Classical Holographic Codes

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based on work with Enrico Brehm (LMU Munich)
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Outline

Holography and AdS/CFT

(gravitational) theory in some region of space (bulk)	$\Leftrightarrow$	(non-gravitational) theory confined to the boundary of that region

['t Hooft '93; Susskind '94]

Holography and AdS/CFT

(gravitational) theory in some region of space (bulk) $\Leftrightarrow$ (non-gravitational) theory confined to the boundary of that region

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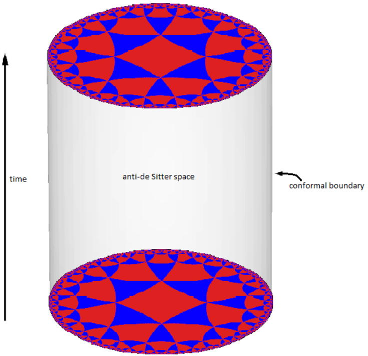
Explicit example: AdS/CFT

(quantum) gravity on $d+1$-dim. asymptotic AdS

$\Leftrightarrow$

d-dim. CFT on the boundary

[Maldacena '98]



Operator representations and subregion duality

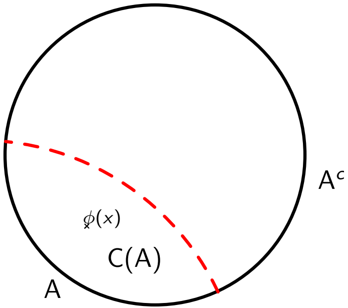


Figure: Equal-time slice of AdS.

Operator representation:

- there exists a operator rep. of $\phi(x)$ on A if $x \in C(A)$:

$$\phi(x) = \int_{D(A)} dX \, K_\phi(x, X) \mathcal{O}(X)$$

$D(A)$: domain of dependence
$C(A)$: causal wedge

Operator representations and subregion duality

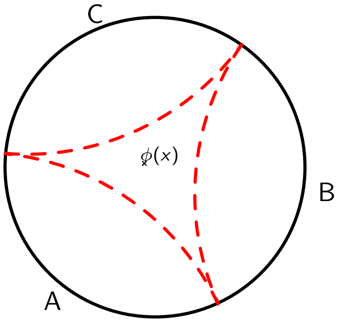


Figure: Equal-time slice of AdS.

subregion duality:

- there exists no operator rep. of $\phi(x)$ on A, B, or C alone.
- however, on AB, BC, or AC

$$\phi(x) = \int_{D(AB)} dX\, K_\phi(x, X)\mathcal{O}(X)$$

Ryu-Takayanagi formula

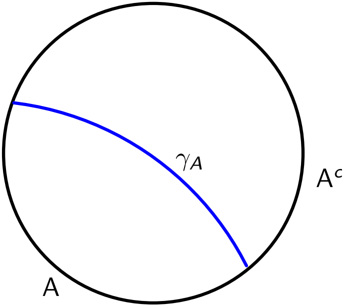


Figure: Equal-time slice of AdS.

Ryu-Takayanagi (RT) formula:

[Ryu, Takayanagi '06]

- entanglement entropy of ρ_A:

$$S(\rho_A) = \frac{\text{Area}(\gamma_A)}{4G}$$

γ_A: minimal surface

Toy models for AdS/CFT

- What is a toy model?
  - captures some characteristic features of a theory
  - there might be distinct toy models for the same theory
  - does not capture all features

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- Toy models for AdS/CFT:

- SYK model:
  - chaos
  - scrambling

[Sachdev, Ye '93]

[Kitaev '15]

- **Quantum error correcting codes:**
  - subregion duality
  - RT formula

[Almheiri, Dong, Harlow '15]

[Pastawski, Yoshida, Harlow, Preskill '15]

[Hayden, Nezami, Qi, Thomas, Walter, Yang '16]

Quantum error-correcting codes

Idea: Adding redundancy, i.e., protect "information" (logical qudit) by encoding it in a larger system (physical qudits). $\Rightarrow$ encode qudit states into entanglement.

Example: $\mathcal{H}_{\tilde{T}} = \mathbb{C}^3 \to \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

$$|\tilde{0}\rangle_{\tilde{T}} = \frac{1}{\sqrt{3}} \left(|000\rangle_{ABC} + |111\rangle_{ABC} + |222\rangle_{ABC} \right) ,$$

$$|\tilde{1}\rangle_{\tilde{T}} = \frac{1}{\sqrt{3}} \left(|012\rangle_{ABC} + |120\rangle_{ABC} + |201\rangle_{ABC} \right) ,$$

$$|\tilde{2}\rangle_{\tilde{T}} = \frac{1}{\sqrt{3}} \left(|021\rangle_{ABC} + |102\rangle_{ABC} + |210\rangle_{ABC} \right) .$$

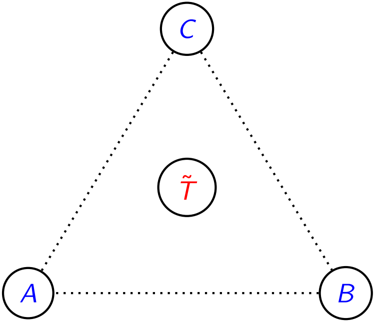

Figure: One logical qutrit $\tilde{T}$ is encoded into three physical qutrits A, B and C.

[Cleve, Gottesman, Lo '99]

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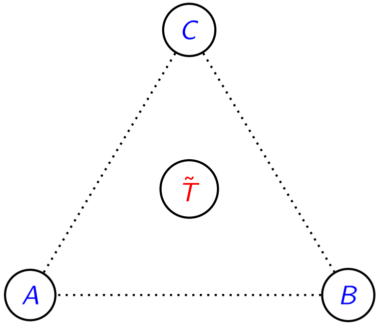
Then there exist unitaries such that

$$U_{AB} |\tilde{i}\rangle_{\tilde{T}} = |i\rangle_A \otimes |\chi\rangle_{BC} ,$$

$$|\chi\rangle = \frac{1}{\sqrt{3}} \left(|00\rangle + |11\rangle + |22\rangle \right) .$$



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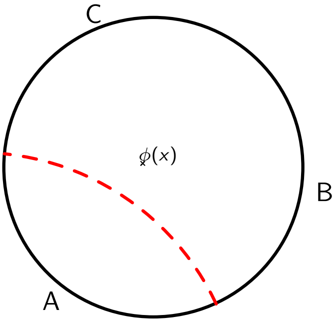
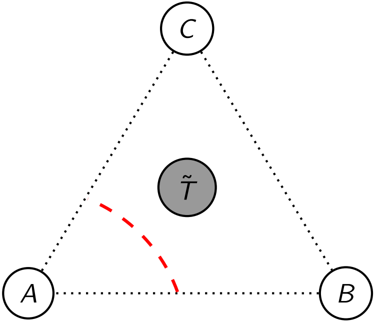
Quantum error correction and AdS/CFT





$$\phi(x) = \int_{D(BC)} dX\, K_\phi(x, X)\mathcal{O}(X)$$

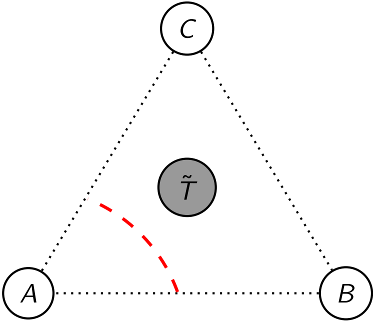
$$\begin{aligned}\tilde{O}|\tilde{i}\rangle_{\tilde{T}} &= U^\dagger_{BC}\, O_B\, U_{BC}|\tilde{i}\rangle_{\tilde{T}} \\ &= U^\dagger_{BC}\, O_B\, (|i\rangle_B \otimes |\chi\rangle_{AC}) \,.\end{aligned}$$

[Almheiri, Dong, Harlow '15]

Quantum error correction and AdS/CFT

Features:

- bulk reconstruction
- subregion duality
- lattice RT formula



[Almheiri, Dong, Harlow '15; Pastawski, Yoshida, Harlow, Preskill '15; Hayden, Nezami, Qi, Thomas, Walter, Yang '16; ...]

Classical holographic codes

- holographic description of entanglement in AdS/CFT [Ryu, Takayanagi '06]
- tensor networks are discussed as tools to build spacetime from entanglement [Swingle '09]
- toy models based on quantum error correction rely on entanglement [Almheiri, Dong, Harlow '15]

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$\Rightarrow$ Is there any chance to obtain models with holographic features without employing entanglement?

$\Rightarrow$ Or even classical holographic models?

$\Rightarrow$ Classical holographic codes [Brehm, BR '16]

Construction of classical holographic codes

Goal: find mapping between bulk and boundary such that:

- bulk can be reconstructed,
- sub-region duality exists,
- version of RT formula holds.

Approach:

- tile AdS space,
- insert one dof (bit) in each tile,
- define mappings

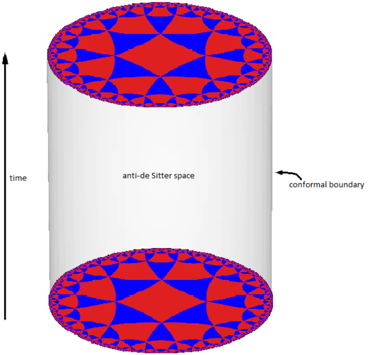

Figure: AdS.

Construction of classical holographic codes

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Approach:

- tile AdS space,
- insert one dof (bit) in each tile,
- define mappings ⇒ network,
- study properties.

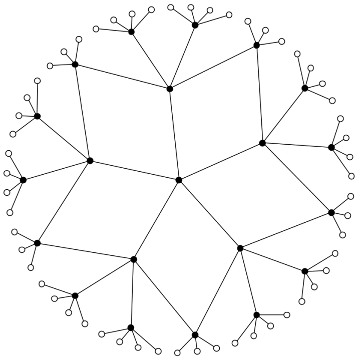


Figure: Network on tiling of AdS.

[c.f. tensor networks: [Pastawski, Yoshida, Harlow, Preskill '15]]

Construction of mappings

Mappings originate from a single set

$$\mathfrak{S} = \{s_k \,|\, k = 1, .., N\}$$

with

- s_k: strings of six bits $\quad (s_k \stackrel{\text{e.g.}}{=} 001111)$,
- discrete uniform probability distribution on $\mathfrak{S}$,
- $\mathfrak{S}$ has the property that substrings are (almost) maximally correlated.

Example:

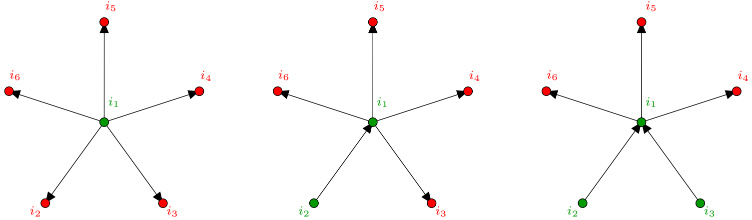
$$\mathfrak{S} = \{000000, 001111, 010110, 011001, 100101, 101010, 110011, 111100\} \,.$$

Construction of mappings

The probability density of the outcome of the mappings for a given input string s_{in} is defined by the conditional probabilities

$$p_{\text{out}}(s_{\text{out}} \mid s_{\text{in}}),$$

where $s_{\text{in}} \cup s_{\text{out}} \in \mathfrak{S}$, e.g., $000000, 001111 \in \mathfrak{S}$.



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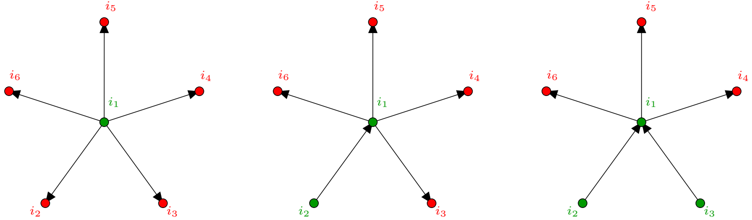


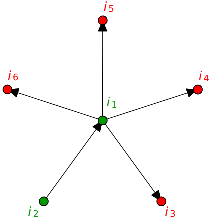
Properties of mappings:

(I) knowledge of three neighboring edge bits gives full information about the complementary bits;

(II) no information about any other single bit can be obtained by the knowledge of one bit.

Construction of mappings

Example: $2 \to 4$ -mapping



$$\mathfrak{S} = \{000000, 001111, 010110, 011001, 100101, 101010, 110011, 111100\}\,.$$

$$\tilde{0}0_e \to p(0000) = p(1111) = \frac{1}{2}\,, \quad \tilde{1}0_e \to p(0101) = p(1010) = \frac{1}{2}\,,$$
$$\tilde{0}1_e \to p(0110) = p(1001) = \frac{1}{2}\,, \quad \tilde{1}1_e \to p(1100) = p(0011) = \frac{1}{2}\,.$$

Features of classical holographic codes

We show that

- bulk input can be reconstructed,
- subregion duality exists,
- a version of the RT formula holds:

$$I_{cl}(A, A^c) = |\gamma_A| \,.$$

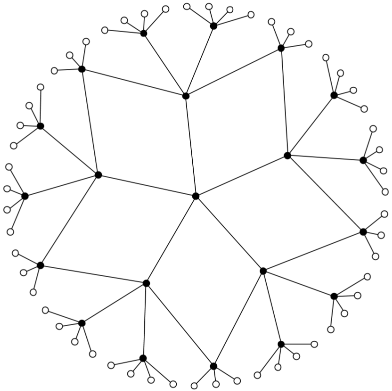


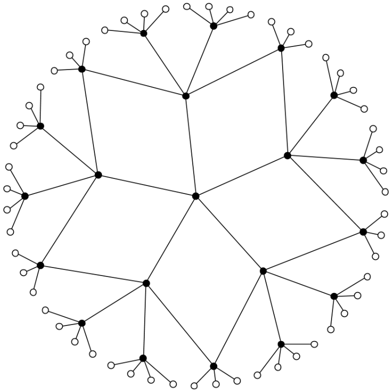
Figure: Network to realize a classical holographic code.

Reconstructing the bulk

Question: Given some subset of the boundary, which bulk bits can be reconstructed?

Remember:

(I) the knowledge of three neighboring edge bits gives full information about the three complementary bits.



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Reconstructing the bulk

(I) the knowledge of three neighboring edge bits gives full information about the three complementary bits.

A bulk bit can be reconstructed

$\Leftrightarrow$

It is contained in the correlation wedge $C(A)$

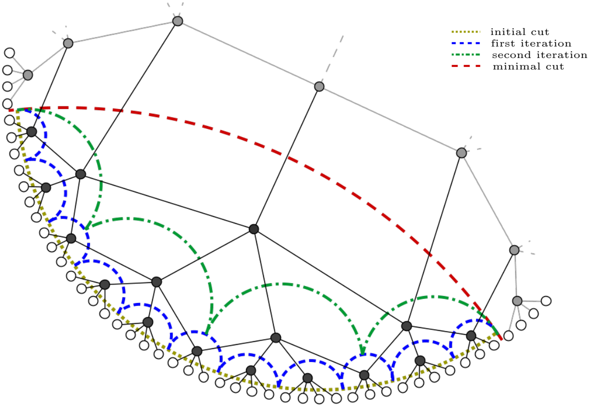
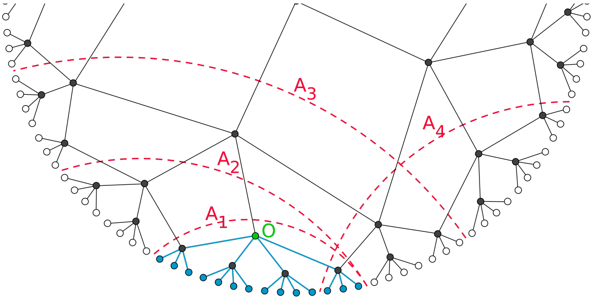


Figure: Algorithm to construct the minimal cut.

Representation of operations: subregion duality

- bit flip O
- boundary regions A_i



A bulk operation O can be represented on A_i

$$\Leftrightarrow$$

It is supported in the correlation wedge $C(A_i)$

$\Rightarrow$ subregion duality

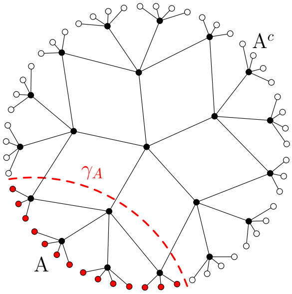
A version of the RT formula

The classical mutual information between a (connected) subregion A on the boundary and its complement A^c is given by the length of the minimal cut γ_A through the network

$$I_{cl}(A, A^c) = |\gamma_A| \, ,$$

where

$$I_{qu/cl}(A, B) = S(A) + S(B) - S(A, B) \, .$$



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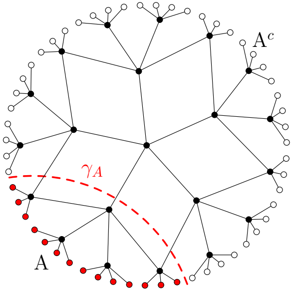

Tensor networks: [Pastawski, Yoshida, Harlow, Preskill '15]

$$S(\rho_A) = \frac{\text{Area}(\gamma_A)}{4G} \overset{\text{network}}{=} |\gamma_A| \qquad \overset{\text{pure state}}{\Rightarrow} \qquad I_{qm}(A, A^c) = 2|\gamma_A|$$

Conclusions

We introduced *classical holographic codes* and analyzed their properties:

- a version of the Ryu-Takayanagi formula holds,
- bulk inputs contained in the correlation wedge $C(A)$ can be reconstructed from the data in A,
- a operation O, acting on any bulk input contained in $C(A)$, can be represented by multiple bit flips in A.

These properties are due to the "correlation structure" and can exist even classically in the absence of quantum correlations, like entanglement.

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A version of the RT formula

$$I_{cl}(A, A^c) = |\gamma_A| \,. \qquad (1)$$

Idea of proof:

1. Show upper bound:

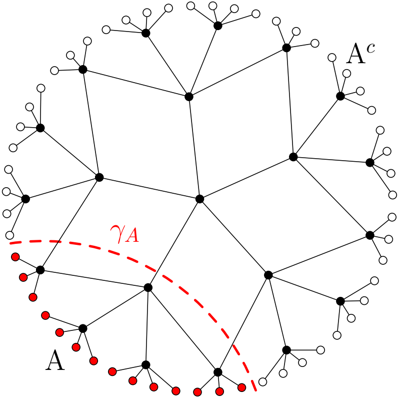
$$I_{cl}(A, A^c) \leq |\gamma_A| \,.$$

2. Show that there are no correlations between edges crossing γ_A.

⇒ Each edge contributes one bit to $I_{cl}(A, A^c)$.

⇒ Upper bound is saturated.

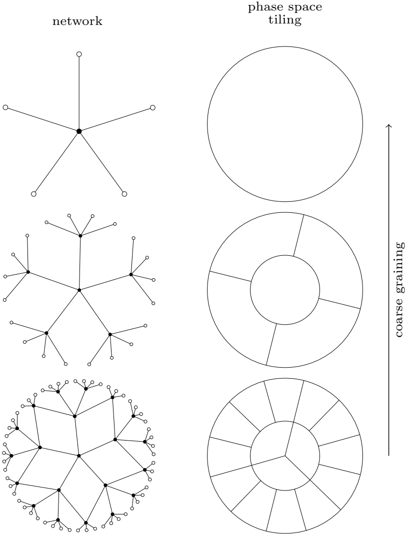
⇒ formula (1)



A possible physical interpretation

bulk direction as a coarse graining parameter for an effective description of the boundary

- interpolates between the microscopic description (at the boundary of AdS) and the macroscopic description (in the center of AdS)
- analogy to the renormalization group flow in AdS/CFT:
  RG flow from the UV to the IR fix point



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