CATEGORICAL KÄHLER GEOMETRY

TQFT @ Lisbon
June 25th, 2021

Fabian Haiden

joint work in progress with:
L. Katzarkov, M. Kontsevich, P. Pandit

Aim: Theory of Kähler geometry for derived categories
- / $\mathbb{C}$ (Archimedean)
- / Novikov field $k((t^{\mathbb{R}}))$ (non-Archimedean)

which unifies

1. B-model: [deformed] Hermitian Yang–Mills
2. A-model: special Lagrangians, LMCF
3. Quiver representations & their GIT

⋮

DUY (Donaldson '85, Uhlenbeck–Yau '86) – theorem

X .... compact Kähler, $E \to X$ ... holomorphic vector bundle

E admits HYM metric $\Leftrightarrow$ E is slope polystable

hermitian metric h on E
with associated connection D
with curvature $F = F_D$
such that:

constant

$$\boxed{F \wedge \omega^{n-1} = \lambda \omega^n}$$

geometric PDE problem

$$\mu(E) := \frac{1}{\mathrm{rk}(E)} \underbrace{\int_X c_1(E) \wedge \omega^{n-1}}_{\deg(E)}$$

E is stable $:\Leftrightarrow$ $F \in \mathrm{Coh}(E)$
$0 \neq F \subsetneq E \Rightarrow \mu(F) < \mu(E)$

polystable $:\Leftrightarrow$ direct sum of stable bundles of same slope

algebraic criterion

Remarks:

- definition of polystability depends only on Kähler class $[\omega] \in H^2(X;\mathbb{R})$, whereas definition of HYM metric depends on Kähler metric
- the HYM metric is essentially unique (unique up to rescaling for stable E)
- Donaldson defines function on ∞-dim space of Hermitian metrics on fixed $E \to X$, (in terms of relative characteristic classes) whose critical points are HYM metrics.
  The corresponding gradient flow is shown to converge (to the HYM metric) if E polystable

Thomas – Yau – Joyce conjecture

By mirror symmetry expect analogous A-side version of this story ....

Initially proposed by Thomas, Thomas–Yau, later made more precise by Joyce.

Conjecture: X ... CY manifold, $\mathcal{F}(X)$ ... Fukaya category
then $\exists$ Bridgeland stability condition on $\mathcal{F}(X)$
with (1) central charge $Z(L) := \int_L \Omega^{n,0}$
(2) semistable objects of phase $\phi \in \mathbb{R}$:
special lagrangians $L \subset X$: $\operatorname{Arg}(\Omega|_L) = \phi$

Analogy, loosely based on mirror symmetry:

complex manifold (X, J)	—	symplectic manifold (M, ω)
Kähler form $\omega^{1,1}$	—	$\{$ compatible cx. structure holomorphic $\Omega^{n,0}$
holomorphic bundle E with metric h $\}$	—	lagrangian submanifold $L \subset M$ [+ unitary connection]
varying h	—	Hamiltonian isotopy of L
HYM condition	—	special lagrangian condition
slope stability	—	Bridgeland stability

Bridgeland stability conditions

$\mathcal{C}$... triangulated category , $cl : K_0(\mathcal{C}) \to \Gamma \cong \mathbb{Z}^n$ fixed

Definition A stability condition on $(\mathcal{C}, \Gamma, cl)$ is

① additive map $Z : \Gamma \to \mathbb{C}$ (central charge)

② full $\oplus$-closed subcategories $\mathcal{C}_\phi \subset \mathcal{C}$, $\phi \in \mathbb{R}$

(semistable objects of phase ϕ)

such that:

1) $\mathcal{C}_\phi[1] = \mathcal{C}_{\phi+\pi}$

2) $\phi_1 < \phi_2$, $A_i \in \mathcal{C}_{\phi_i} \Rightarrow \mathrm{Hom}(A_2, A_1) = 0$

3) $\forall\, E \in \mathcal{C}$ $\exists$ tower of exact triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \cong E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

with $0 \neq A_i \in \mathcal{C}_{\phi_i}$, $\phi_1 > \phi_2 > \dots > \phi_n$

(Harder – Narasimhan filtration)

4) $0 \neq A \in \mathcal{C}_\phi \Rightarrow Z(A) := Z(cl(A)) \in \mathbb{R}_{>0} e^{i\phi}$

5) $\exists$ norm $\|\cdot\|$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, $C > 0$:

$$\|cl(A)\| \leq C\, |Z(A)| \quad , \quad A \in \mathcal{C}_\phi$$

(support property)

Bridgeland's result: Set $Stab(\mathcal{C}) = Stab(\mathcal{C}, \Gamma, cl)$ of all stability conditions on fixed $\mathcal{C}$ has natural topology such that

$$Stab(\mathcal{C}) \longrightarrow Hom(\Gamma, \mathbb{C})$$

is local homeomorphism. ($\Rightarrow Stab(\mathcal{C})$ is complex mfd)

BUT: 1) Typically hard to show $Stab(\mathcal{C}) \neq \emptyset$, let alone determine all of $Stab(\mathcal{C})$

2) Geometric origins forgotten....

Analogy:

Kähler class	stability condition
Kähler metric	?

"categorical Kähler metric"

Finite-dimensional example: quiver representations

$Q = (Q_0, Q_1, s, t)$ quiver 

vertices, arrows, source, target

Representation of Q / $\mathbb{C}$ given by

① vector spaces E_i, $i \in Q_0$ (require $\dim E_i < \infty$)

② linear maps $\phi_\alpha : E_i \to E_j$ for $\alpha : i \to j$ arrow

metric on representation E is just Hermitian metric h_i on each E_i

For stability, fix $z_i \in \mathbb{C}$, $Im(z_i) > 0$, $i \in Q_0$

$$Z(E) := \sum z_i \dim E_i \in \mathbb{C}$$

Theorem (A.D. King)

E .... representation of Q

E has harmonic metric $\iff$ E is slope polystable of phase $\operatorname{Arg} Z(E)$

($\longleftrightarrow$ slope $-\mathrm{Re}\,Z(E)/\mathrm{Im}\,Z(E)$)

depends on h

$$\operatorname{Arg}\Big(\sum_{\alpha\in Q_1} [\phi_\alpha^*, \phi_\alpha] + \sum_{i\in Q_0} z_i \, pr_{E_i} \Big) = \phi \, 1_E$$

Arg for normal operators

Proof is application of Kempf-Ness principle from GIT

GIT quotient = symplectic quotient

for $\bigoplus_{\alpha: i\to j} \mathrm{Hom}(E_i, E_j) \circlearrowleft \prod_{i\in Q_0} GL(E_i)$

- Get Kähler metric on moduli of polystable objects X_{ps} (from description as symplectic reduction)
- $Q \rightsquigarrow$ path algebra $\mathbb{C}Q$

$$\mathrm{Rep}(Q) \cong \mathrm{Mod}(\mathbb{C}Q)$$

need this description for metric on X_{ps}

abstract abelian category with Z is enough for stability

$\Rightarrow$ forgets "noncommutative Kähler metric"

Also, everything extends to $D^b(\mathrm{Rep}\, Q)$

... chain complexes

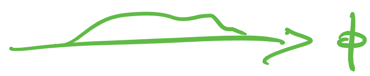
Categorical Kähler metrics (tentative)

$\mathcal{C}$ ... triangulated category

DATA: • for each $E \in \mathcal{C}$ a space $Met(E)$ of metrics on E (functorial for isos in $\mathcal{C}$)

– In the non-Archimedean case $\mathcal{C} / k((t^{\mathbb{R}}))$ can be more precise: want $\tilde{\mathcal{C}} / k[[t^{\mathbb{R}}]]$ with
$\mathcal{C} \cong \tilde{\mathcal{C}} \otimes_{k[[t^{\mathbb{R}}]]} k((t^{\mathbb{R}}))$ ... general fiber
then $Met(E) = hofib_E(\tilde{\mathcal{C}} \to \mathcal{C})$, i.e. lift to $\tilde{\mathcal{C}}$

• for $h \in Met(E)$ a finite measure μ_h on $\mathbb{R}$ with bounded support



<u>Define:</u>

$$m(E,h) := \int_{\mathbb{R}} \mu_h \qquad \text{(mass)}$$

$$Z(E,h) := \int_{\mathbb{R}} e^{i\phi} \mu_h \qquad \text{(central charge)}$$

$\Rightarrow$ "BPS inequality" $|Z| \leq m$

metric h is harmonic of phase ϕ $\Leftrightarrow$ $\mathrm{supp}(\mu_h) = \{\phi\}$
($\Rightarrow$ $Z = m e^{i\phi}$)

May want to keep track of additional structure, e.g. flow on $Met(E)$.

AXIOMS:

- $\mu_h = 0$ for some $h \in Met(E) \Rightarrow E = 0$
- $Z(E,h)$ induces additive map $K_0(\mathcal{C}) \to \mathbb{C}$
- If $E_1, E_2 \in \mathcal{C}$, $h_i \in Met(E_i)$ with

$$Supp(\mu_{E_1}) < Supp(\mu_{E_2})$$

then $Hom(E_2, E_1) = 0$

- If $E \in \mathcal{C}$ there is filtration

$$\begin{tikzcd}
0 \arrow[rr] && E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] && E_n \cong E \arrow[dl] \\
& A_1 \arrow[ul, dashed] &&&& A_n \arrow[ul, dashed]
\end{tikzcd}$$

and $h_i \in Met(A_i)$ harmonic of phase ϕ_i

$$\phi_1 \geq \phi_2 \geq \dots \geq \phi_n$$

Examples: ① A-model: $\mathcal{C}$ = Fukaya category over Novikov ring $k[[t^{\mathbb{R}}]]$

given lagrangian $L \subset M$, $\phi : L \to \mathbb{R}$ lift of $Arg(\Omega|_L)$

$$\mu_L := \phi_* |\Omega|_L|$$

② Quiver:

$$\mu_{E,h} := Tr(\text{Spectral measure}(Arg\, P)\,|P|)$$

where $P := \sum_\alpha [\phi_\alpha^*, \phi_\alpha] + \sum z_i pr_{E_i}$

③ Tautological example: $\mathcal{C}$ is triangulated category with stability condition

$$Met(E) = \text{filtrations of } E$$

$$\begin{tikzcd}
E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \cong E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

Dirac measure at ϕ

$$\mu_{E,(E_i)_i} := \sum_{i=1}^{n} \sum_{\phi \in \mathbb{R}} \left| Z(\phi\text{-component of HN-filt. of } A_i) \right| \cdot \delta_\phi$$