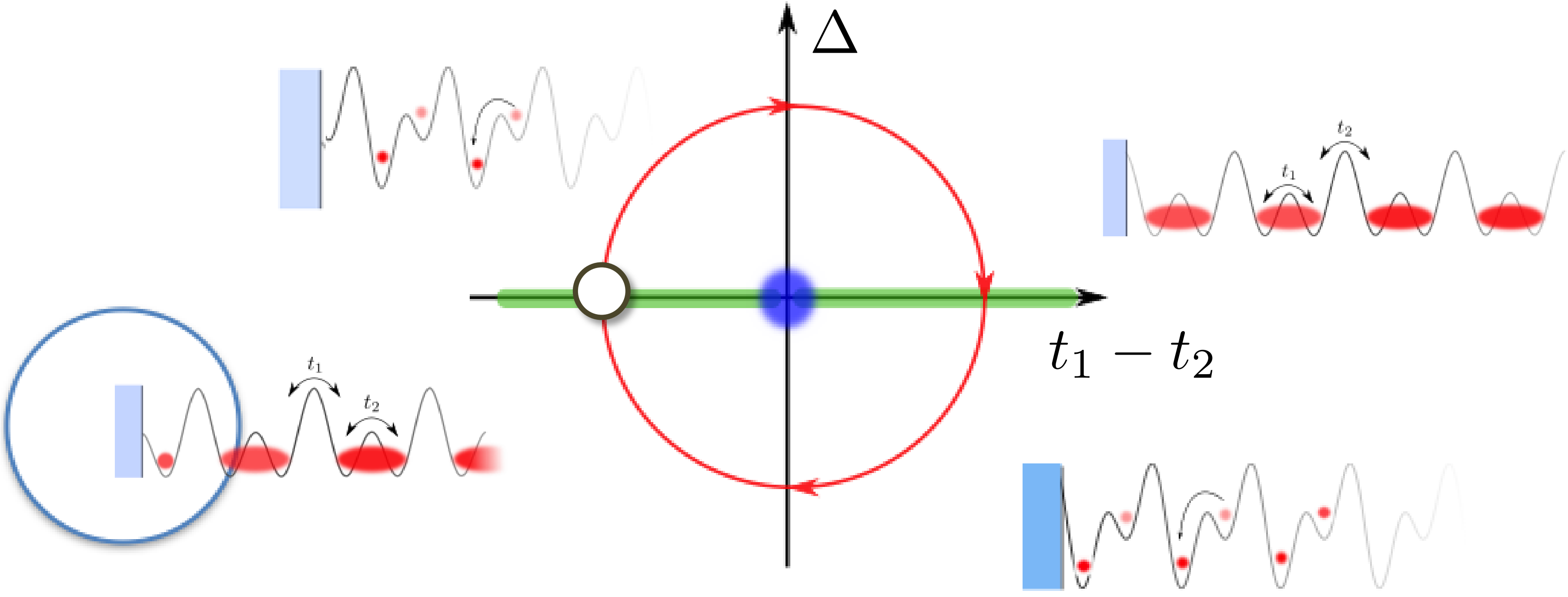


Charge pumps (Thouless)



Charge pumps (Thouless)

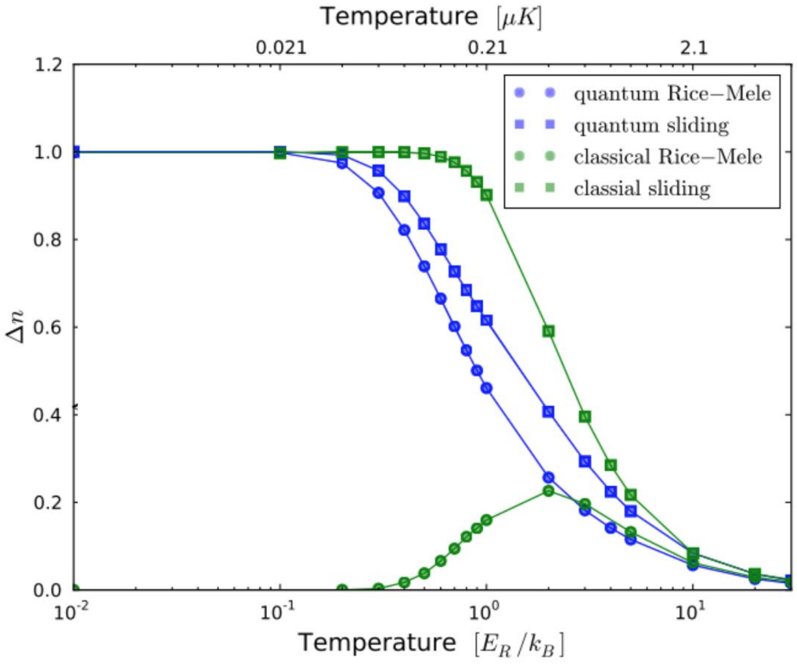
geometric phase

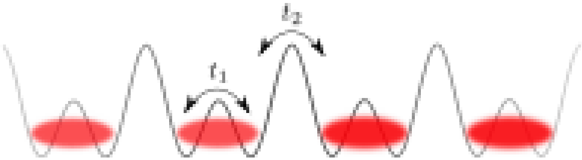
topological pumps

$$\Delta \cancel{\varphi_{\mathrm{Zak}}} \quad \longleftrightarrow \quad \Delta n$$

mixed states

Charge pumps (Thouless)

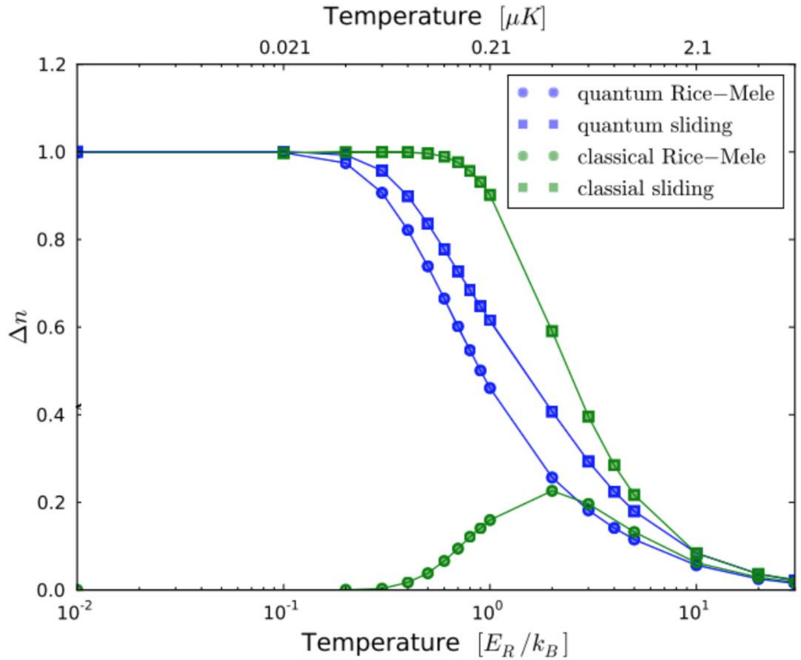


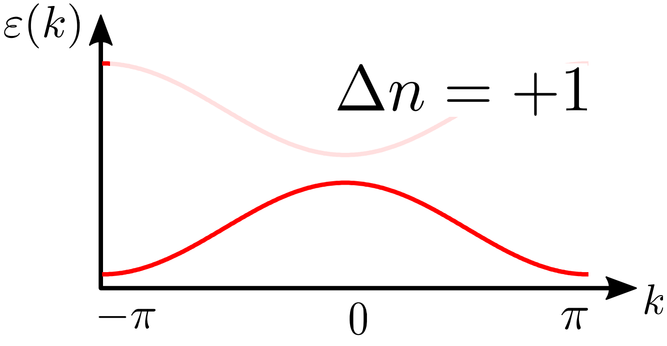


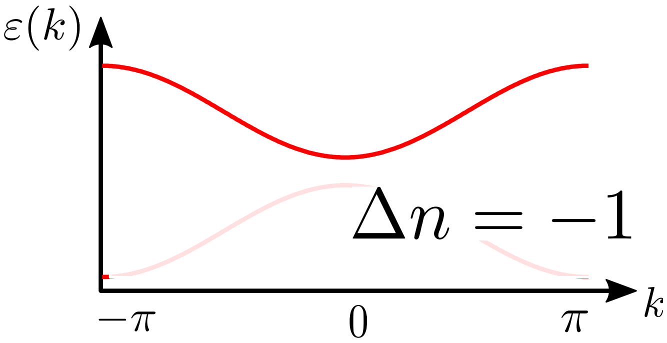
particle transport no longer quantized

Wang, Troyer, Dai, PRL (2013)

Charge pumps (Thouless)







Wang, Troyer, Dai, PRL (2013)

Charge pumps (Thouless)

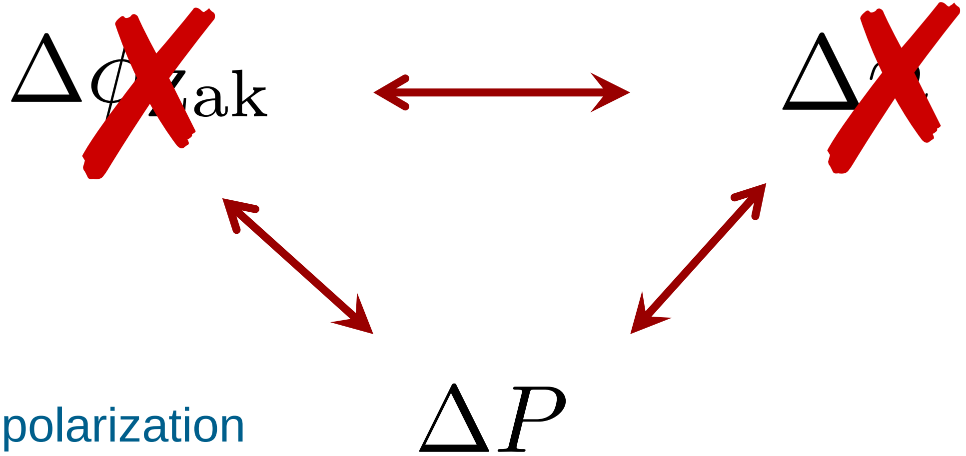
geometric phase

topological pumps

$$\Delta \xcancel{\phi_{\mathrm{Z}}}\mathrm{ak} \longleftrightarrow \Delta n$$

mixed states

Polarisation

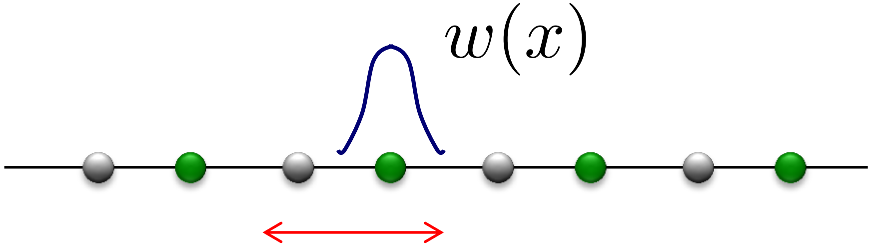


King-Smith, Vanderbildt PRB (1983)

$$\boxed{\Delta\phi_{\text{Zak}} = \frac{2\pi}{a}\Delta P}$$

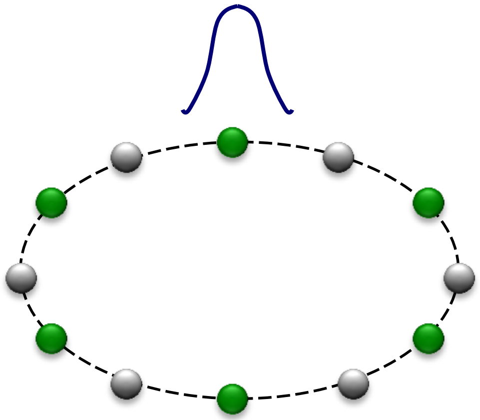
Many-body polarization

$$P = \int dx\, w^*(x)\, x\, w(x)$$



R. Resta PRL (1998)

$$\boxed{P = \frac{1}{2\pi} \mathrm{Im} \ln \left\langle \exp\left\{ i \frac{2\pi}{L} \hat{X} \right\} \right\rangle}$$



$$\hat{X} = \sum_j j\, \hat{c}_j^\dagger c_j \qquad (a = 1)$$

Ensemble Geometric Phase

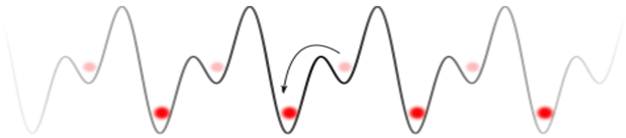
$$\varphi_{\mathrm{Zak}} = \mathrm{Im}\ln\langle\psi_0|\hat{T}|\psi_0\rangle \qquad \hat{T} = e^{i\frac{2\pi}{L}\hat{X}}$$

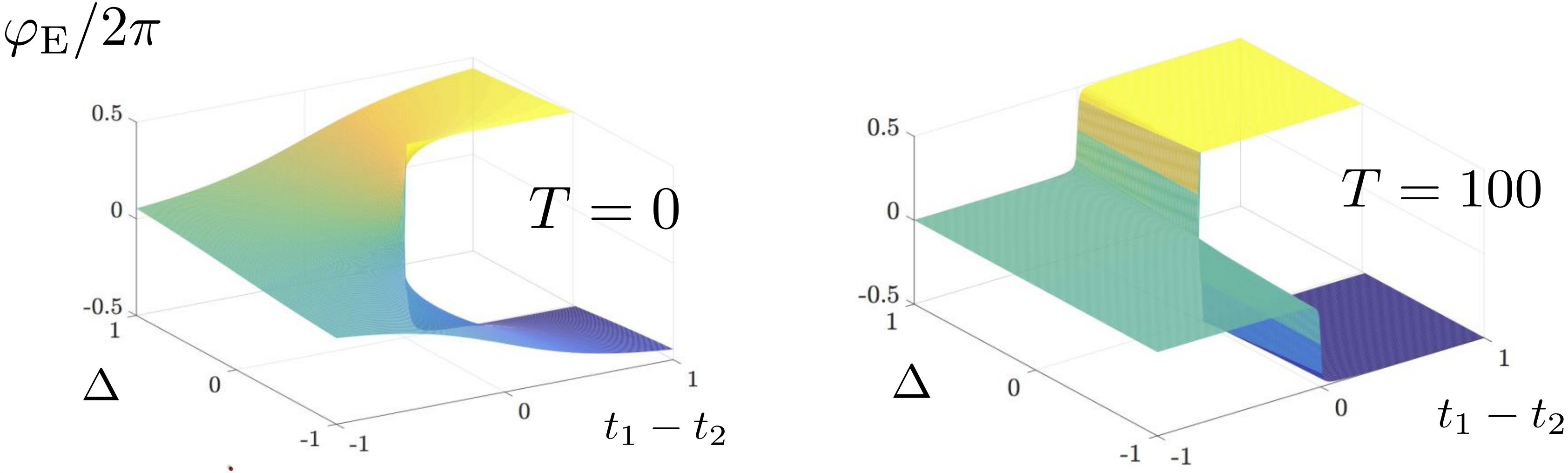
momentum-shift operator in x

$$\boxed{\varphi_{\mathrm{E}} = \mathrm{Im}\ln\mathrm{Tr}\left(\rho\hat{T}\right)}$$

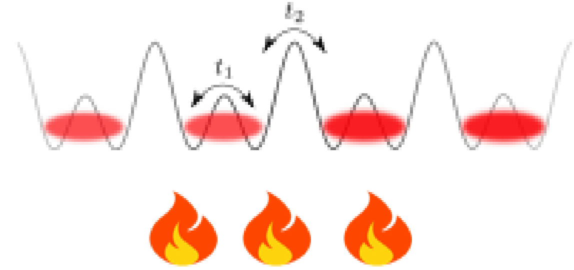
D. Linzner et al. PRB (2016); Ch. Bardyn et al. PRX (2018)

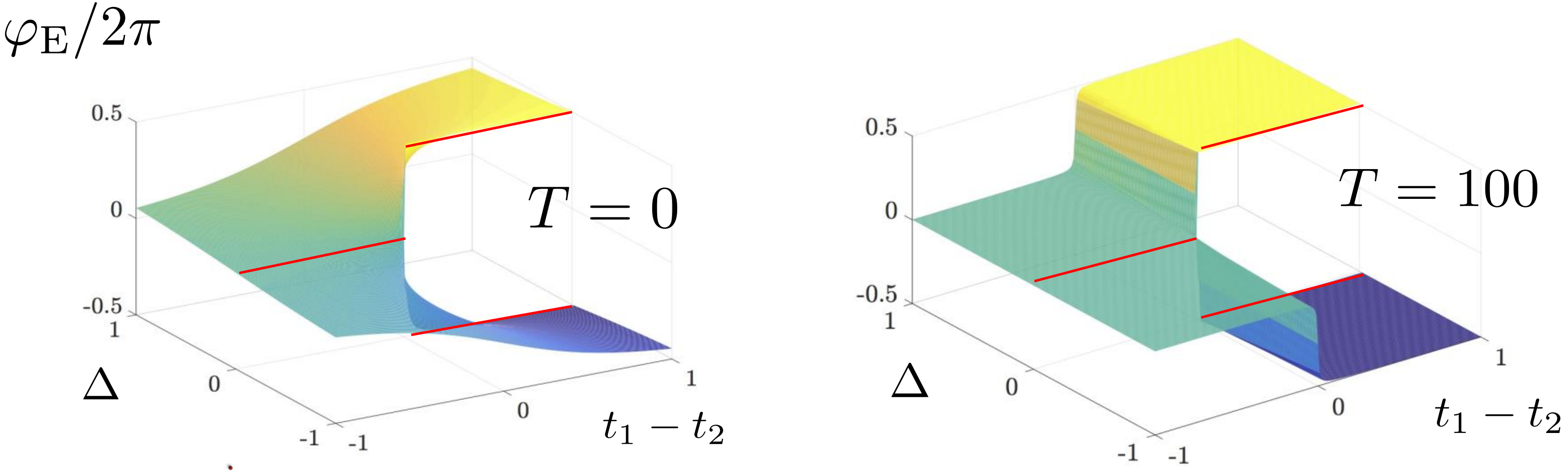
Finite-T Rice-Mele model



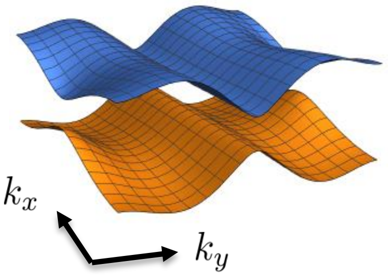


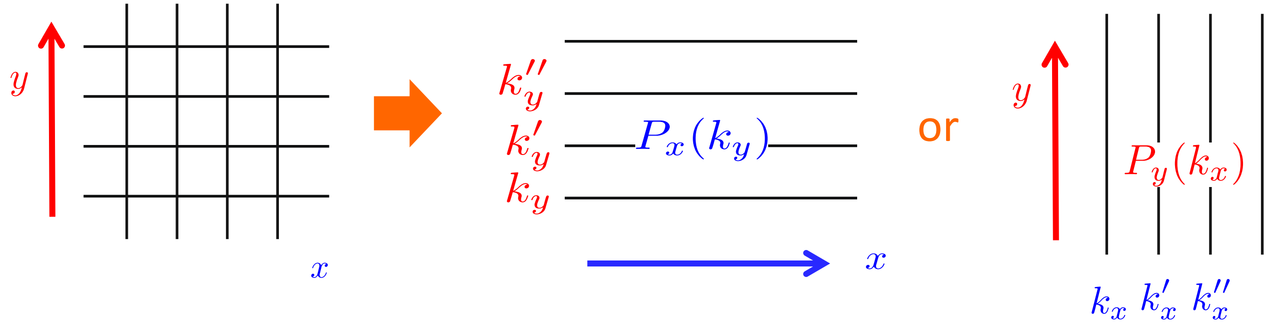
Finite-T SSH model





Finite-T Qi-Wu-Zhang model

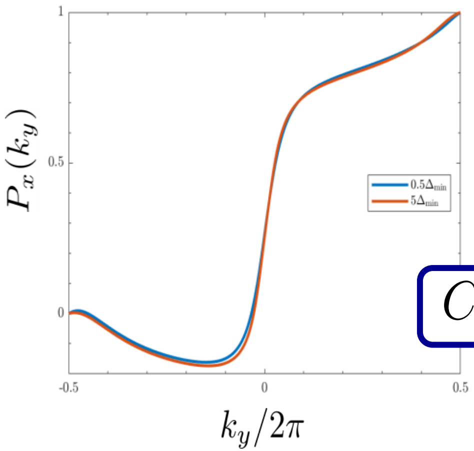


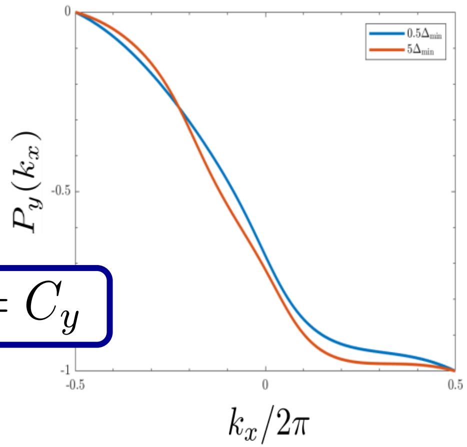


$$C_x = \int_{\mathrm{BZ}} \mathrm{d}k_y \frac{\partial P_x(k_y)}{\partial k_y}$$

or

$$C_y = -\int_{\mathrm{BZ}} \mathrm{d}k_x \frac{\partial P_y(k_x)}{\partial k_x}$$

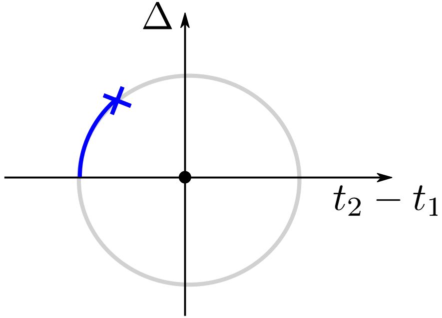


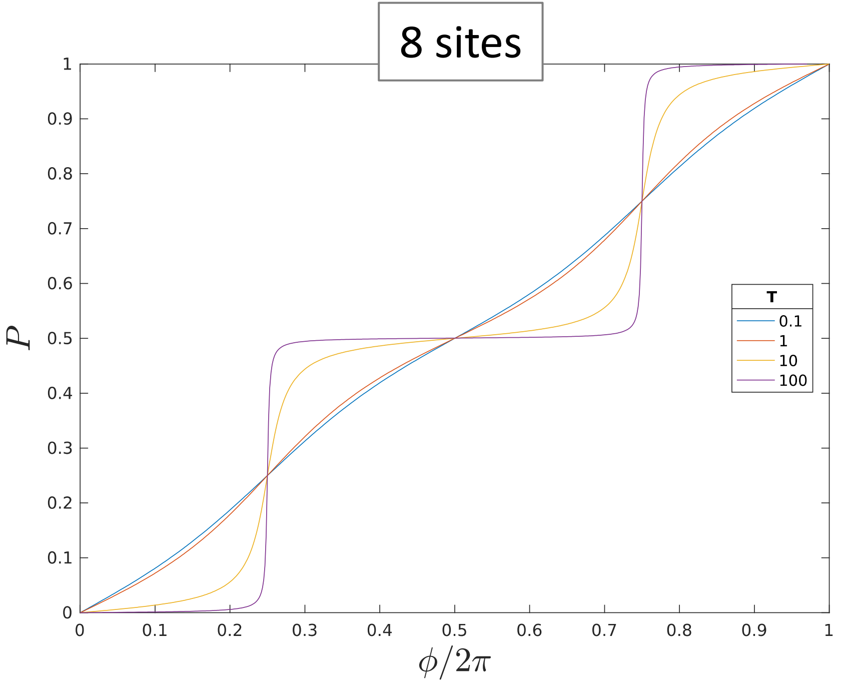


$$\boxed{C_x = C_y}$$

EGP and fictitious Hamiltonian

- finite-T Rice Mele model (Thouless charge pump)





Bardyn, Wawer, Altland, Fleischhauer, Diehl (PRX 2018)

EGP and fictitious Hamiltonian

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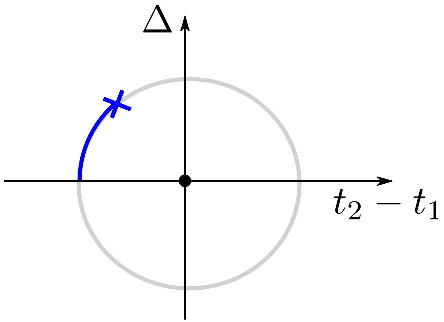

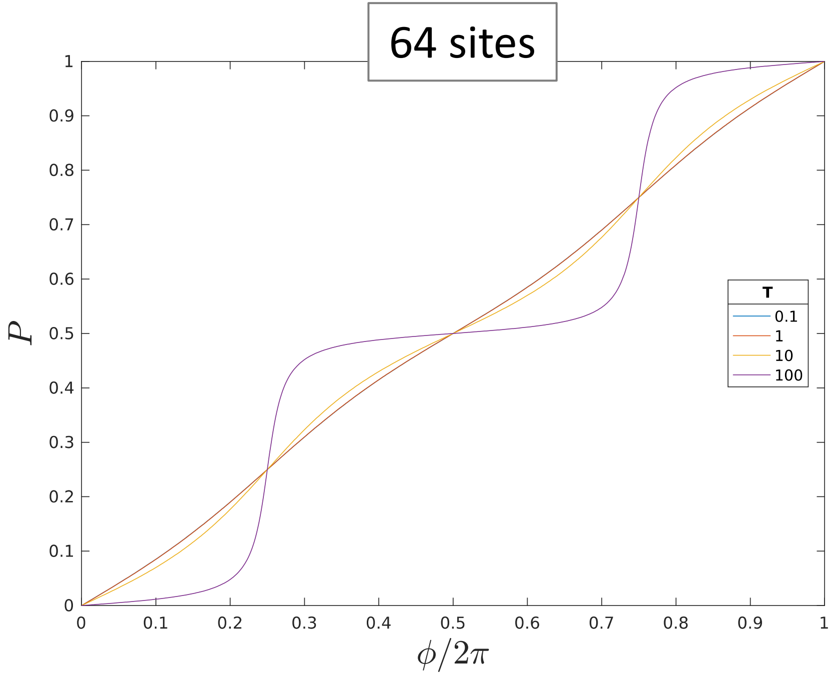

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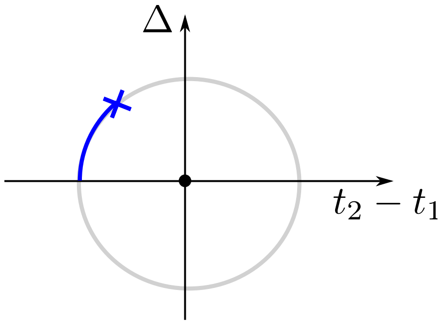

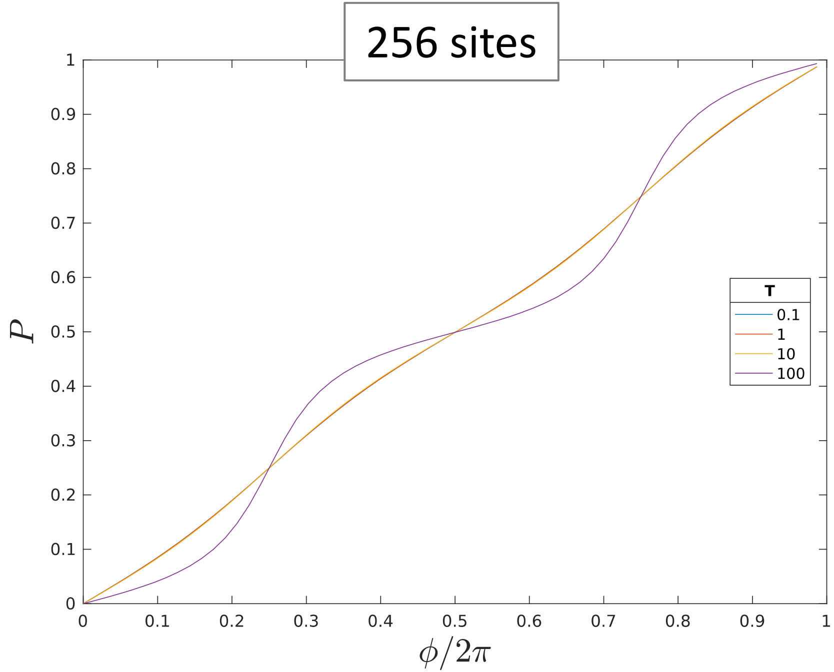

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EGP and fictitious Hamiltonian

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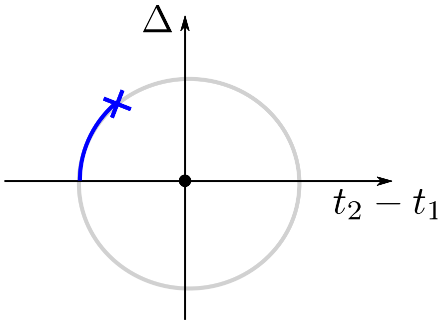
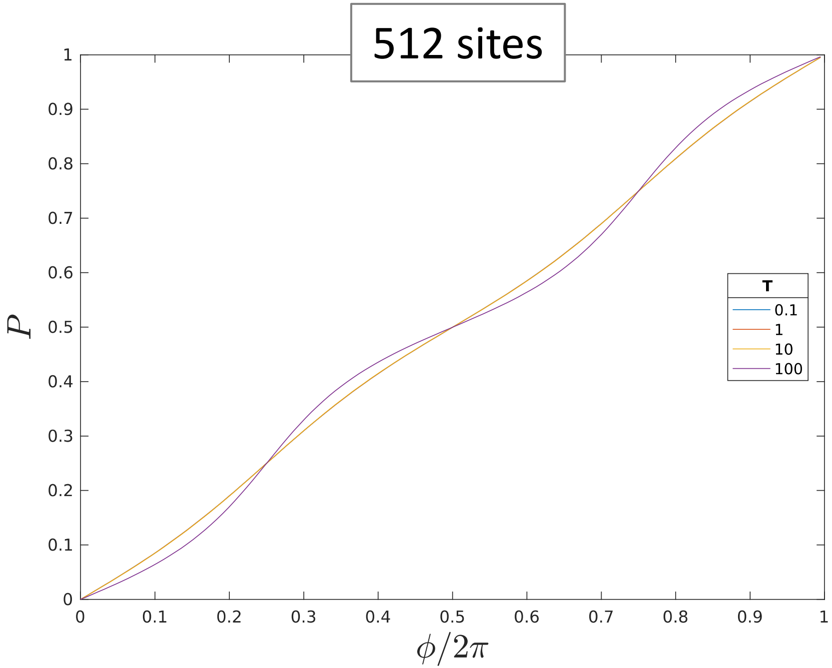




Bardyn, Wawer, Altland, Fleischhauer, Diehl (PRX 2018)

EGP and fictitious Hamiltonian

$$\boxed{P\left(\rho_{\mathrm{ss}}\right) = P\left(|\psi\rangle\langle\psi|\right) + \mathcal{O}\left(L^{-1}\right)}$$

$|\psi\rangle$ ground state of $$\mathcal{H}_{\mathrm{fict}} = \sum_{ij} G_{ij}\, c_i^\dagger c_j$$

φ_{EGP} = Zak phase of $|\psi\rangle$

Bardyn, Wawer, Altland, Fleischhauer, Diehl (PRX 2018)

Topological classification

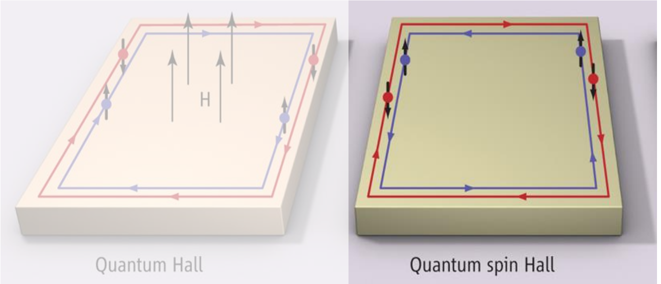
$$\mathcal{H}_{\mathrm{fict}} = \sum_{ij} G_{ij}\, c_i^\dagger c_j$$

- symmetries of ficticious Hamiltonian classify topology
- topological phase transitions
  - (I) closing of the purity gap = gap of ficticious Hamiltonian

    thermal equilibrium: $\underline{\underline{G}} = \beta\,(\underline{\underline{H}} - \mu)$
  - (II) closing of the damping gap (criticality)

Z_2 number

TR-symmetric



Kramers degeneracy

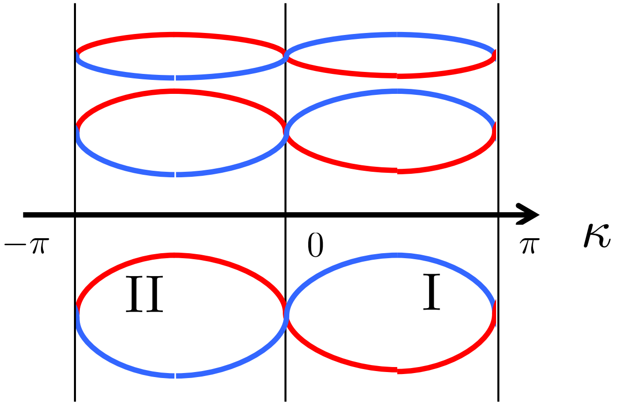
- time-reversed Bloch eigenstate is again an eigenstate

$$|u_{\mathrm{II}}(-\vec{k})\rangle = e^{i\chi(\vec{k})}\,\hat{\mathcal{T}}\,|u_{\mathrm{I}}(\vec{k})\rangle$$

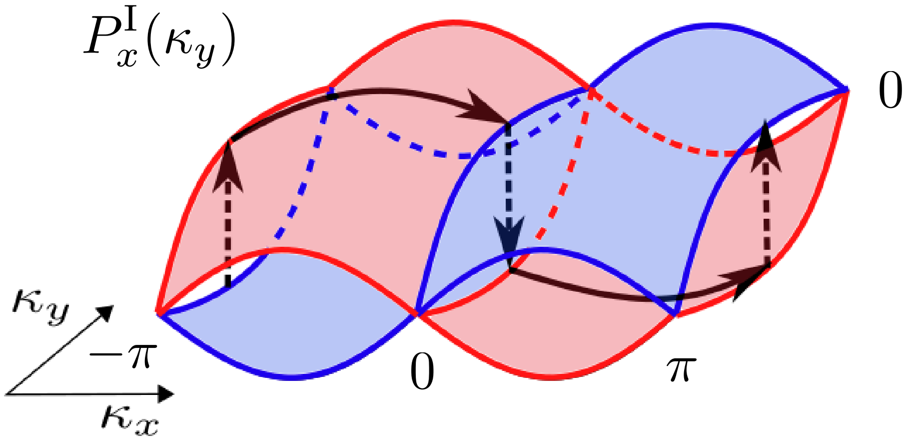
energy bands come in pairs

- total Chern number vanishes

$$C = \frac{1}{2\pi}\int_{\mathrm{BZ}} d\kappa_y \frac{\partial P_{\mathrm{tot}}(\kappa_y)}{\partial \kappa_y} = 0$$



Continuous TR polarization



Z$_2$ invariant: winding of continuous TR polarization over half Brillouin zone

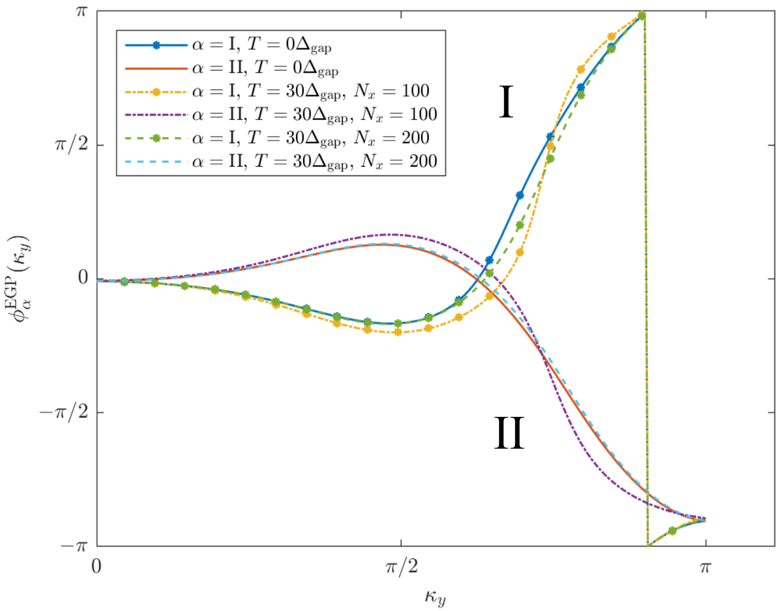
$$\nu_2 = \int_0^{\pi} d\kappa_y \frac{\partial P_\theta(\kappa_y)}{\partial \kappa_y}$$

$$P_\theta = P^{\mathrm{I}} - P^{\mathrm{II}}$$

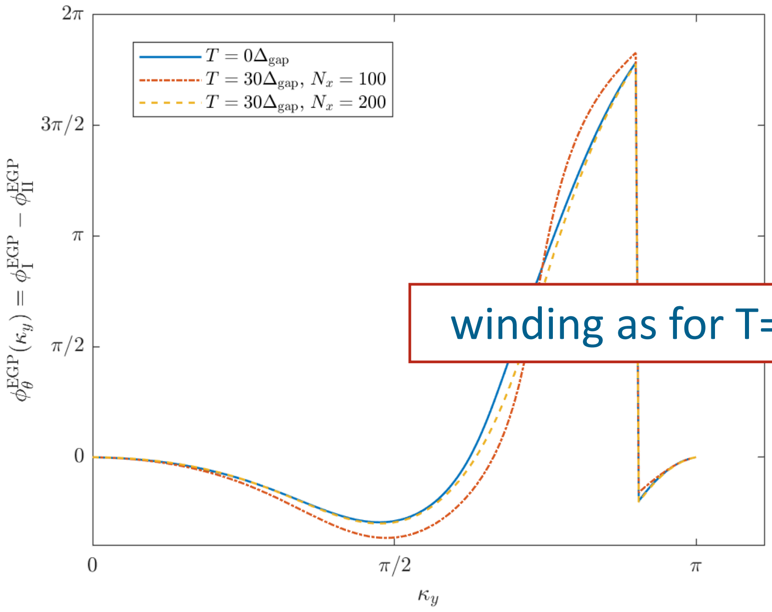
$$P^{\mathrm{I}}(\kappa_y) \sim \arg \prod_{\kappa_x=-\pi}^{0^-} \langle u^u(\kappa_x+\delta\kappa) | u^u(\kappa_x) \rangle \times \prod_{\kappa_x=0^+}^{\pi} \langle u^l(\kappa_x+\delta\kappa) | u^l(\kappa_x) \rangle$$

Kane Mele model at finite T

individual EGPs



time-reversal EGP



winding as for T=0

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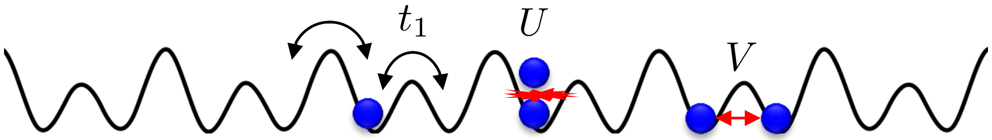
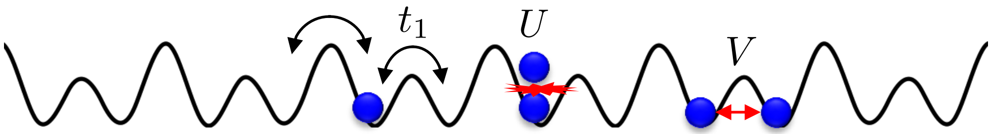
Interacting systems

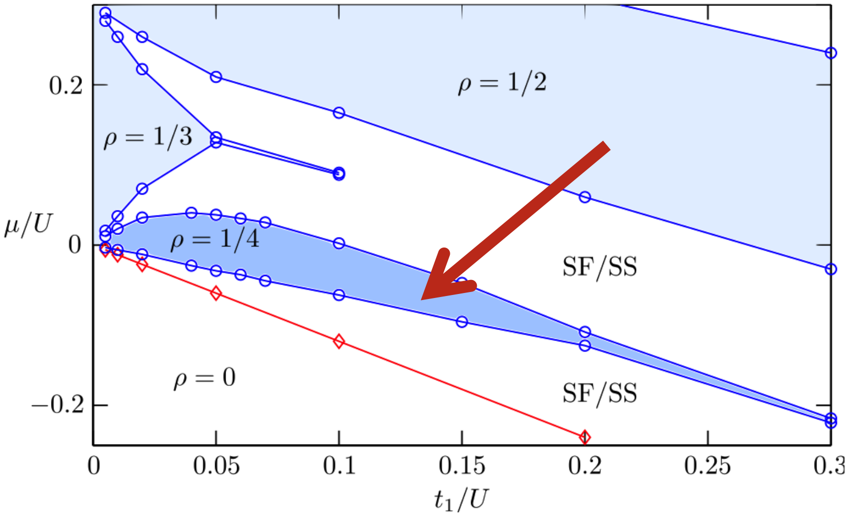


Extended superlattice Bose-Hubbard model

model



$$H = -t_1 \sum_{\text{odd}} \hat{a}_i^\dagger \hat{a}_{i+1} - t_2 \sum_{\text{even}} \hat{a}_i^\dagger \hat{a}_{i+1} + \frac{U}{2} \sum_{i} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \; + V \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_i \hat{a}_j^\dagger \hat{a}_j$$



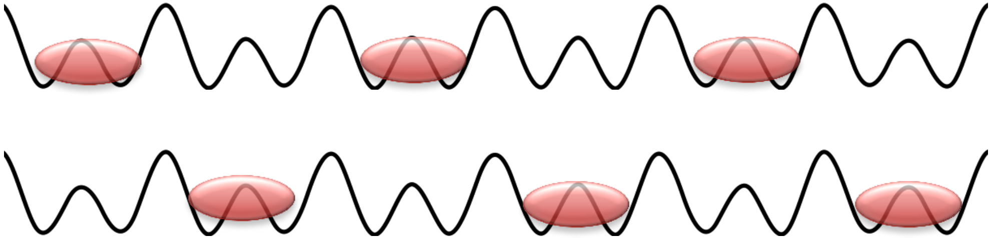
insulating phase with fractional filling

fractional filling & topological order

- atomic limit

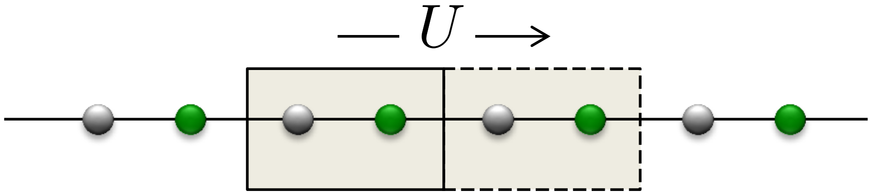
$$\rho = 1/4$$

degenerate CDWs



$$|\Phi_0^{(\pm)}\rangle = \mathrm{CDW}_1 \pm \mathrm{CDW}_2$$

Degeneracy & Wilson loop



$$\hat{T} = e^{i\frac{2\pi}{L}\hat{X}} \qquad U\hat{T}U^{-1} = \hat{T}\,e^{i2\pi N/L} \qquad \hat{X} = \sum_j j\,\hat{n}_j \qquad \langle\hat{T}\rangle = 0$$

- Wilson loop

$$\nu_{\text{tot}} = \frac{1}{2\pi}\oint d\lambda\,\frac{\partial}{\partial\lambda}\,\text{Im}\ln\det \mathsf{W}(\lambda)$$

$$\mathsf{W}(\lambda) = \mathcal{P}\exp\left\{i\int_0^{2\pi} d\theta\;\mathsf{A}(\theta)\right\} \qquad \mathsf{A}_{\mu\nu}(\theta) = i\langle\Phi_0^\mu|\partial_\lambda\Phi_0^\nu\rangle$$

d-fold degeneracy

Niu, Thouless, Wu, PRB 1985

$$|\Phi_0^{(1)}\rangle \overset{1}{\rightarrow} |\Phi_0^{(2)}\rangle \overset{2}{\rightarrow} \cdots \rightarrow |\Phi_0^{(d)}\rangle \overset{d}{\rightarrow} |\Phi_0^{(1)}\rangle$$

$$\nu_{\rm tot} = \frac{1}{2\pi}\int_0^{2\pi} d\lambda \int_0^{2\pi d} d\theta \ \mathrm{Im}\langle\Phi_0^\mu|\partial_\theta\Phi_0^\mu\rangle$$

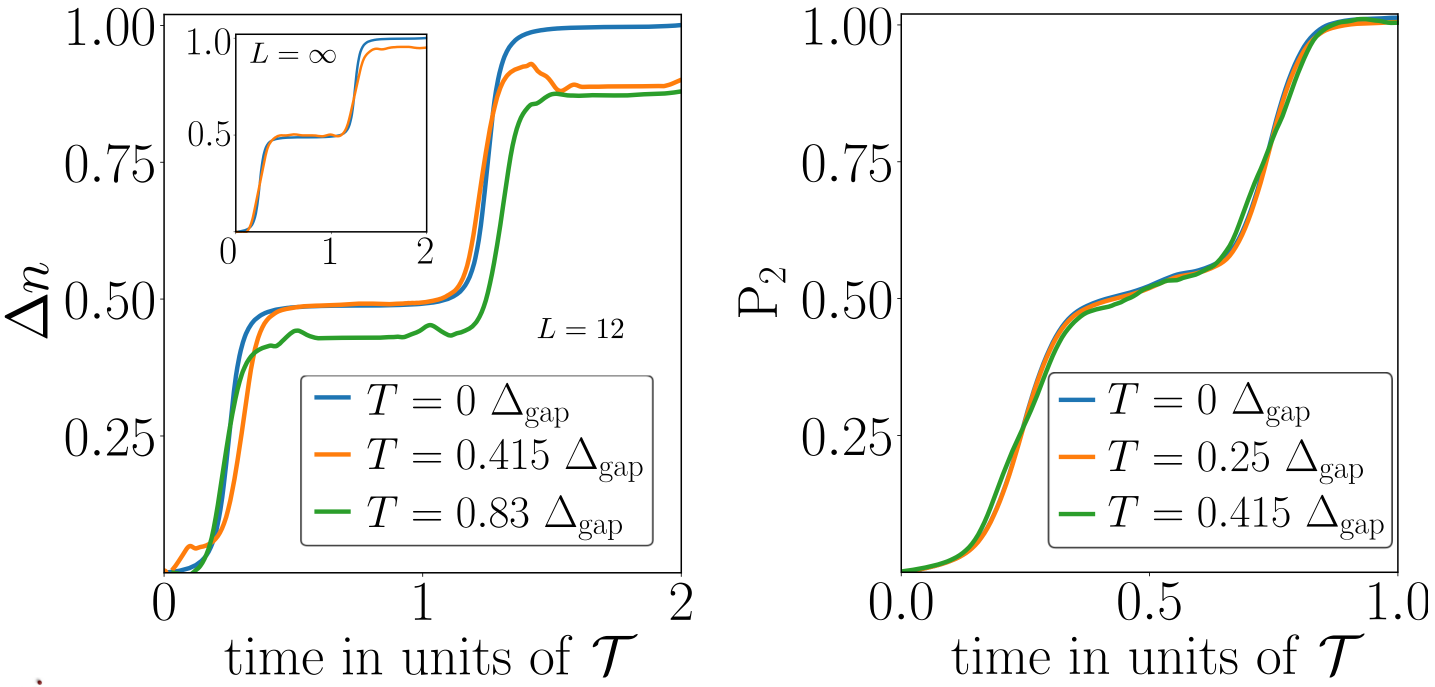
$$\hat{T} \rightarrow \hat{T}^d$$

Extended SL-Bose-Hubbard model

$d = 2$

transport

generalized EGP $P^{(2)}$



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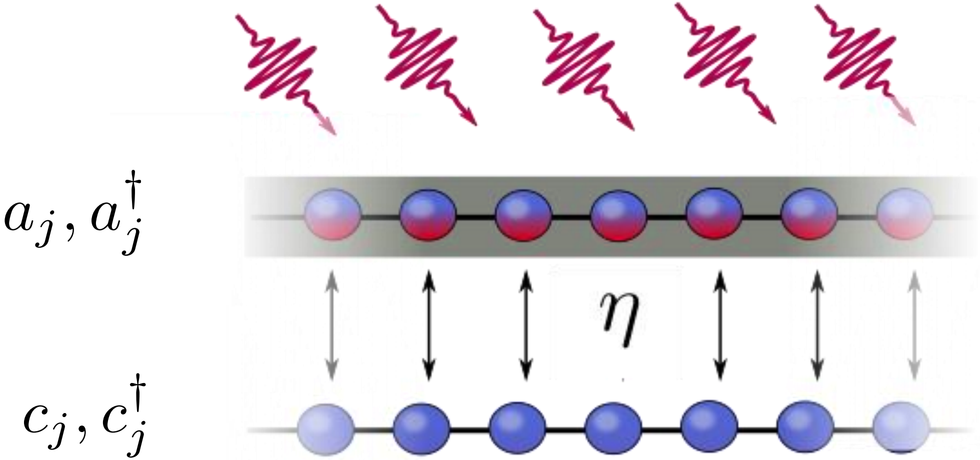
R. Unanyan et al. PRL (2020)

Measurable consequences?

Topology transfer

- coupling of open (finite-T) system to closed auxiliary fermion system at T = 0



$T = 0$

$$H = -\eta \sum_{k,\alpha,\alpha'} c^\dagger_{\alpha k} c_{\alpha' k} \, a^\dagger_{\alpha k} a_{\alpha' k}$$

Topology transfer

- dynamics of auxiliary fermions in mean-field approximation

$$H = -\eta \sum_{k,\alpha,\alpha'} c^\dagger_{\alpha k} c_{\alpha' k}\, a^\dagger_{\alpha k} a_{\alpha' k}$$

$$a^\dagger_n a_m \;\rightarrow \langle a^\dagger_n a_m \rangle \;\sim\; \mathsf{G}_{mn} \longleftarrow$$

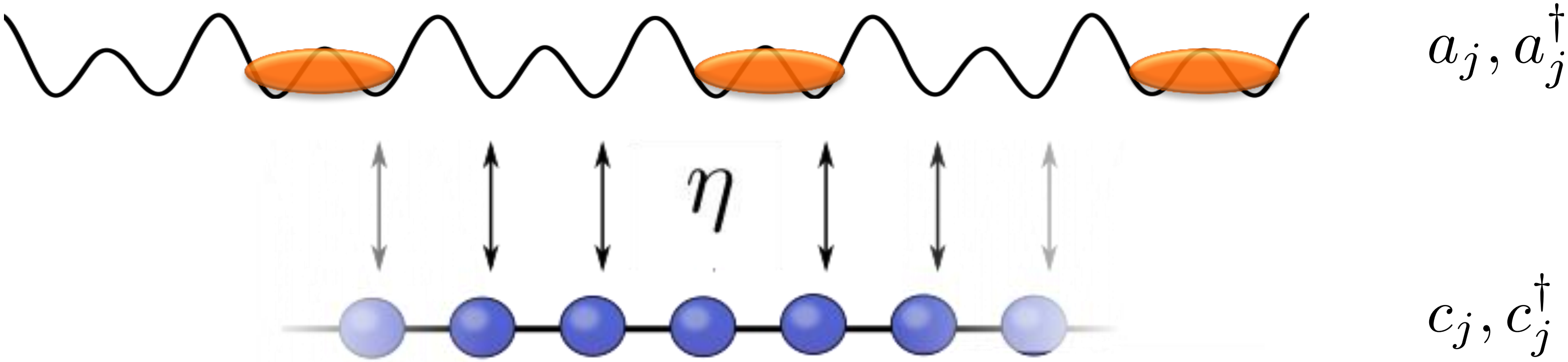
fictitious Hamiltonian

$$H \sim -\eta \sum_{k,\alpha,\alpha'} \hat{c}^\dagger_{\alpha k} \hat{c}_{\alpha' k}\, \mathsf{G}_{\alpha\alpha'}(k)$$

auxiliary system at T=0 → quantized transport induced by topology transfer

Topology transfer at T=0

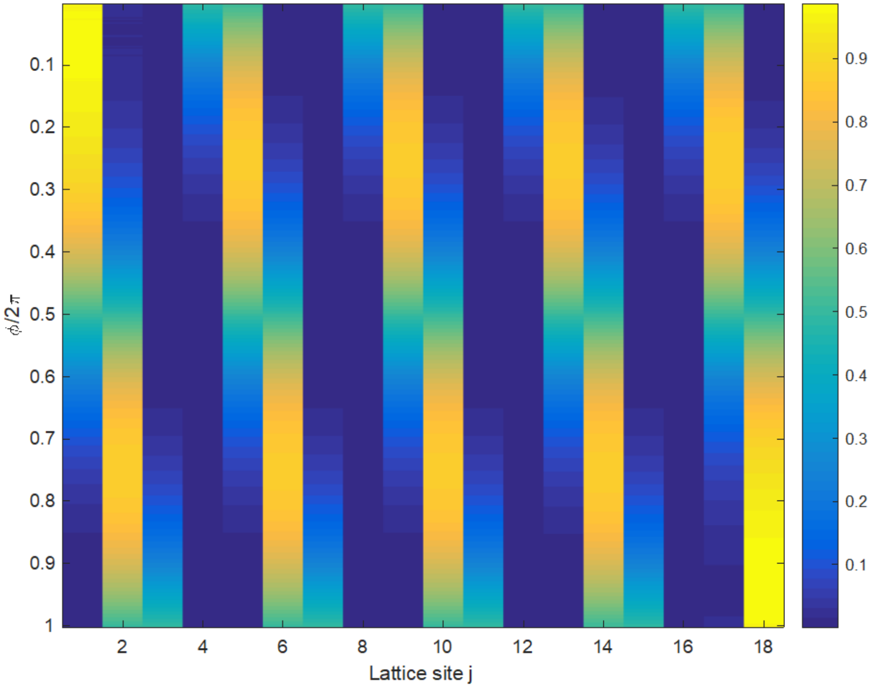
- coupling of Ext. SL-BHM to auxiliary fermion chain

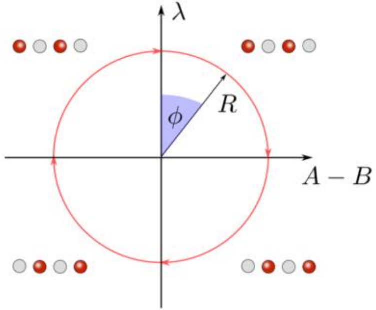


L. Wawer et al. arxiv 2009.04149

Topology transfer at T=0

- charge transport in boson system

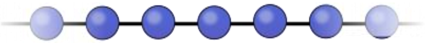


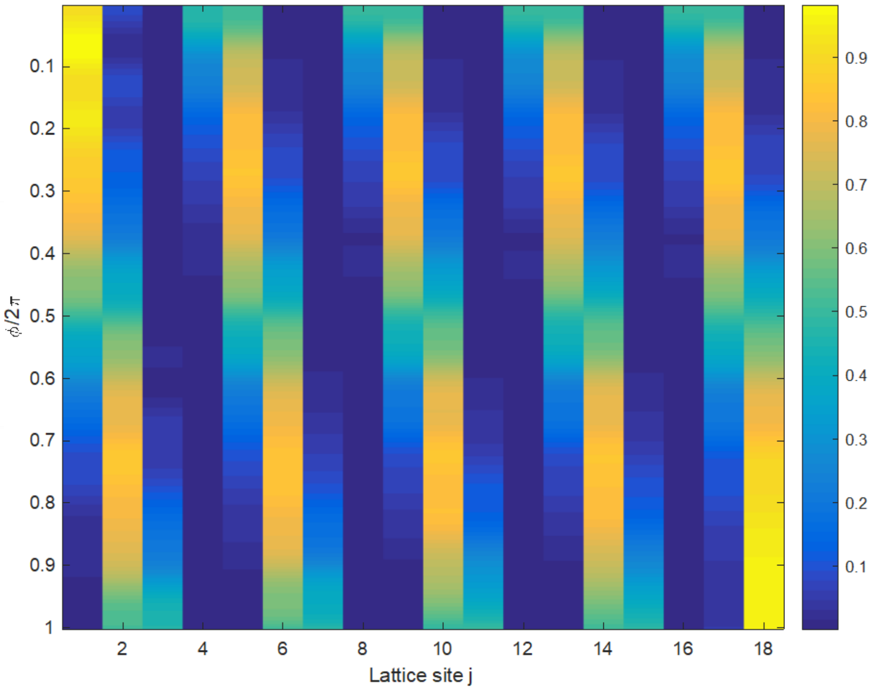


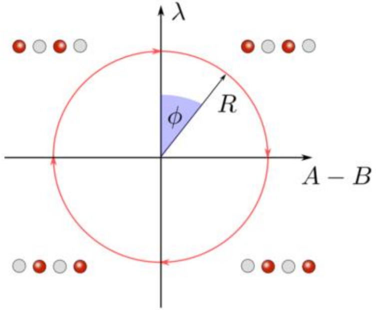
L. Wawer et al. arxiv 2009.04149

Topology transfer at T=0

- charge transport in auxiliary system







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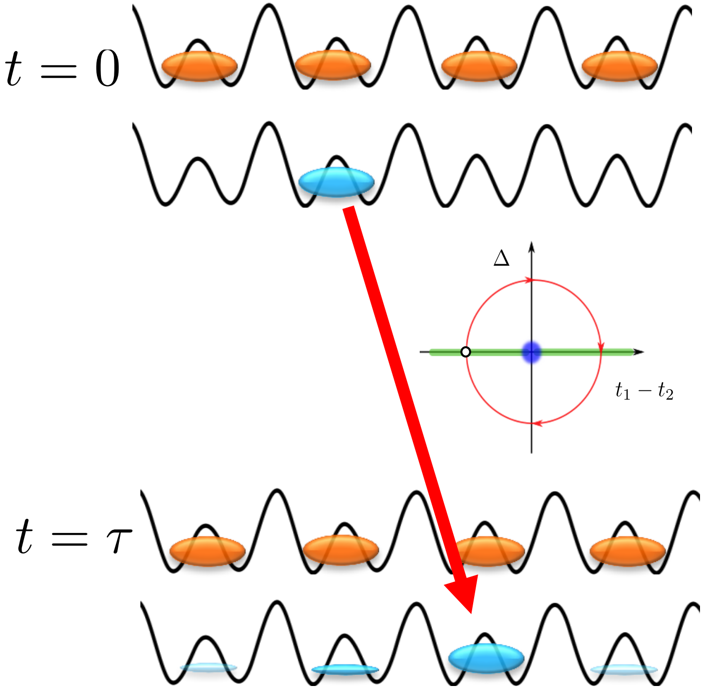
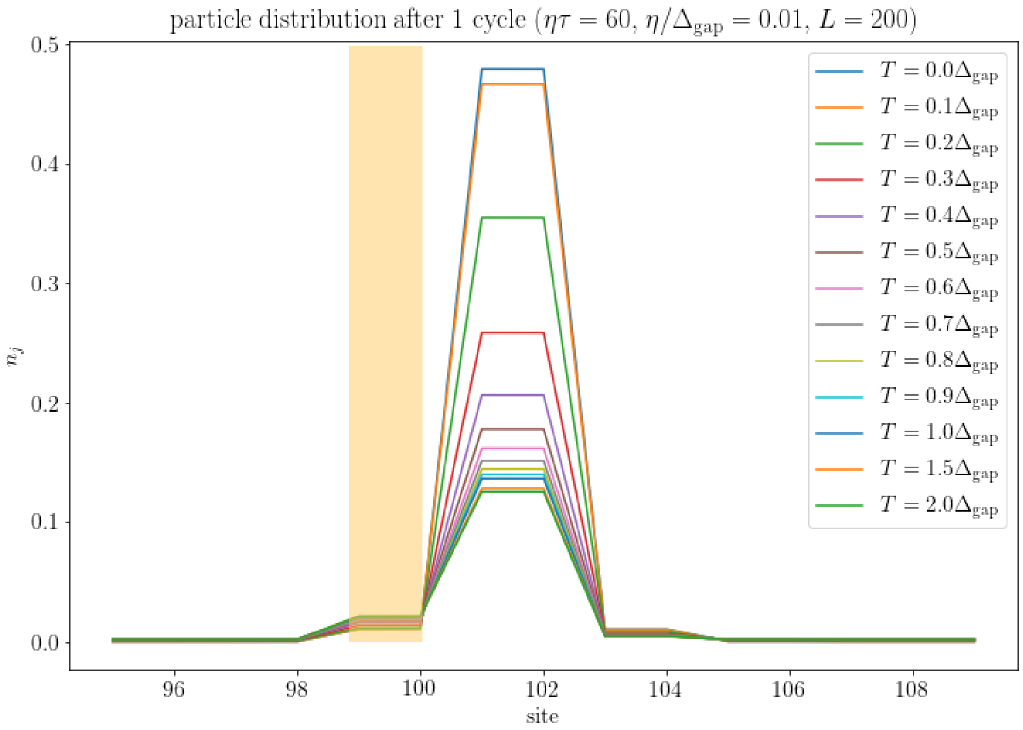
PHYSIK

Topology transfer at T > 0

Rice Mele model at T > 0

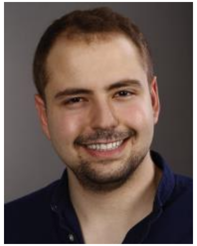




Summary

- Topology of Gaussian mixed states of fermions governed by symmetries of fictitious Hamiltonian (single-particle correlations)
- Z and Z_2 topological invariants in 1+1 and 2D:
  ensemble geometric phase = Zak phase of fictitious Hamiltonian
- Extension to interacting systems with fractional topological charges
- Measurable consequences: quantized transport through topology transfer

Thanks to



Max Kiefer

Lukas
Wawer

Razmik
Unanyan

Charles Bardyn
(Geneva)

Sebastian Diehl
(Cologne)

Alex Altland
(Cologne)

Dominik Linzner
Rui Li
Christopher Mink

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Thanks !