Adiabatic quantum transport

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Quantum Matter meets Math

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Plan

- ▷ The adiabatic principle
- ▷ Parallel transport
- ▷ An index for charge transport
- ▷ Local charge fluctuations
- ▷ Anyons

Adiabatic principle

Evolution equation

$$\frac{d}{dt}\varphi(t) = L_p(\varphi(t)) \qquad \varphi(0) = \varphi_0$$

for

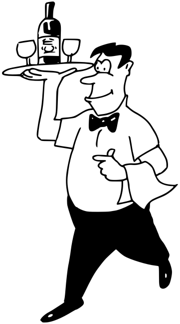
$$L : \mathcal{M} \times \mathcal{H} \to \mathcal{H}$$

where

- $\mathcal{H}$ is a Banach space
- $\mathcal{M}$ is the manifold of parameters

Stable fixpoints

$$\varphi(t) \rightharpoonup \varphi_p \qquad (t \to \infty)$$



Adiabatic principle

Driving

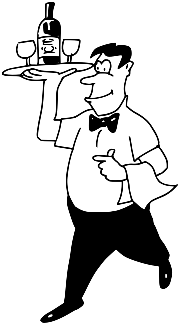
$$t \mapsto p_t$$

namely

$$\frac{d}{dt}\varphi(t) = L_{p_t}(\varphi(t)) \qquad \varphi(0) = \varphi_{p_0}$$

Adiabatic principle:

If the driving is slow, the solution $\varphi(t)$ shadows the instantaneous fixpoint φ_{p_t}



Thouless pump

In quantum mechanics:

- $\varphi(t) \in \mathcal{H}$, a Hilbert space
- Schrödinger's equation

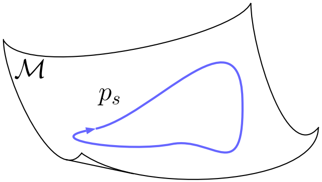
$$\frac{d}{dt}\varphi(t) = -\mathrm{i}H_p\varphi(t)$$

for a linear $H_p = H_p^*$.

A quantum pump

- Normalized ground states $\Omega_p \in \mathcal{H}$
- Spectral gap
- Closed smooth path

$$p : [0,1] \to \mathcal{M} \qquad p_0 = p_1$$



Geometry

Adiabatic time

$$s = \epsilon t \qquad 0 < \epsilon \ll 1$$

accordingly,

$$\epsilon \frac{d}{ds} \varphi_\epsilon(s) = -\mathrm{i} H_{p_s} \varphi_\epsilon(s)$$

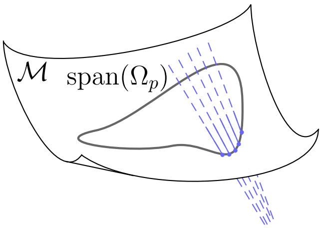
Adiabatic theorem (Kato,...)

$$\varphi_\epsilon(s) \to \Omega_{p_s} \qquad (\epsilon \to 0)$$

The parallel transport in a vector bundle

$$s \mapsto \Omega_{p_s}$$

is geometric (Berry, Simon,...)



Parallel transport & charge transport

Physical consequences:

- Quantum Hall effect

$$\text{Hall conductance} = \text{adiabatic curvature}$$

as a consequence (Thouless, Niu, Avron, Seiler,...)

$$\text{Hall conductance} \in \mathbb{Z}$$

- For quantum pumps

$$2\pi \,\text{Charge transport} = \oint_{\mathrm{int}(p)} \text{Adiabatic curvature}$$

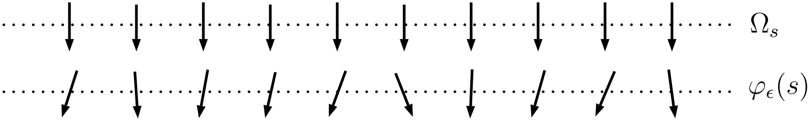
(..., Brouwer, Graf,...)

Orthogonality catastrophe

Family of quantum systems – atoms in a crystal

$$\mathcal{H} = \otimes_{x \in \Lambda} \mathcal{H}^{\{x\}}, \qquad \dim(\mathcal{H}^{\{x\}}) < \infty$$

After adiabatic evolution:



$$\langle \varphi_\epsilon(s), \Omega_s \rangle \to 0 \qquad (|\Lambda| \to \infty)$$

Adiabatic theorem, quantized transport, in the thermodynamic limit?

$$\lim_{\epsilon \to 0} \lim_{|\Lambda| \to \infty}$$

Local parallel transport

- ▷ Weak-* topology: only expectation values
- ▷ Local parallel transport (Hastings,...)

$$\Omega_s = U_{\|}(s)\Omega_0$$

and

$$A_X \text{ supported in } X \quad \Rightarrow \quad U_{\|}(s)^* A_X U_{\|}(s) \text{ supported in } X$$

- ▷ Higher order perturbation theory to control times of order ϵ^{-1}

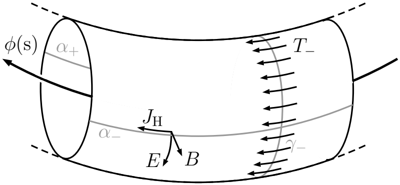
Theorem. [B-De Roeck-Fraas]
If $s \mapsto H_s$ is smooth, then uniformly in the volume

$$|\langle \varphi_\epsilon(s), A_X \varphi_\epsilon(s)\rangle - \langle \Omega_s, A_X \Omega_s\rangle| \leq C\|A_X\||X|^2\epsilon$$

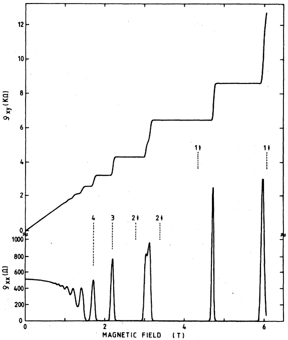
see also Monaco-Teufel (fermions), Henheik-Teufel (infinite volume)

More on the Hall effect

A Hall setting:



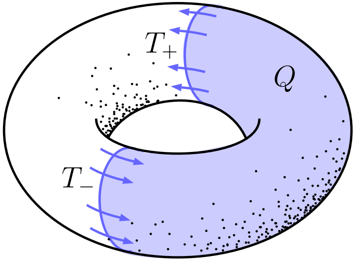
Quantized conductance:



v.Klitzing–Dorda–Pepper (1980)

$$\sigma_H = \frac{J_H}{E} = n\frac{e^2}{h}, \qquad n \in \mathbb{Z}$$

Finite volume setting



Flux Hamiltonians

$$H_\phi, P_\phi \quad \text{s.t.} \quad H_0 = H_{2\pi}$$

$U(1)$-charge

$$Q_Z = \sum_{x \in Z} q_x \qquad \mathrm{Spec}(q_x) \subset \mathbb{Z}$$

Parallel transport $U_\|$ over cycle $\phi = 0 \to \phi = 2\pi$

$$U_\| P U_\|^* = P$$

Local charge conservation

$$T = U_\|^* Q U_\| - Q = T_- + T_+$$

Index

Theorem. [B-Bols-De Roeck-Fraas]
Let $p = \mathrm{Rank}(P)$. *Then*

$$\text{(i)} \quad \mathrm{Ind}_P(U_\parallel) = \mathrm{Tr}(P(U_\parallel^* Q U_\parallel - Q)_-) \in \mathbb{Z}_{(\mathcal{O}(L^{-\infty}))}$$

$$\text{(ii)} \quad 2\pi\sigma_{\mathrm{H}} = p^{-1}\mathrm{Tr}(P(U_\parallel^* Q U_\parallel - Q)_-) + \mathcal{O}(L^{-\infty})$$

- Interacting electrons
- (i) holds for any adiabatic quantum pump
- In fact, (i) holds for any local U such that $[P, U] = 0$
- Fractional Hall conductance

Noninteracting fermions

- ▷ Unique ground state Ω_{F}, the Fock vacuum (filled Fermi sea):

$$P = |\Omega_{\mathrm{F}}\rangle\langle\Omega_{\mathrm{F}}|$$

- ▷ Hence $p = 1$: integer charge transport
- ▷ Index of projections (1d)

$$\begin{aligned}\mathrm{Ind}_P(U) &= \langle\Omega_{\mathrm{F}}, (\Gamma(u)^* \mathrm{d}\Gamma(q)\Gamma(u) - \mathrm{d}\Gamma(q))\Omega_{\mathrm{F}}\rangle \\ &= \mathrm{tr}(p_{\mathrm{F}}(u^* q u - q)) \\ &= \mathrm{tr}(u^*(p_{\mathrm{F}} q p_{\mathrm{F}})u - (p_{\mathrm{F}} q p_{\mathrm{F}}))\end{aligned}$$

for p_{F} the Fermi projection

- ▷ Charge: $q = \chi_{[0,\infty)}$, and

$$p_{\mathrm{F}} q p_{\mathrm{F}} = \mathrm{projection} + \mathrm{compact}$$

Linear response

Charge transport is driven (Faraday's law) by

$$E = -(\text{flux change}) = -\epsilon\phi'(s)$$

Linear response is exact:

Theorem. [B-De Roeck-Lange-Fraas]
For the Schrödinger propagator U_ϵ

$$p^{-1}\mathrm{Tr}(P(U_\epsilon^* Q U_\epsilon - Q)_-) = 2\pi\sigma_{\mathrm{H}} + \mathcal{O}(\epsilon^\infty)$$

see also Klein-Seiler

Block diagonalization

The linear map on $\mathcal{A}$

$$A \mapsto \mathcal{I}(A) := \widehat{W}(-\mathrm{ad}_H)(A) = \int_{-\infty}^{\infty} W(t)\mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} \mathrm{d}t$$

for $W \in L^1(\mathbb{R};\mathbb{R}) \cap L^\infty(\mathbb{R};\mathbb{R})$ such that

$$\widehat{W}(\xi) = -\mathrm{i}\xi^{-1} \qquad (|\xi| > \mathrm{gap})$$

is the inverse of $-\mathrm{i}\,\mathrm{ad}_H$

$$O - \mathcal{I}(\mathrm{i}[O, H]) = 0$$

whenever $O = PO(1-P) + (1-P)OP$

$$\begin{pmatrix} POP & POP^\perp \\ P^\perp OP & P^\perp OP^\perp \end{pmatrix}$$

Block diagonalization

- Generator of parallel transport

$$P' = \mathcal{I}(\mathrm{i}[P', H]) = \mathrm{i}\mathcal{I}([H', P]) = \mathrm{i}[\mathcal{I}(H'), P]$$

- For any operator A

$$\overline{A} = A - \mathcal{I}(\mathrm{i}[A, H])$$

  is block diagonal

$$[\overline{A}, P] = 0$$

- Locality

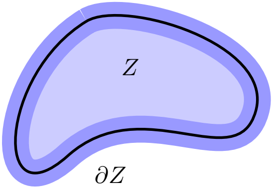
$$\mathrm{supp}(B) = X \implies \mathrm{supp}(\mathcal{I}(B)) \sim X$$

  (because $|W(t)| = \mathcal{O}(|t|^{-\infty})$)

Charge fluctuations

Charge conservation

$$\mathrm{supp}([H, Q_Z]) = \partial Z$$



Then,

- $K_{\partial Z} = \mathcal{I}([H, Q_Z])$ is supported on ∂Z
- $\overline{Q}_Z = Q - K_{\partial Z}$ is so that

$$[P, \overline{Q}_Z] = 0$$

Wilson loops

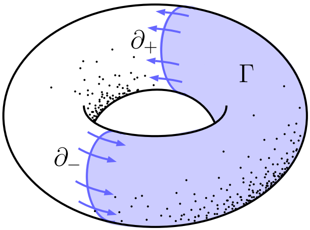
$$\begin{aligned} \mathrm{e}^{-2\pi \mathrm{i} \overline{Q}_Z} &= \mathrm{e}^{-2\pi \mathrm{i}(Q_{\partial Z} - K_{\partial Z})} \mathrm{e}^{-2\pi \mathrm{i}(Q_Z - Q_{\partial Z})} \\ &= \mathrm{e}^{-2\pi \mathrm{i}(Q_{\partial Z} - K_{\partial Z})} \in \mathcal{A}_{\partial Z} \end{aligned}$$

since $\mathrm{Spec}(Q_Z - Q_{\partial Z}) \subset \mathbb{Z}$.

Loops and boundaries

For half-space Q

$$V = \mathrm{e}^{-2\pi\mathrm{i}\overline{Q}} = \mathrm{e}^{-2\pi\mathrm{i}\overline{Q}_-}\mathrm{e}^{-2\pi\mathrm{i}\overline{Q}_+} = V_-V_+$$



With the gap

$$PV_-V_+P = PV_-PV_+P$$

and hence

$$[P, V_\pm] = 0$$

but $[P, \overline{Q}_-] \neq 0$.

Importantly

$$\partial_- \cup \partial_+ = \partial\Gamma$$

but ∂_- is not a boundary

Loops and boundaries

If

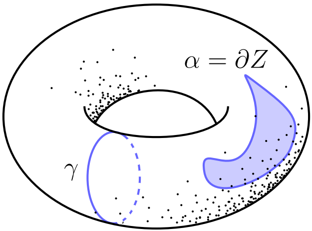
$$P\overline{Q}_Z P \propto P$$

then

- α is a boundary

  $$V_\alpha P = P\mathrm{e}^{2\pi \mathrm{i}\overline{Q}_Z} P = \mathrm{e}^{2\pi \mathrm{i} P\overline{Q}_Z P} \propto P$$

  V_α acts trivially on $\mathrm{Ran} P$
- γ is not a boundary,
  V_γ may act nontrivially on $\mathrm{Ran} P$

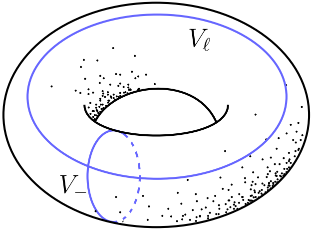


Algebra of loops

The index theorem can be phrased as

$$V_\ell^* V_- V_\ell V_-^* P = \mathrm{e}^{2\pi \mathrm{i} \frac{q}{p}} P$$

Rational rotation algebra

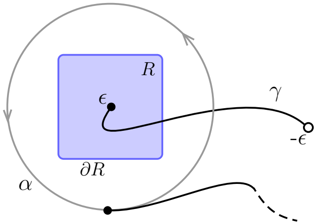


Charged anyons

For an open path γ

$$\varphi = V_\gamma \Omega \qquad (\Omega = P\Omega)$$

is a state of a pair of excitations



▷ Fractional charge (Laughlin, Saminadayar, Reznikov,...)

$$\langle \varphi, Q_R \varphi \rangle - \langle \Omega, Q_R \Omega \rangle = \langle \Omega, (V_\gamma^* Q_R V_\gamma - Q_R)\Omega \rangle = \frac{q}{p}$$

▷ Berry phase, aka Braiding (..., Wen, Fröhlich,...)

$$V_\alpha \varphi = V_\gamma (V_\gamma^* V_\alpha V_\gamma V_\alpha^*)\Omega = \mathrm{e}^{2\pi\mathrm{i}\frac{q}{p}} \varphi$$