Holographic complexity

Javier Martín García

Based on:
J. L. F. Barbón & J. M-G. 1510.00349
& Ongoing work...
1 Introduction
   The holographic dictionary
   Quantum Information

2 Quantum entanglement and computational complexity
   Entanglement
   Entanglement in the spacetime picture
   Computational complexity
   Complexity in the spacetime picture

3 Complexity of topological AdS black holes
   General features
   Non-extremal regime
   Near-extremal regime

4 The Action/complexity proposal
   From volume to action
   Action for hyperbolic black holes
   Comparison with C/V duality

5 Conclusions
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5 Conclusions
Holography

- Quantum gravity in d+1 dimensions must be described by a non-gravitational theory in d dimensions.
  - Translation between both theories should be possible: DICTIONARY
  - Not so easy to find precise examples of holographic theories.
- Successful example: AdS/CFT
  - Finding the entries of the dictionary is manageable.

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★ 1997: Today:
\[ \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2}. \]
**Dictionary:**

**Chapter QI**

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# Introduction

The holographic dictionary

Quantum Information

# Quantum entanglement and computational complexity

Entanglement

Entanglement in the spacetime picture

Computational complexity

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# Complexity of topological AdS black holes

General features

Non-extremal regime

Near-extremal regime

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# Conclusions
Quantum entanglement

- Entanglement in QM
  - Failure of making the decomposition $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
  - Quantified by entanglement entropy $S_A = -\text{Tr}_A[\rho_A \log \rho_A]$
  - Characterization of operator correlations
    $$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \iff \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle = 0.$$  

- Tensor Network
  - 2-Qubit entangled state $|\psi\rangle = \sum_{i,j} \alpha_{ij} |\psi_i\rangle_A |\psi_j\rangle_B$.
  - General entangled state $|\psi\rangle = \sum_{\mu_1,\ldots,\mu_n} c_{\mu_1,\ldots,\mu_n} |\psi_{\mu_1}\rangle_1 \ldots |\psi_{\mu_n}\rangle_n$.
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\[ \begin{array}{c}
  \bar{\alpha}_{ij} \\
  i \nearrow j
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Entanglement in the spacetime picture

- Tensor network for a CFT
  - Scale invariance $\Rightarrow$ Tree-like structure TN
  - New (non-physical) dimension: Network depth
  - Mimics a discretized hyperbolic space

- Holographic entanglement entropy
  - Entanglement between two regions $\IFF$ number of links cut to separate the TN in two.
  - Ryu-Takayanagi formula $S_A = \frac{\text{Area}(\tilde{A})}{4G}$. 
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- Entanglement and spacetime connectivity
  - Eternal AdS black hole $|\Psi\rangle = \sum_i e^{-\beta E_i/2} |E_i\rangle \otimes |E_i\rangle \implies$ Two entangled BH connected by a wormhole
  - Classically connected spacetime $\iff$ Superposition of disconnected ones
  - ER=EPR: Any entangled system is connected by a wormhole
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Complexity?
Computational complexity

- Computer science: Minimum number of 'simple' operations needed to get a state from a 'simple' reference one. Example: Given a string of $K$ bits, get some state 011010.. from 000000...
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- **Growth of complexity**
  - Quantum complexity keeps growing after thermalization

\[ C \propto e^K \]
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- Quantum complexity
- Growth of complexity
  - Quantum complexity keeps growing after thermalization
  - $\frac{dC}{dt} \propto TS$

\[ C \]

\[ K \]

\[ t_{th} \]

\[ t_{comp} \sim e^K \]
Complexity in the spacetime picture

- Tensor networks
  - Complex states have bigger tensor networks
  - Complexity $\Leftrightarrow$ size of the tensor network

- Volume/complexity relation: The complexity of $|\psi(t)\rangle$ is proportional to the volume of a maximal slice in the dual spacetime that passes through $t$.

- Black holes
  - Growth of quantum complexity is encoded as the growth of the Einstein-Rosen Bridge
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5 Conclusions
General features

- Topological black holes
  - Metric
    \[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + \frac{r^2}{l^2}dH^2_{d-1} \]
    \[ f(r) = -1 + \frac{r^2}{l^2} - \frac{\mu}{r^{d-2}} \]
  - Dual to two CFT’s on a hyperboloid [Emparan]
  - Degenerate system: \( \lim_{T \to 0} S \neq 0 \)

- Slicing conditions
  - Spacelike Cauchy surfaces
  - ’Nice’ slices: Stay away from singularities
  - Asymptotically match constant \( t \) surfaces far away
  - Foliation of the entire exterior region
  - Asymptote to constant \( r_m \) surface in the interior for long times.
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Non-extremal regime

• Features
  * $T_H \gg 1$
  * One horizon. Schwarzschild-Ads-like topology.

• Metric Patching
  * Exterior ($r \gg 1$) \( \implies f_E(r) = -1 + r^2 \implies \) Constant $t$ surface
  * Rindler region ($r \sim r_h$) \( \implies f_R(r) = 4\pi T_H (r - r_h) \implies \)
  Horizontal planes (in $X, T$ coordinates)
  * Interior ($r \ll r_h$)

• Results
  * Exterior: Constant contribution
  * Wormhole:
    \[
    V \sim \sqrt{\frac{d}{1 - 2^{-1/d}}} G_N S T_H t
    \]
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Near-extremal regime

- **Features**
  - $T_H < \frac{1}{2\pi}$
  - Two horizons $r_{\pm}$. Timelike singularity. Degenerate system.

- **Metric Patching**
  - Exterior ($r \gg 1$) $\Rightarrow$ Constant $t$ surface
  - $AdS_{1+1} \times H^{d-1}$ region ($r_h \ll r \ll 1$) $\Rightarrow$ Constant $t$ surface
  - Rindler region ($r \sim r_h$) $\Rightarrow$ Horizontal planes (in X,T coordinates)
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- **Results**
  - Exterior: Constant contribution
  - $AdS_{1+1} \times H^{d-1}$ region: Constant divergent contribution $\sim \log T^{-1}$
  - Wormhole:
    $V \sim G_N S T_H t$
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5. Conclusions
From volume to action

- Unpleasant features of volume/complexity duality
  - Arbitrary scale $\mathcal{C} \sim \frac{V}{G\ell}$, $\ell \sim \ell_{AdS}, r_h, \ldots$
  - Why should the maximal slice play a preferred role?
- A higher dimensional object might solve the problem
  - $\mathcal{C} \sim \frac{V\ell_{AdS}}{G\ell_{AdS}^2} \sim \frac{\mathcal{W}}{G\ell_{AdS}^2} \sim \frac{\Lambda}{G} \int \sqrt{g}dV \sim A$
  - Consider all possible foliations
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The Action/complexity proposal

- The complexity of \( |\psi(t)\rangle \) is given by the on-shell action of its gravitational dual evaluated on the WdW patch corresponding to \( t \).

\[
C = \frac{\mathcal{A}}{\pi \hbar}
\]

[Brown, Roberts, Susskind, Swingle, Zhao]

- Some features
  - Absence of arbitrary scales
  - No preferred foliation
  - Recovers the nice features of V/C
    - BH complexity growth: \( \frac{dC}{dt} = 2M \sim ST \)
    - Shockwave tests [Susskind, Stanford]
  - Connection to Lloyd’s bound

- Drawbacks
  - The YGH term is ill-defined for null surfaces and joints. New prescriptions for these quantities still ambiguous and \textit{ad hoc}. [Lehner, Myers, Poisson, Sorkin...]

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A curiosity

- 'Extended' black hole thermodynamics
  
  Smarr formula: \((d - 2)M = (d - 1)TS - 2PV\)

- Action growth for spherical black holes factorizes in these terms
  [Coach, Fischler, Nguyen]
  
  \(\delta S_{\text{bulk}} \sim PV\delta t\)
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<td><strong>Complexity</strong>$_b$</td>
</tr>
</tbody>
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Toute? Non!

One small set of indomitable hyperbolic black holes still resist to match both prescriptions.
WdW action for hyperbolic black holes

- Complexity for cold BH’s is finite [Myers, Chapman, Marrochio]

- Complexity growth
  
  - Hot black holes: \( \frac{dC}{dt} = 2M_{AdS} \)
  
  - Cold black holes do not compute! \( \frac{dC}{dt} = 0 \)

- For the cold ones ’thermodynamic factorization’ no longer holds

  - \( \delta S_{\text{bulk}} \sim (PV + r_+^d)\delta t \)
  
  - \( \delta S_{\text{bound.}} = 0 \)
  
  - \( \delta S_{\text{joints}} \sim (TS - M_{AdS} - r_+^d)\delta t \)

- Vacua ambiguities

\[
\begin{align*}
\frac{dS}{dt} &= TS - \frac{2PV_+}{d-1} - \frac{(d-2)}{(d-1)}M_{AdS} \\
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\( M_0(\text{Empty AdS}) \)
\( M = 0 \) (Extremal BH)
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Comparison between both prescriptions

<table>
<thead>
<tr>
<th>General features</th>
<th>Complexity/Volume</th>
<th>Complexity/Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition</td>
<td>Preferred slicing</td>
<td>Covariant definition</td>
</tr>
<tr>
<td>Scale</td>
<td>Ambiguous</td>
<td>Fixed</td>
</tr>
<tr>
<td>TN interpretation</td>
<td># of tensors</td>
<td>?</td>
</tr>
<tr>
<td>$dC / dt$ in BH</td>
<td>$\sim ST$</td>
<td>$2M$</td>
</tr>
<tr>
<td>Technicalities</td>
<td>Optimization</td>
<td>Boundary terms</td>
</tr>
</tbody>
</table>

Cold hyp. BH

<table>
<thead>
<tr>
<th>$C$ in $AdS_{1+1}$ throat</th>
<th>$\sim \log T$</th>
<th>Finite</th>
</tr>
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<tbody>
<tr>
<td>$dC / dt$</td>
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</tr>
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</table>
1 Introduction
   The holographic dictionary
   Quantum Information

2 Quantum entanglement and computational complexity
   Entanglement
   Entanglement in the spacetime picture
   Computational complexity
   Complexity in the spacetime picture

3 Complexity of topological AdS black holes
   General features
   Non-extremal regime
   Near-extremal regime

4 The Action/complexity proposal
   From volume to action
   Action for hyperbolic black holes
   Comparison with C/V duality

5 Conclusions
Conclusions & Outlook

- Quantum complexity might be a interesting tool to explore black hole interiors, but its definition is not yet clear
  - Need for a precise definition in the continuum CFT
  - Decide which of the holographic proposals (if any) is the correct one

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Questions

Muito obrigado!