CHAOS & COMPLEXITY IN BLACK HOLE PHYSICS

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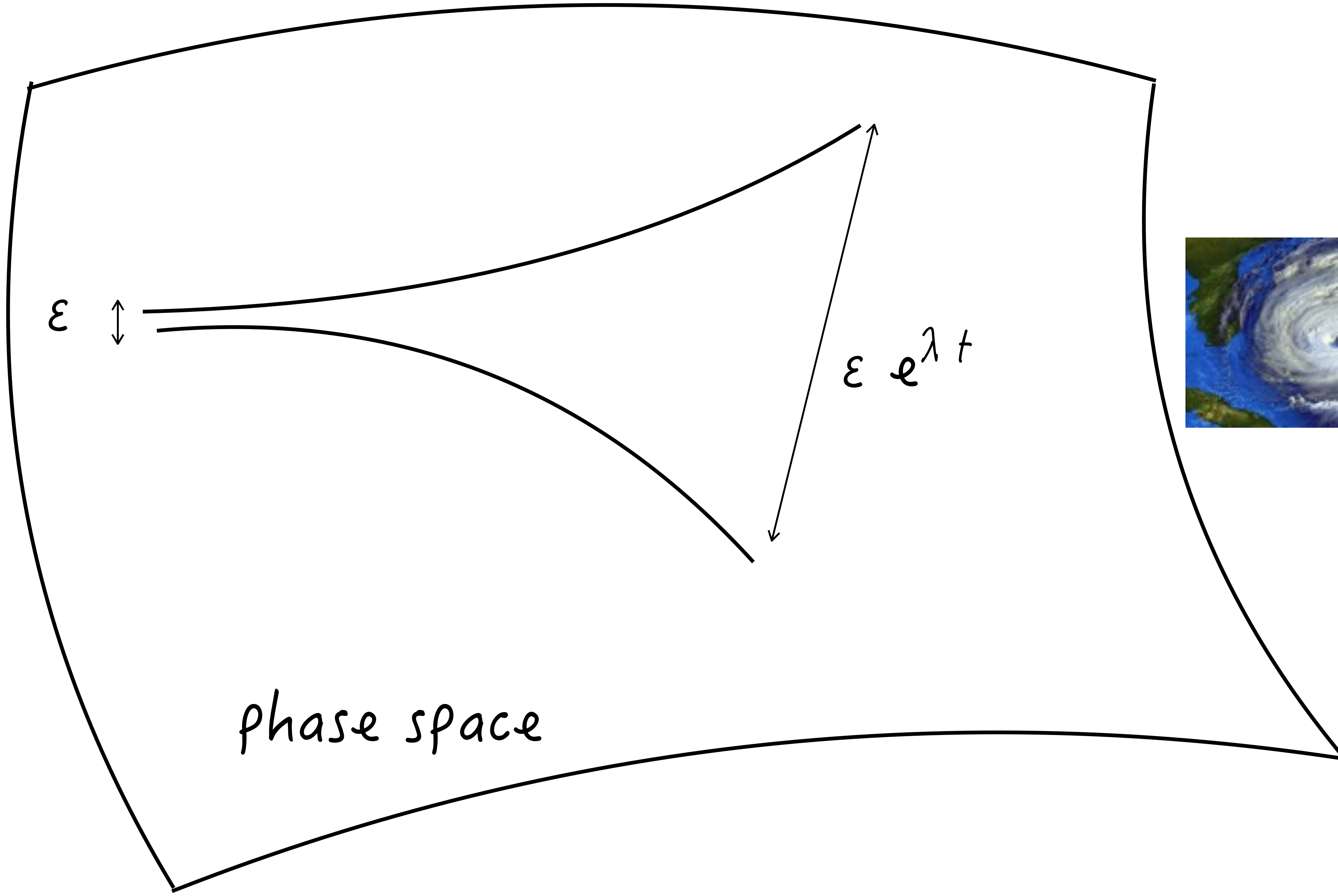
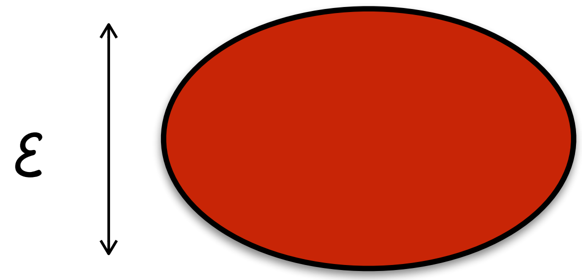
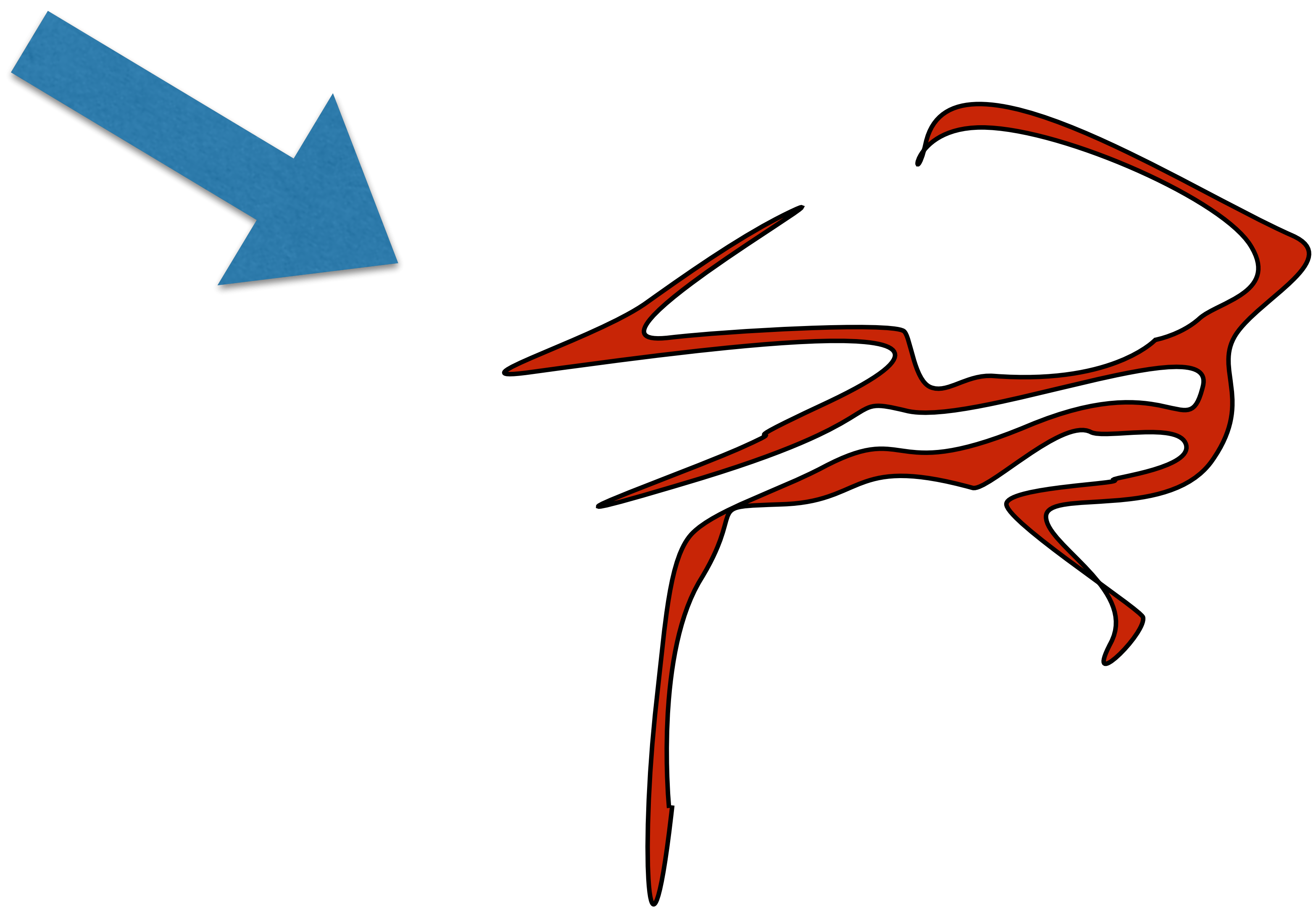
CHAOS



MIXING

$$\partial_t \rho = \{\rho, H\}_{\mathrm{PB}}$$





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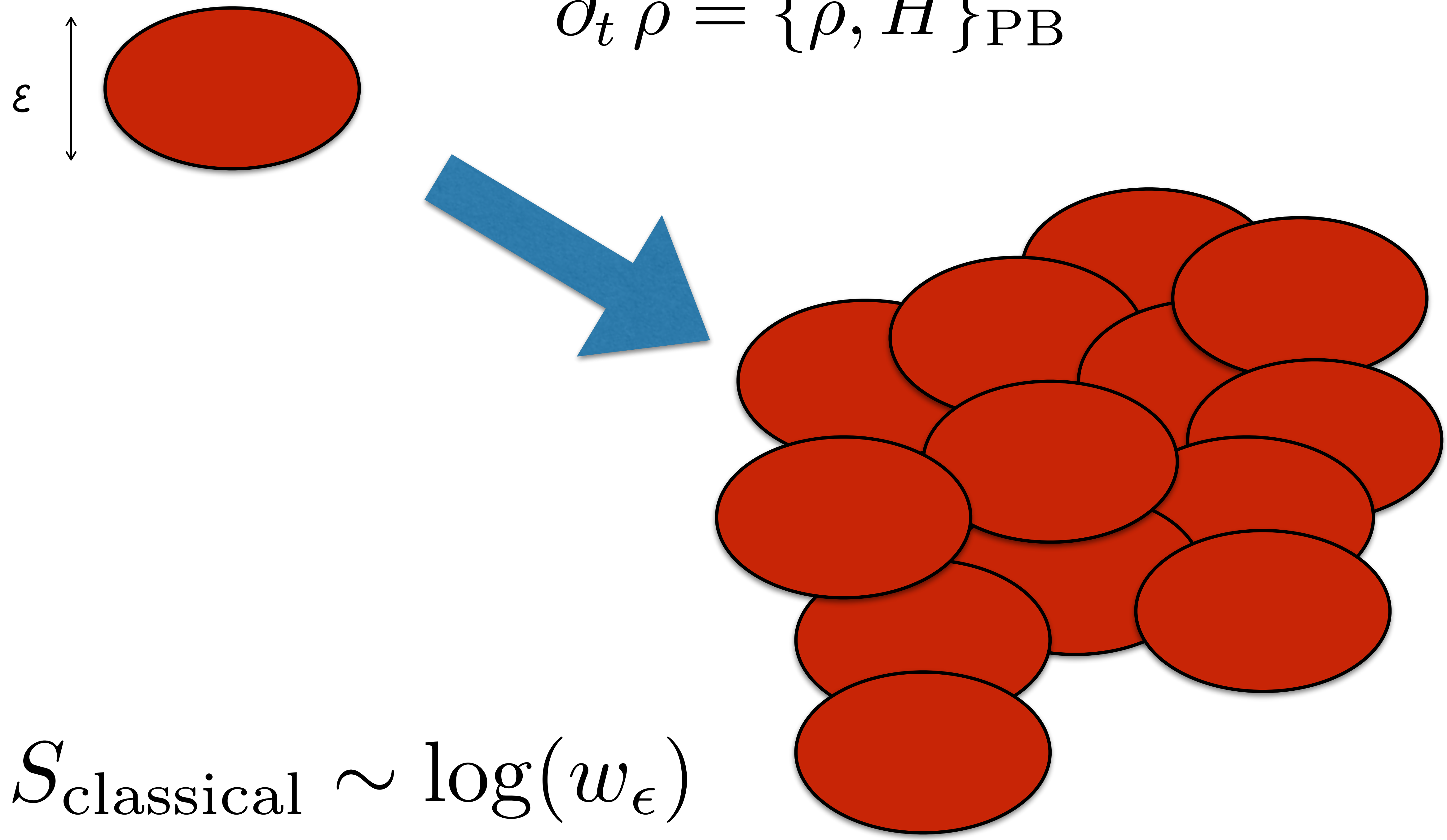


$$S_{\mathrm{classical}} \sim \log(w_\epsilon)$$

MIXING

System "forgets" initial correlations

$$[q(t)q(t+t')]_{average\ in\ t} \sim e^{-\lambda t'}$$

This property is always limited by ergodicity, which implies the phenomenon of Poincaré recurrences

Return is guaranteed after a time proportional to

$$e^{S_{\mathrm{maximal}}} \sim e^{N_{d.o.f.}}$$

Short time → Lyapunov

Long time → Mixing

Very long time → Poincaré recurrences

In quantum theory, these "fractal pictures" in phase space are very misleading because the classical coarse-graining is forced to be

$$\varepsilon^2 > \hbar$$

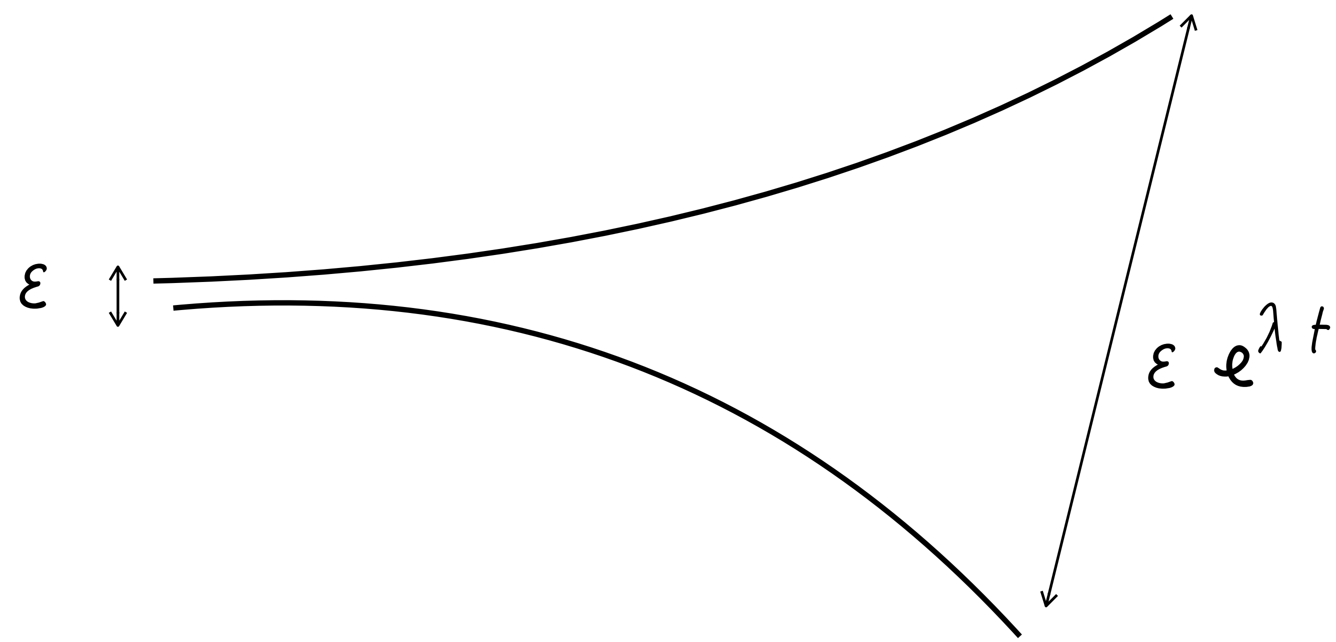
So, they are not useful after the Ehrenfest time scale

$$t_{\mathrm{Ehrenfest}} \sim \frac{1}{\lambda} \log\left(A_{\mathrm{min}}/\hbar\right)$$

$$A_{\mathrm{min}} = \mathrm{min}_{\Sigma} \int_{\Sigma} p\, dq$$

In particular, the classical Poincaré time **exp(S)** is heavily modified quantum mechanically

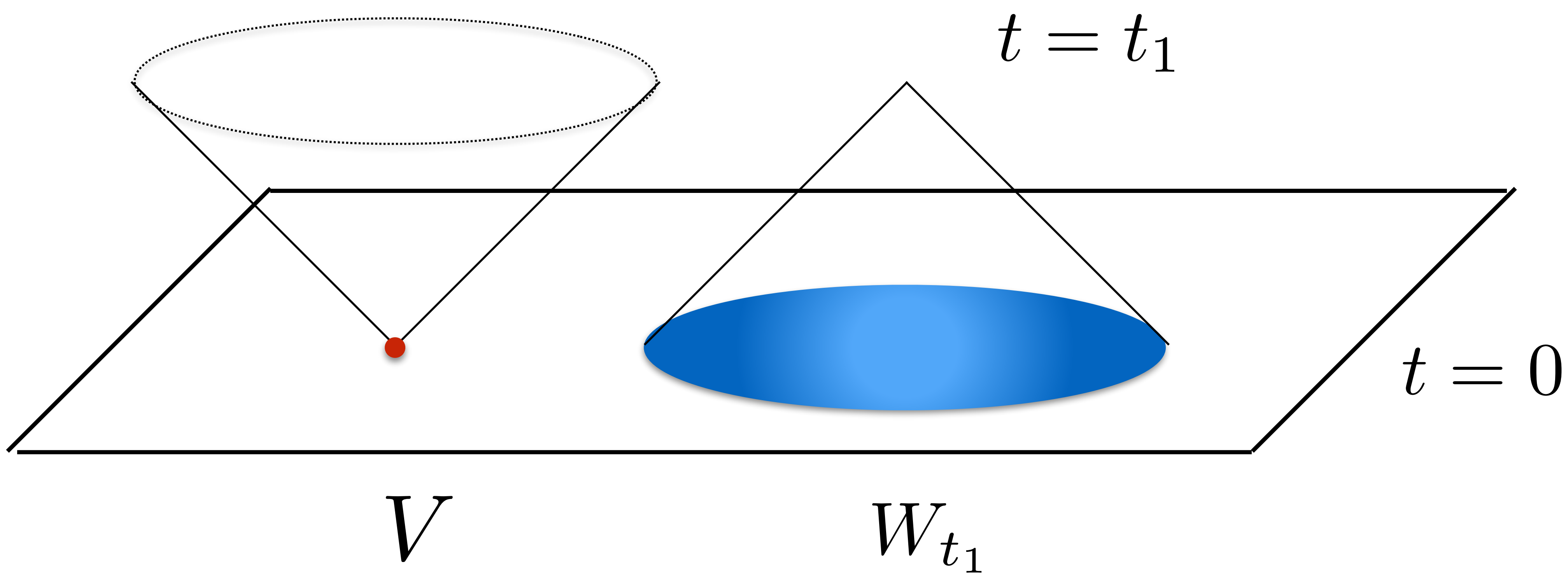
Still, one can generalize the LOCAL chaotic behavior on short time scales

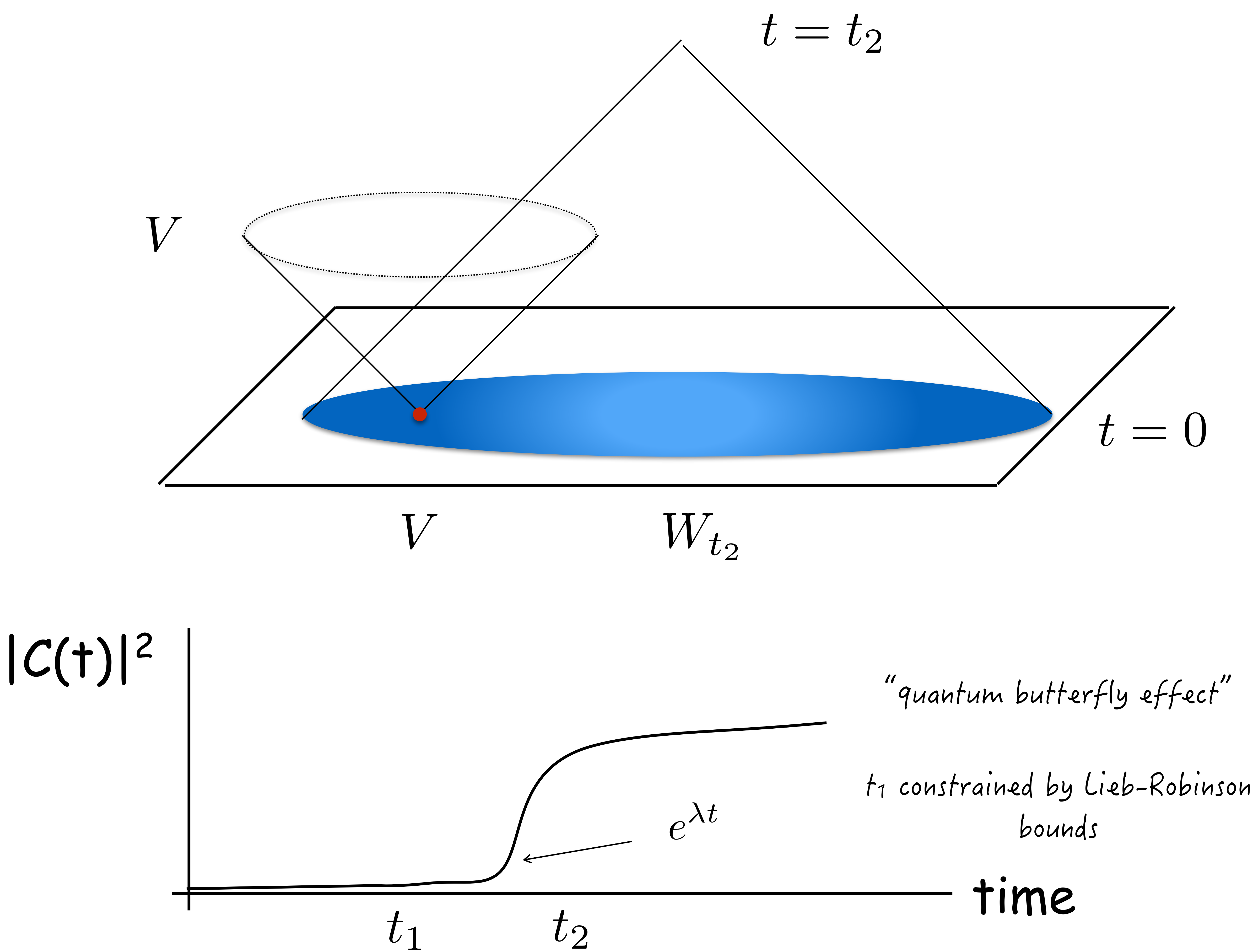


$$\frac{\partial q(t, q_0)}{\partial q_0} = \{q(t), p_0\}_{\mathrm{PB}}$$

Look for exponential growth of quantum commutators

$$\Big| [V, W_t] \Big|^2 \sim e^{\lambda t}$$


$t = t_1$
$t = 0$
V
W_{t_1}



$t = t_2$
V
$t = 0$
V
W_{t_2}
$|C(t)|^2$
"quantum butterfly effect"
t_1 constrained by Lieb-Robinson bounds
$e^{\lambda t}$
time
t_1
t_2

What about late time behavior?

Using a generalized WKB approximation, it was argued that statistical properties of the energy spectrum are related to the behavior of classical periodic orbits in phase space

Gutzwiller, Berry, Voros, $\cdots$

$$\rho(E) \approx \frac{1}{\pi\hbar} \sum_{\gamma_{\text{periodic}}} \frac{T_\gamma}{\sqrt{\text{Det}_\gamma}}\, e^{iS_\gamma/\hbar}$$

$$\rho(E) \approx \frac{\partial}{\partial E}\, e^{S_{\mathrm{eff}}(E)} + \text{fluctuations}$$

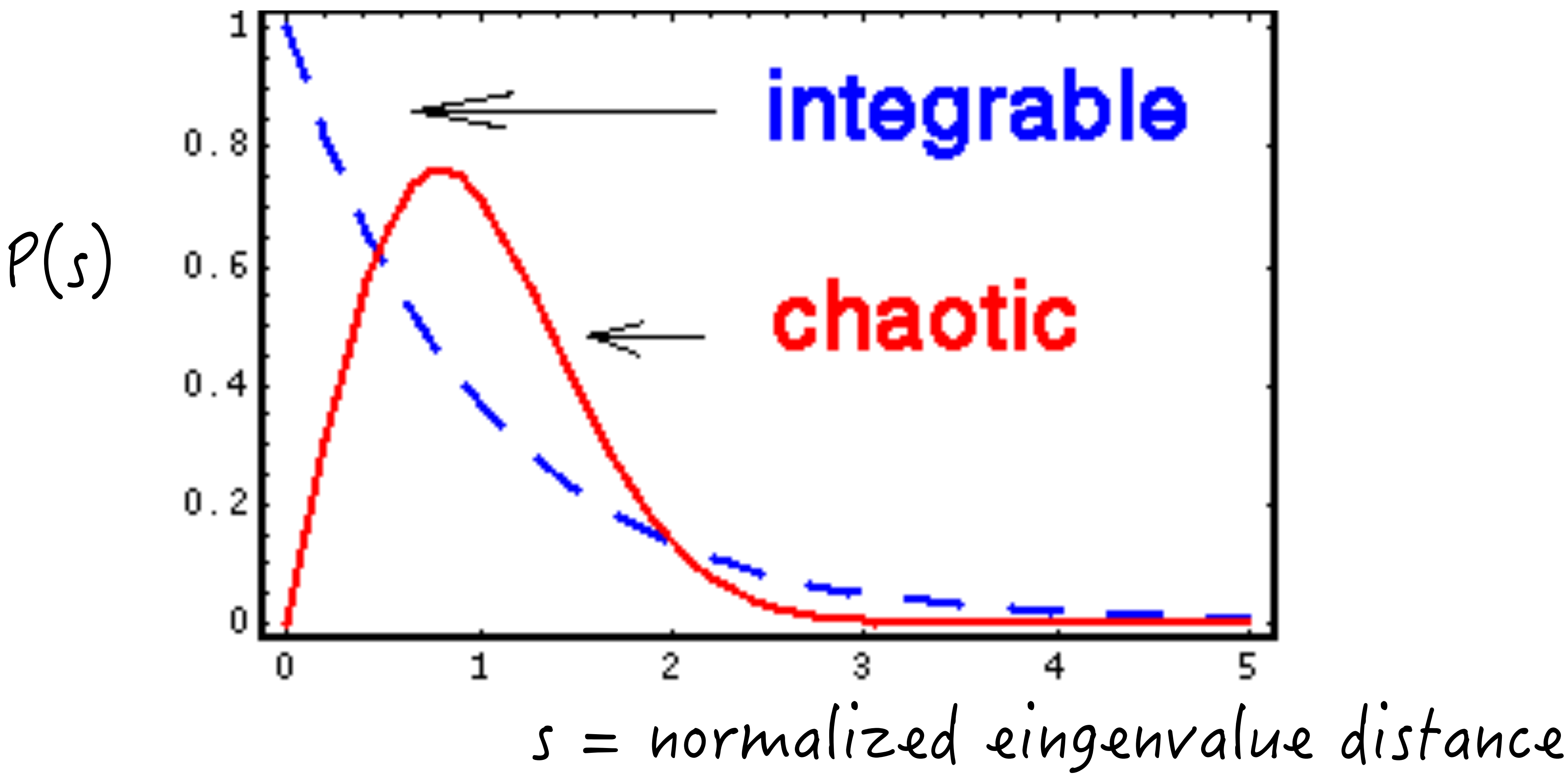
generic (Weyl) — specific

$$S_{\mathrm{eff}} = \log(\text{phase space volume})$$

in units of $\hbar$

Looking at model systems, it was established that chaotic systems have statistical level repulsion characteristic of

random matrices



Dyson, Wigner, ...

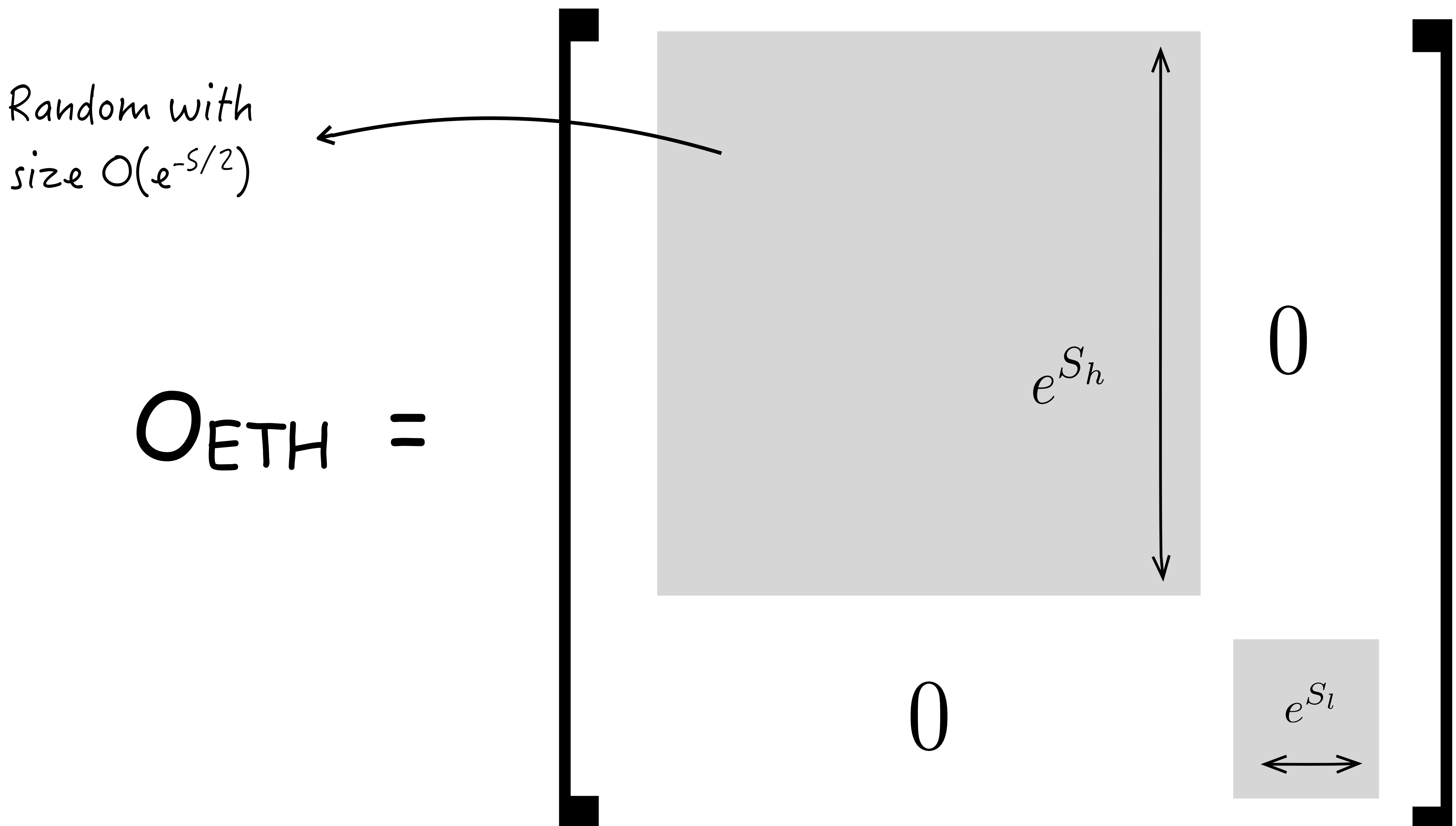
Bohigas, Giannoni & Schmit

ETH Paradigm of quantum chaos

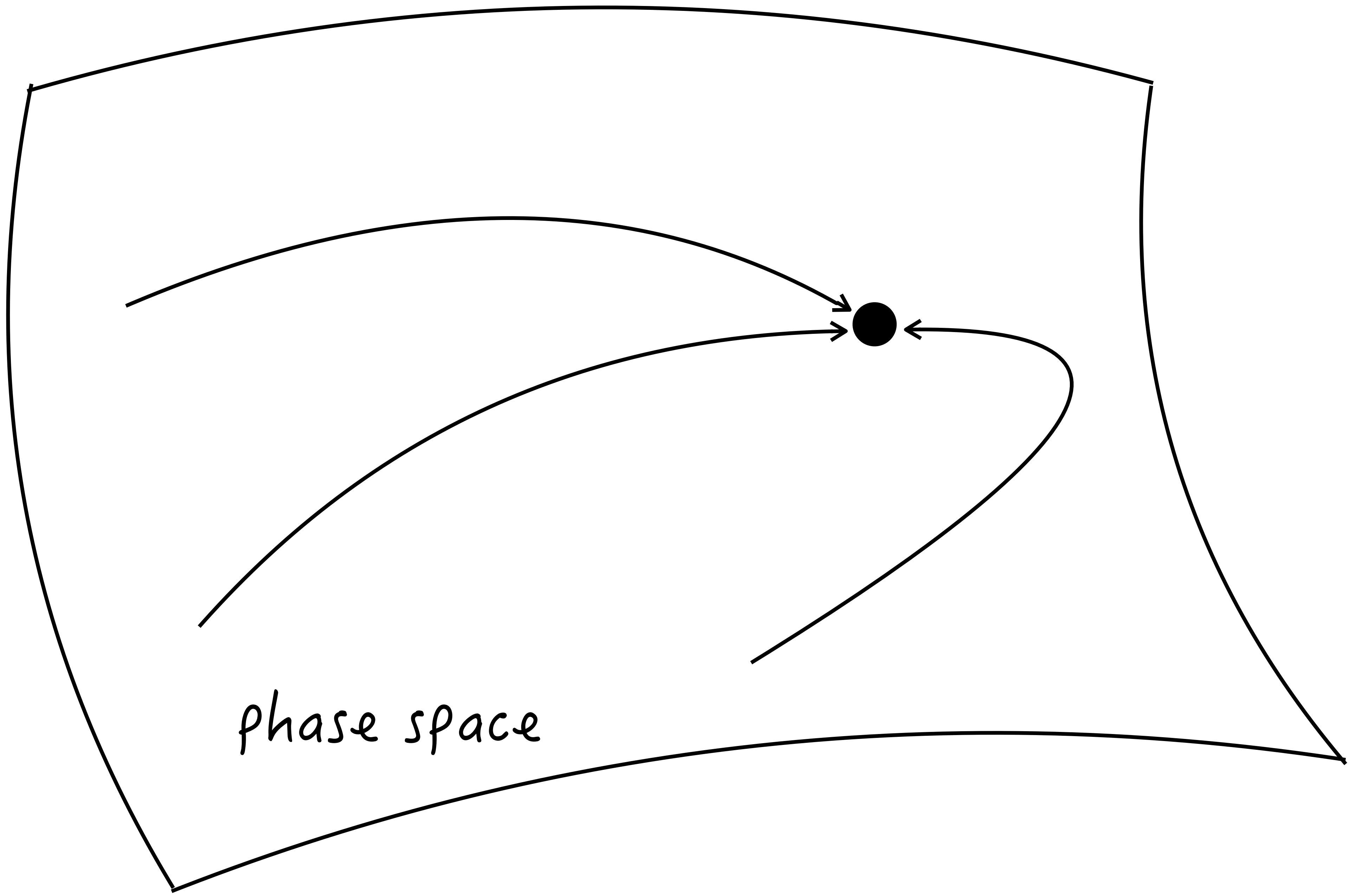
High energy eigenstates have random matrix statistics

Generic observables diagonalize in bases uncorrelated with the Hamiltonian

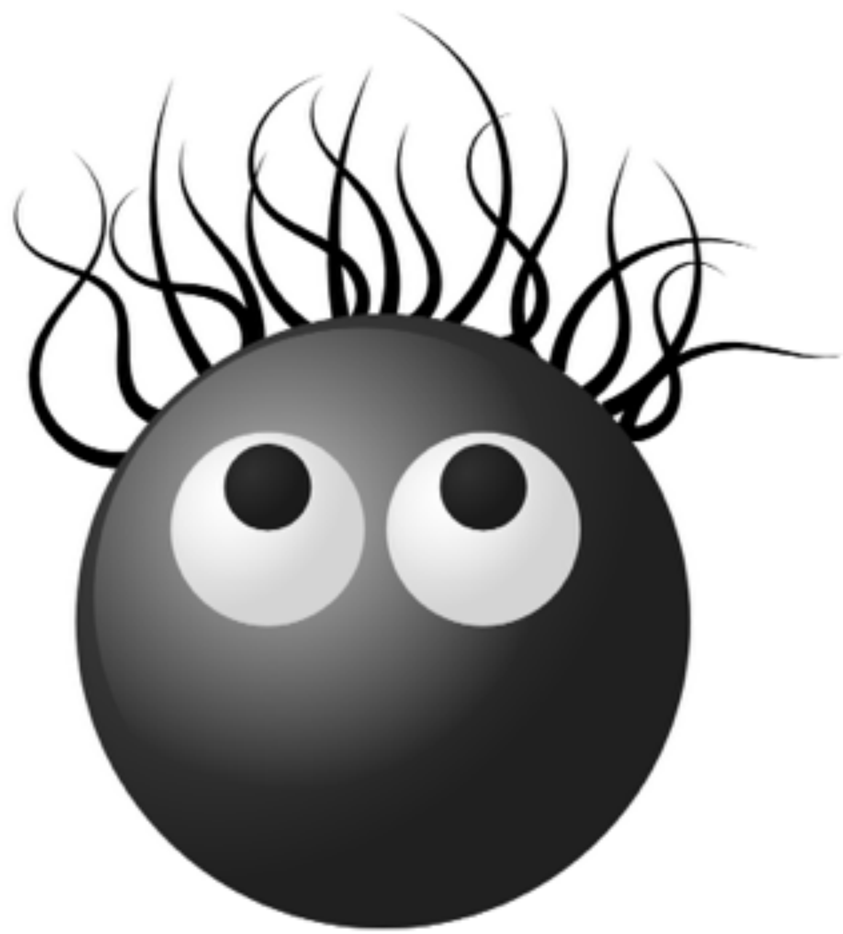
Peres, Deutsch, Srednicki



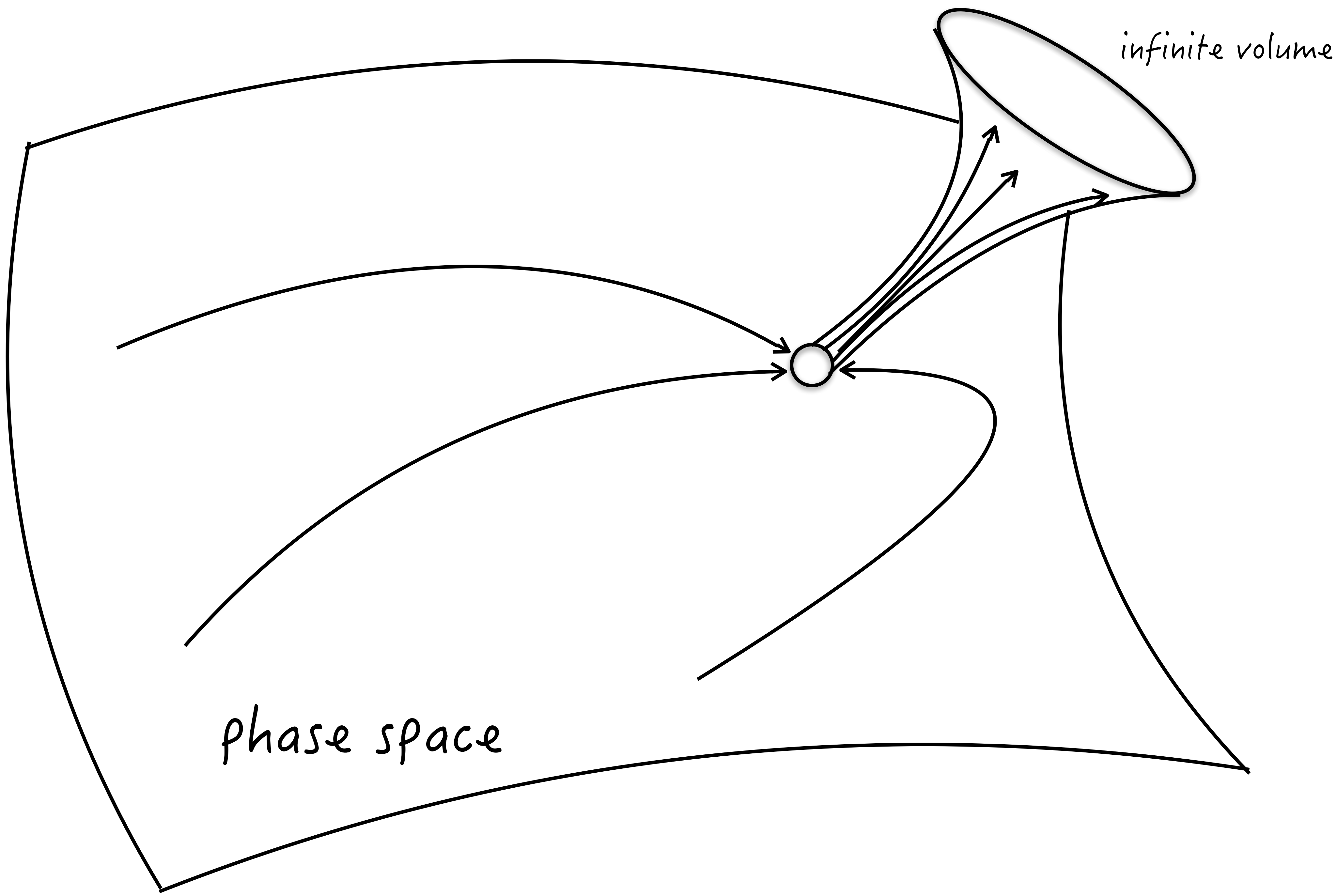
WHAT ABOUT BLACK HOLES?



they look like simple attractors but ...



WHAT ABOUT BLACK HOLES?



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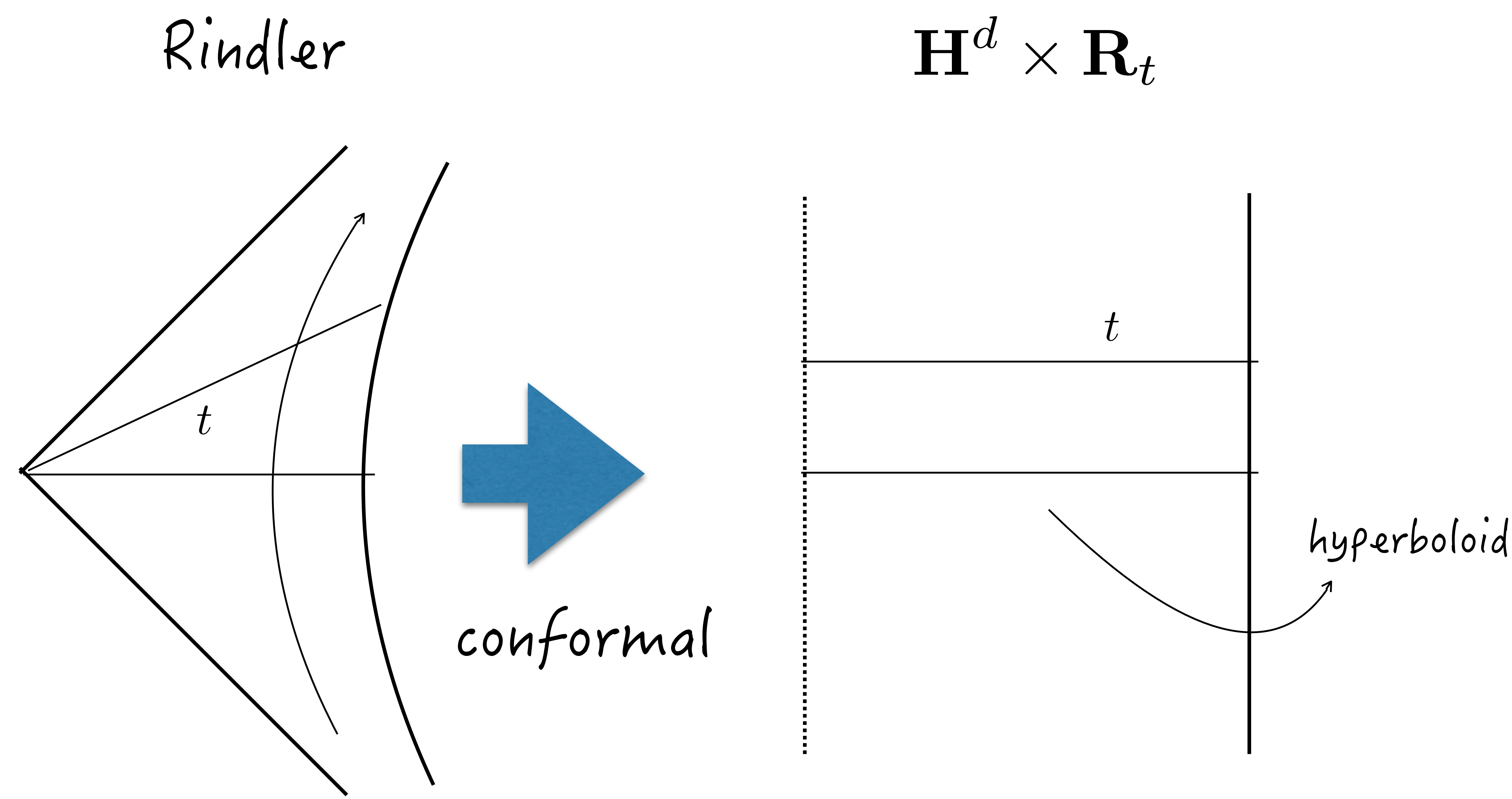
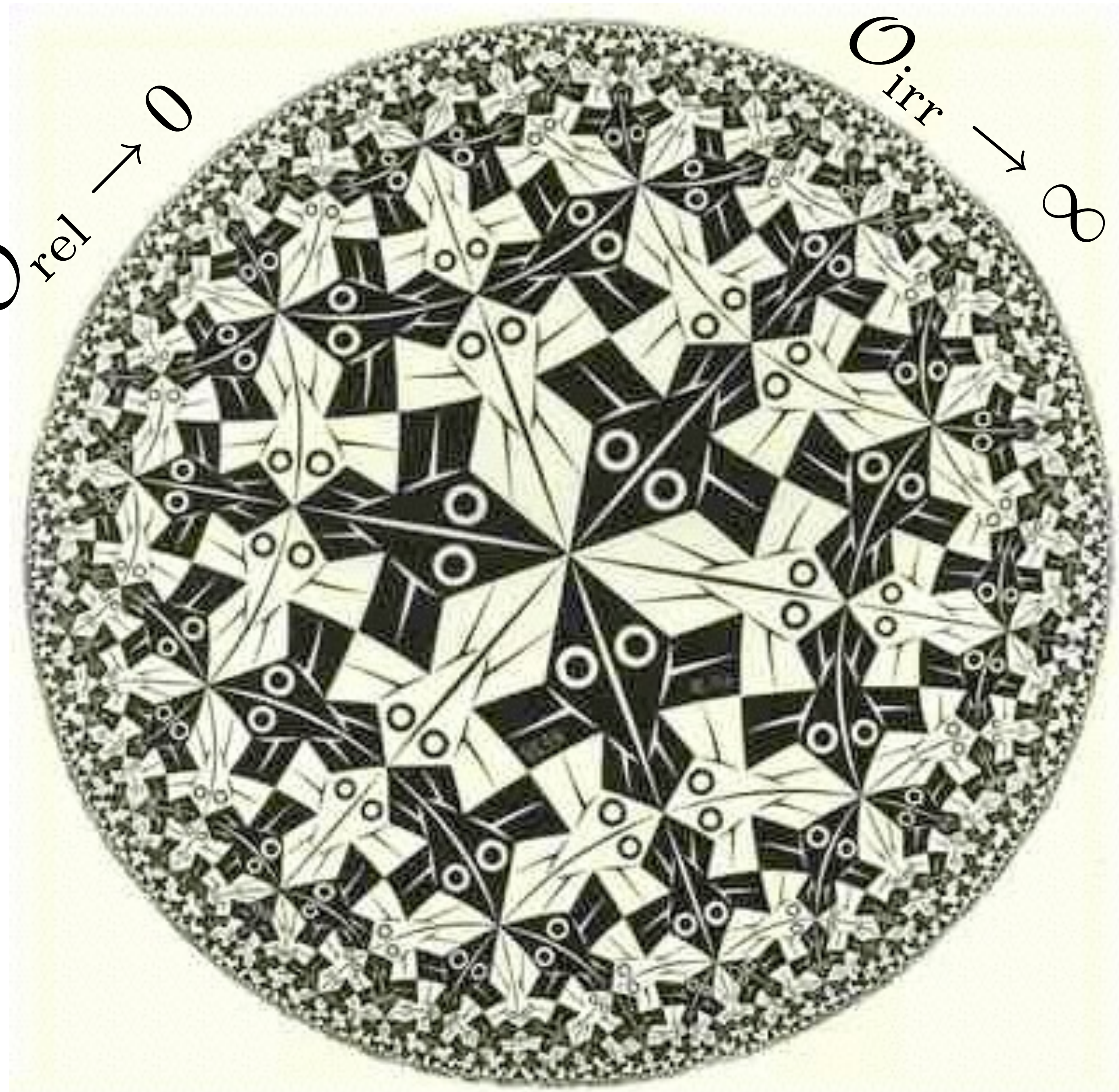


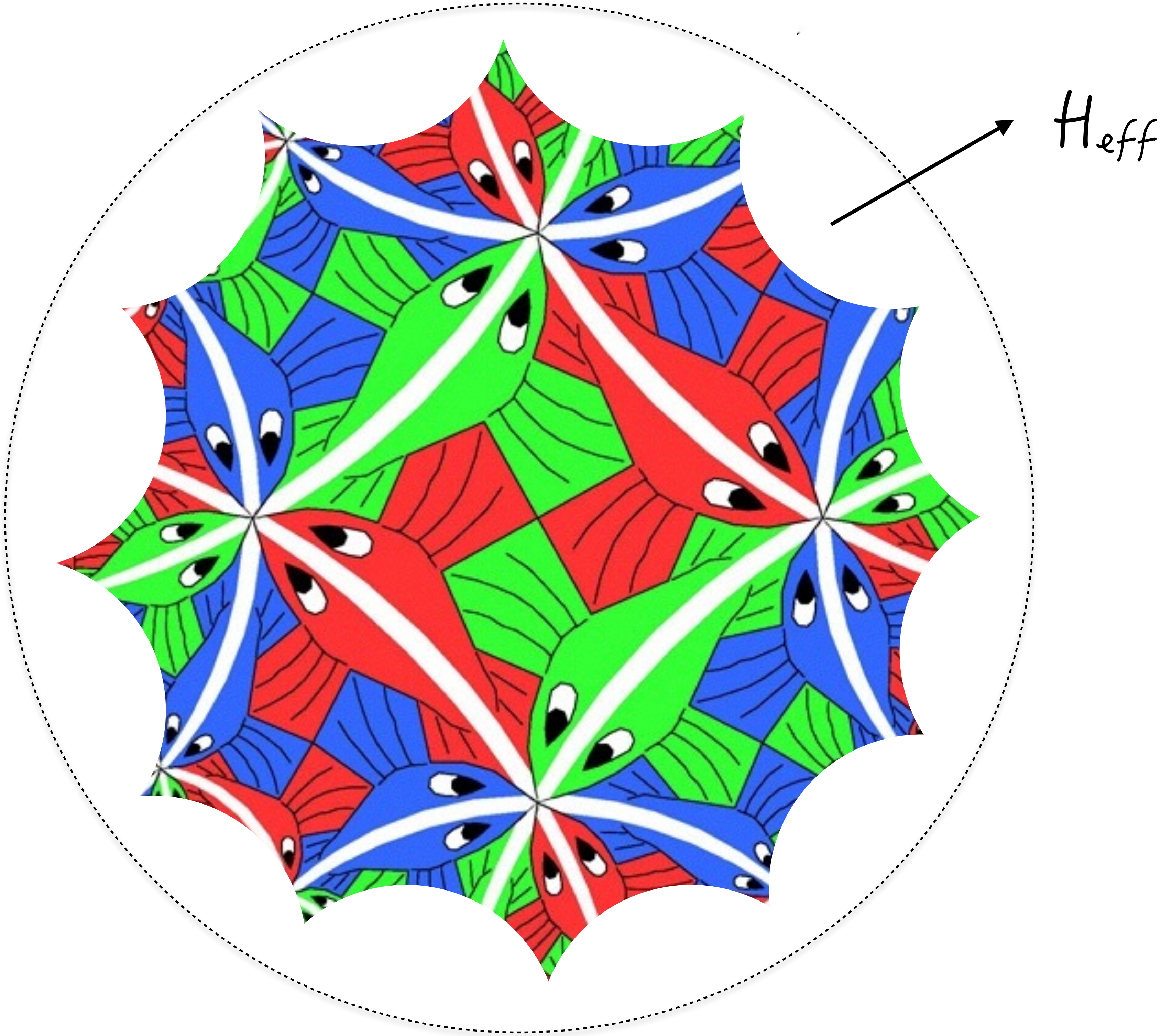
Horizon is mapped to the boundary of the hyperboloid

For non-conformal theories this boundary becomes generically strongly coupled



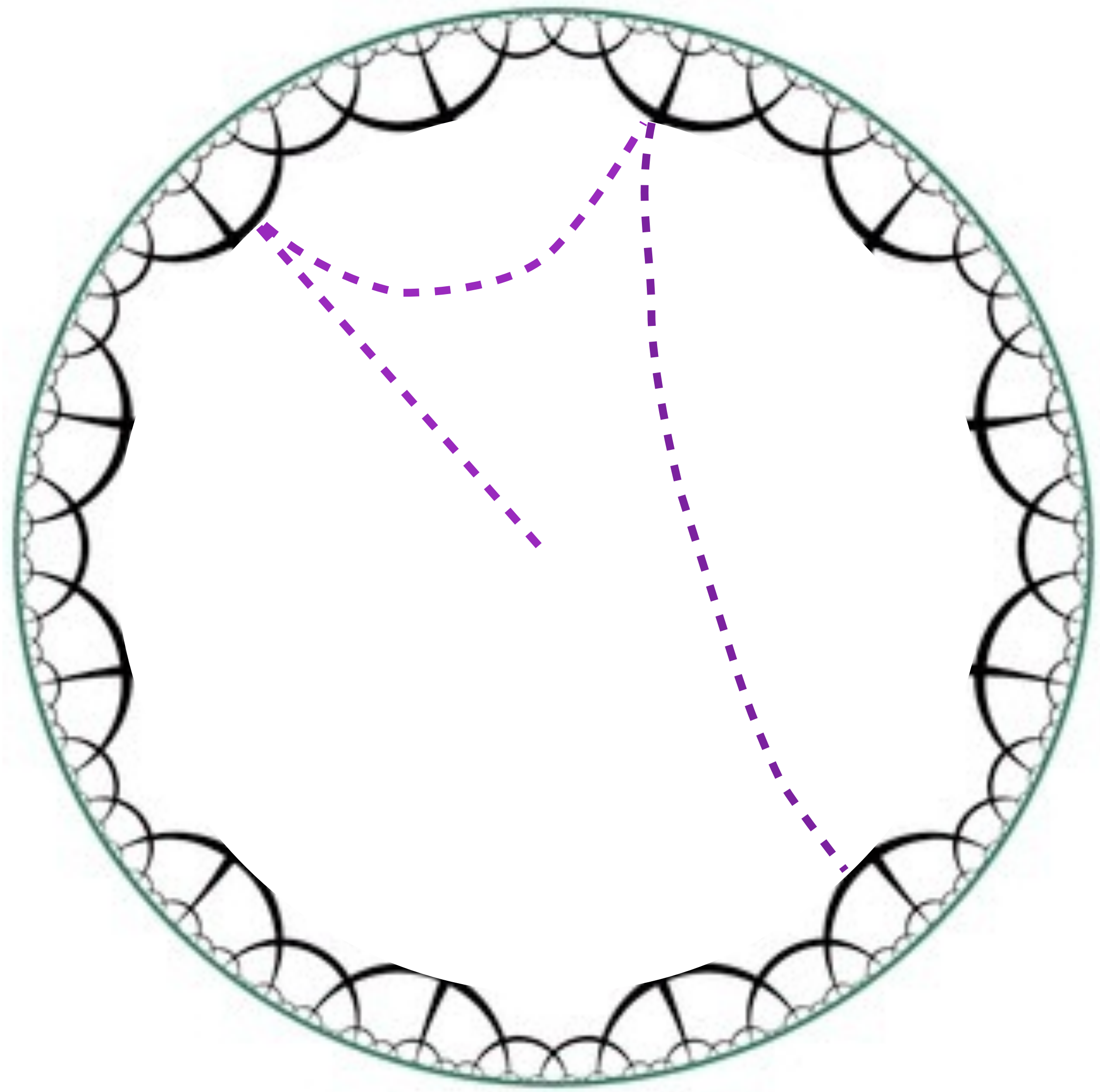
Barbón & Magan

Cutoff defines some stretched horizon



Oldest and simplest: 't Hooft's brick wall

Stretched horizon as a hyperbolic billiard



This chaotic system scrambles in O(1) collisions

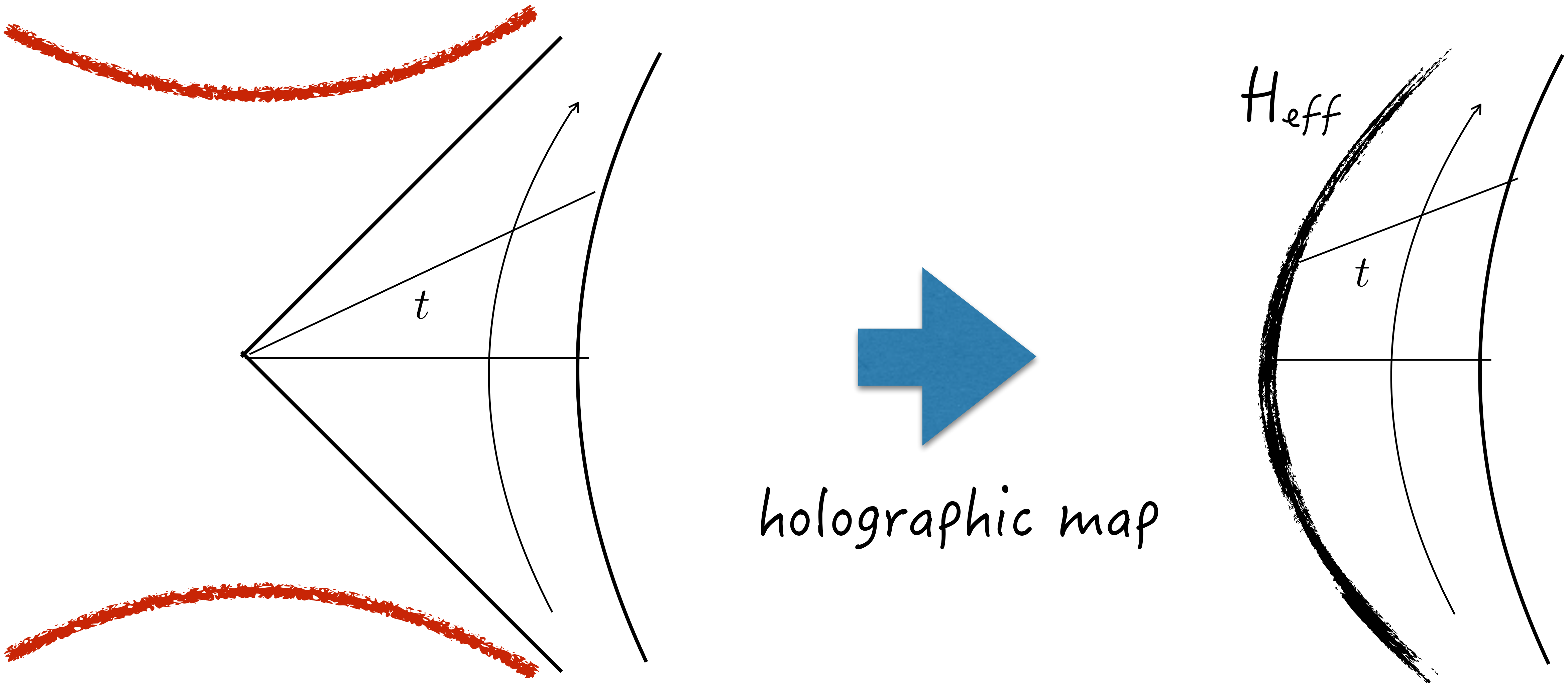
$$t_s \sim t_{\text{Lyapunov}} \sim \text{Diameter} \sim \log N_{\text{scatterers}}$$

$$t_s \sim \frac{\beta}{2\pi} \log S$$

All these models are obviously naive ...

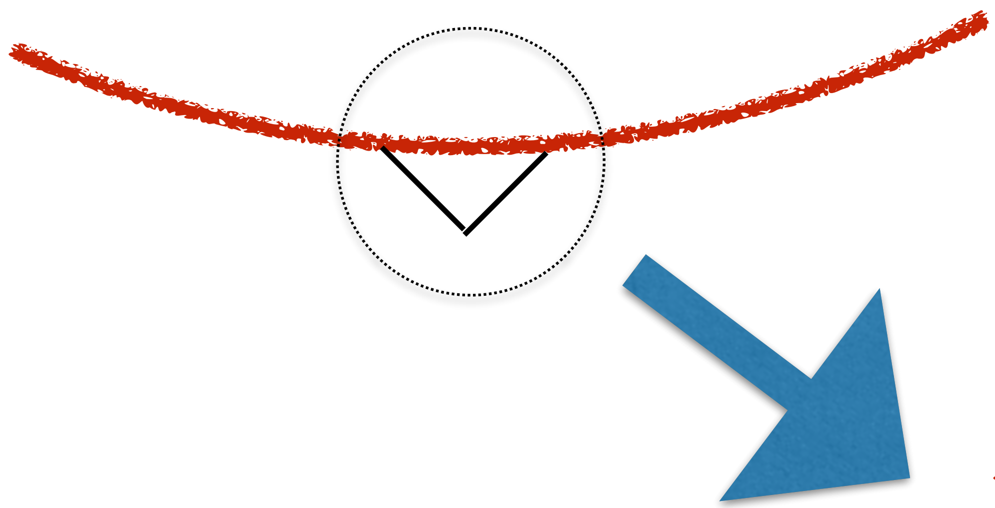
but from these considerations we learn that any microscopic realisation of the holographic Hamiltonian at the stretched horizon must at least satisfy the ETH hypothesis

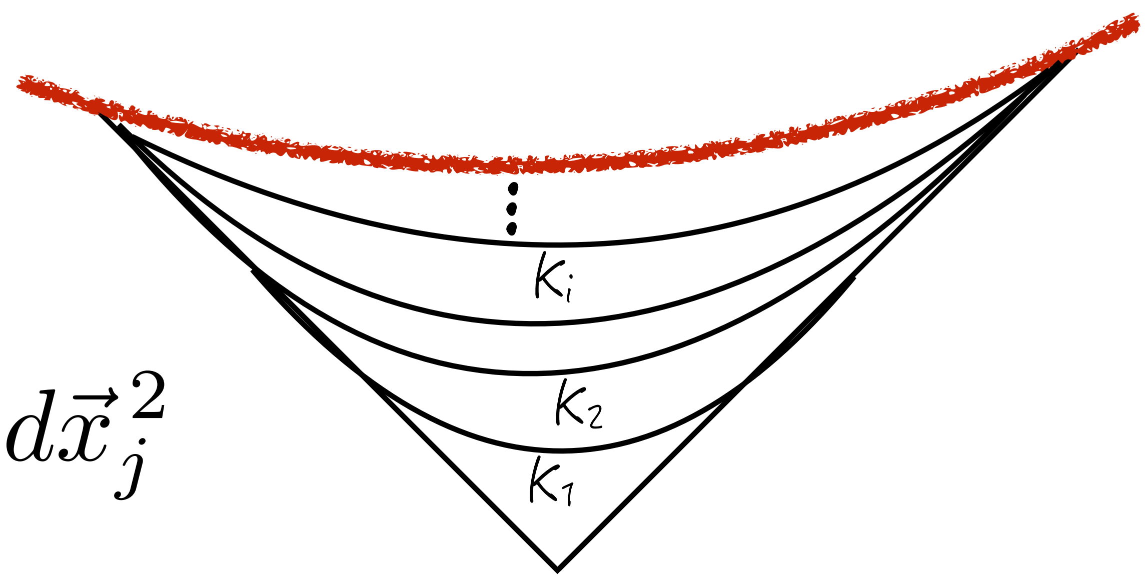
A wild speculation



Hyperbolid billiards are long known to appear near singularities ···

BKL





$$ds^2_{K_i} \approx -dt^2 + \sum_j t^{2p_j}\, d\vec{x}_j^{\,2}$$

The succession of p_j is determined by a hyperbolic billiard game

Could this be a dual of the previous "billiard" at the stretched horizon?

More recently, we have seen a number of more quantitative analysis which introduce ideas from quantum chaos into black hole dynamics

- Quantum Lyapunov exponents
- Contact with ETH paradigm

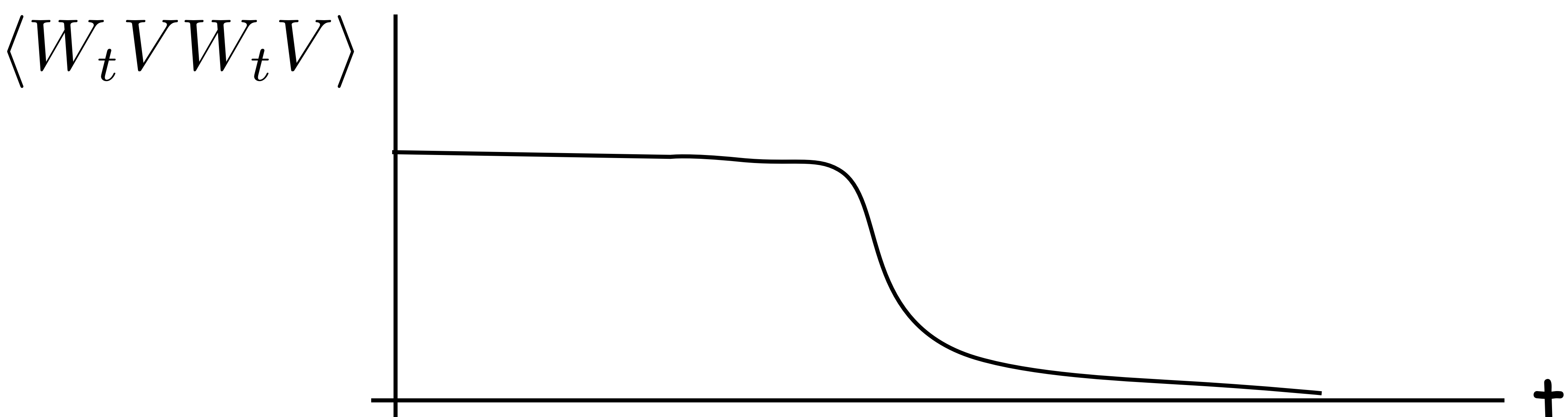
An equivalent characterization of quantum Lyapunov exponents

$$\left\langle \Psi \middle| \Big| [V, W_t] \Big|^2 \middle| \Psi \right\rangle \sim \langle \chi^+ | \chi^- \rangle + \text{const}$$

$$|\chi^+\rangle = V\, W_t\, |\Psi\rangle$$

$$|\chi^-\rangle = W_t\, V\, |\Psi\rangle$$

looks like a scattering amplitude



For a **thermal state**, a bound on the Lyapunov exponent was derived

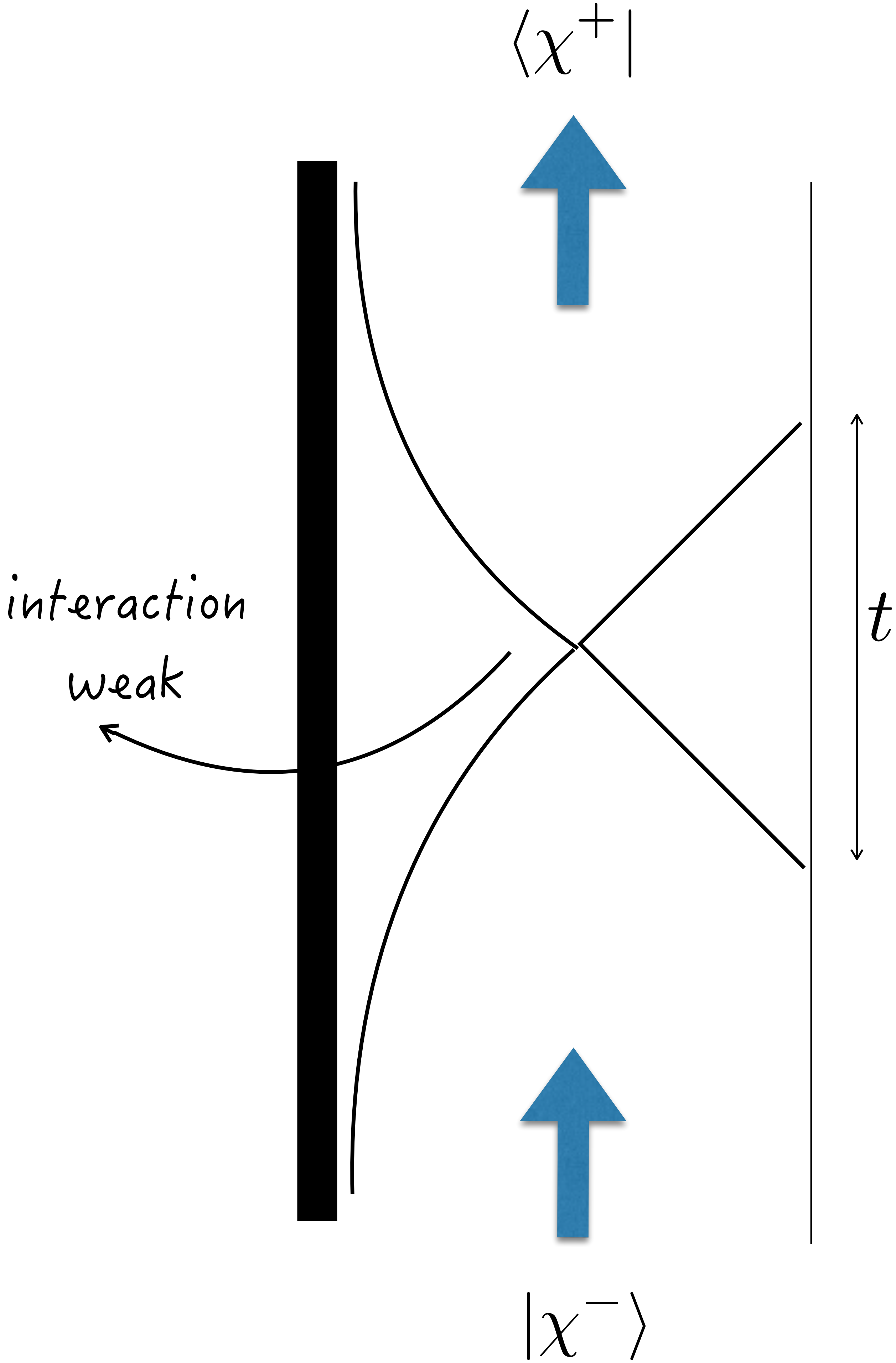
Maldacena, Shenker & Stanford

$$\lambda \leq 2\pi T$$

from analyticity properties of

$$\mathrm{Tr}\left[\rho_\beta\, W_t\, V\, W_t\, V\right]$$

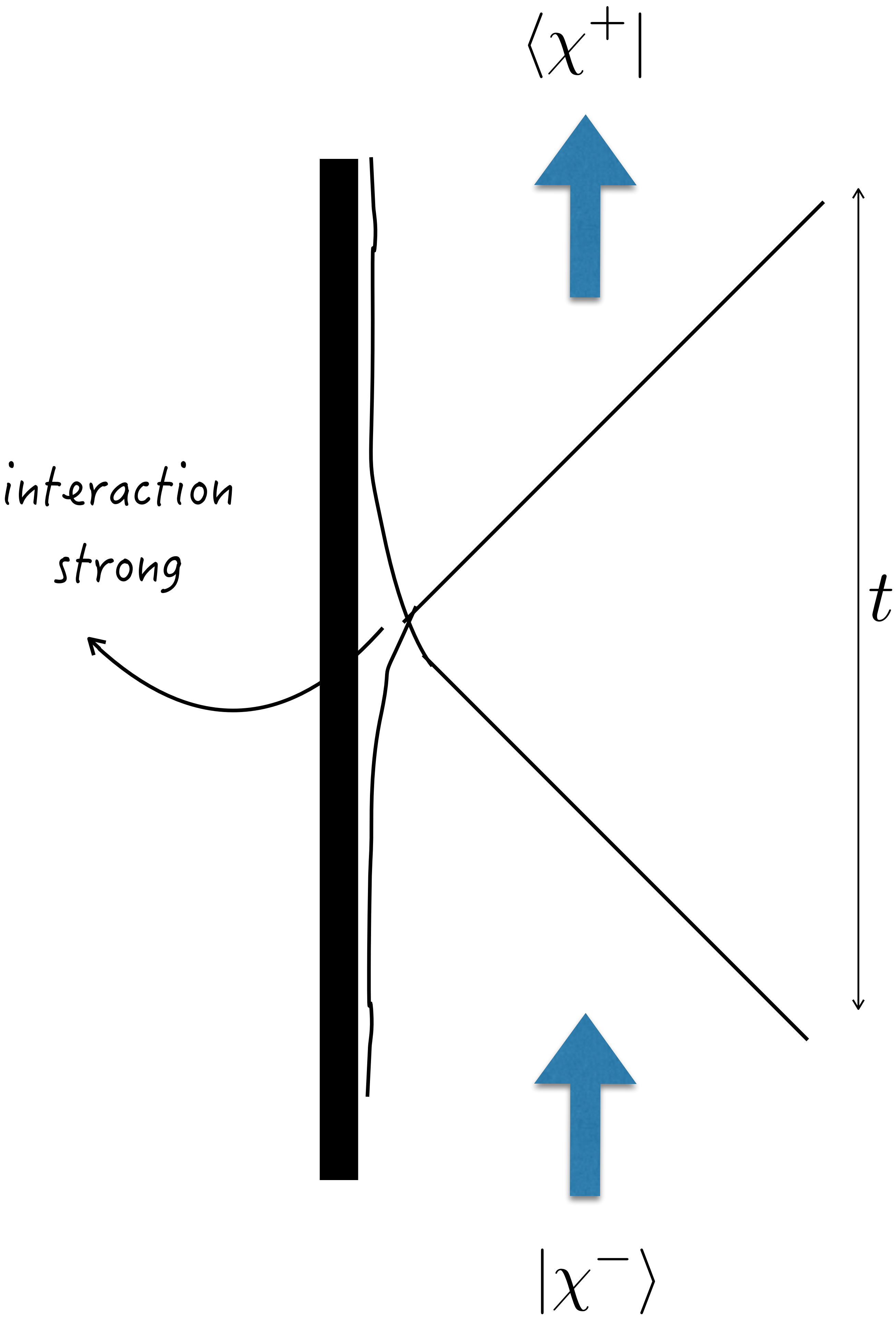
AdS/CFT examples were worked
out in advance, suggesting saturation of the bound



Shenker, Stanford, $\cdots$

$t < t_{\text{scrambling}}$

$\langle \chi^+ | \chi^- \rangle \approx 1$



Shenker, Stanford, ⋯

t-translations act as near-horizon boosts

$$t \approx t_{\text{scrambling}}$$

$$\langle \chi^+ | \chi^- \rangle \approx 1 - O(G_N T^2)\, e^{2\pi T t} + \cdots$$

scrambling time is naturally the free fall time to the stretched horizon

The same behavior was famously seen by Kitaev in a microscopic model, the so called SYK random fermion model

$$H_{\mathrm{SYK}} = \sum_{i,j,k,l}^{N} J_{ijkl}\, \chi_i\, \chi_j\, \chi_k\, \chi_l$$

(J_{ijkl} →) random couplings

Extensively studied in the 1/N expansion

Low energy behavior controlled by a Schwarztian effective action

Gravity picture involves nontrivial AdS_2 dynamics

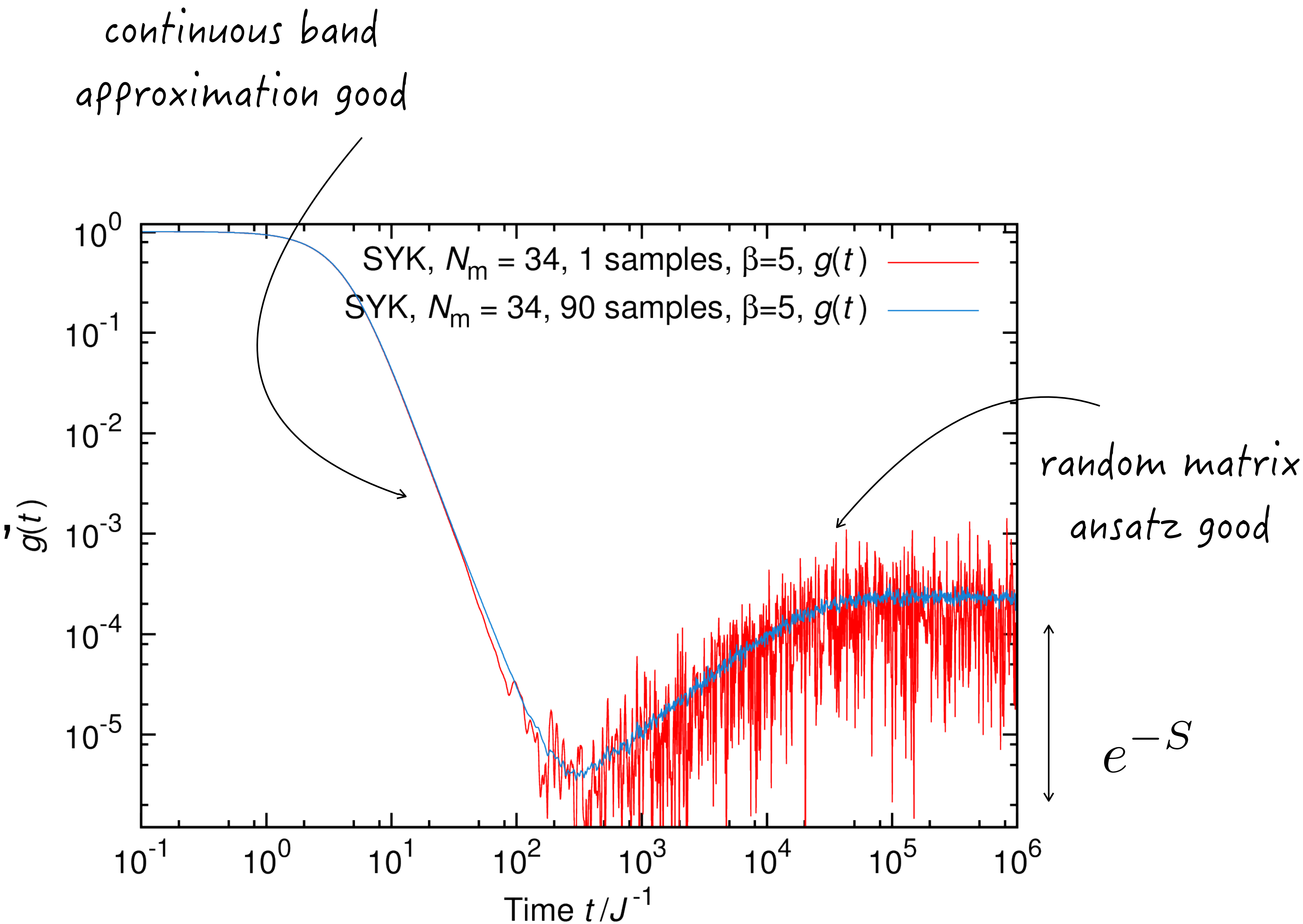
SYK model emerges also as a paradigm for the study of ETH and general chaotic long-time behavior

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, Tezuka

Spectral form factor

$$g(\beta, t) = \left| \frac{Z(\beta + it)}{Z(\beta)} \right|^2$$

$$= Z(\beta)^{-2} \sum_{m,n} e^{-\beta(E_n + E_m)} e^{-i(E_n - E_m)t}$$



continuous band approximation good
random matrix ansatz good
e^{-S}
SYK, N_m = 34, 1 samples, β=5, $g(t)$
SYK, N_m = 34, 90 samples, β=5, $g(t)$
$g(t)$
Time t/J^{-1}

What are the general expectations from correlations of ETH operators?

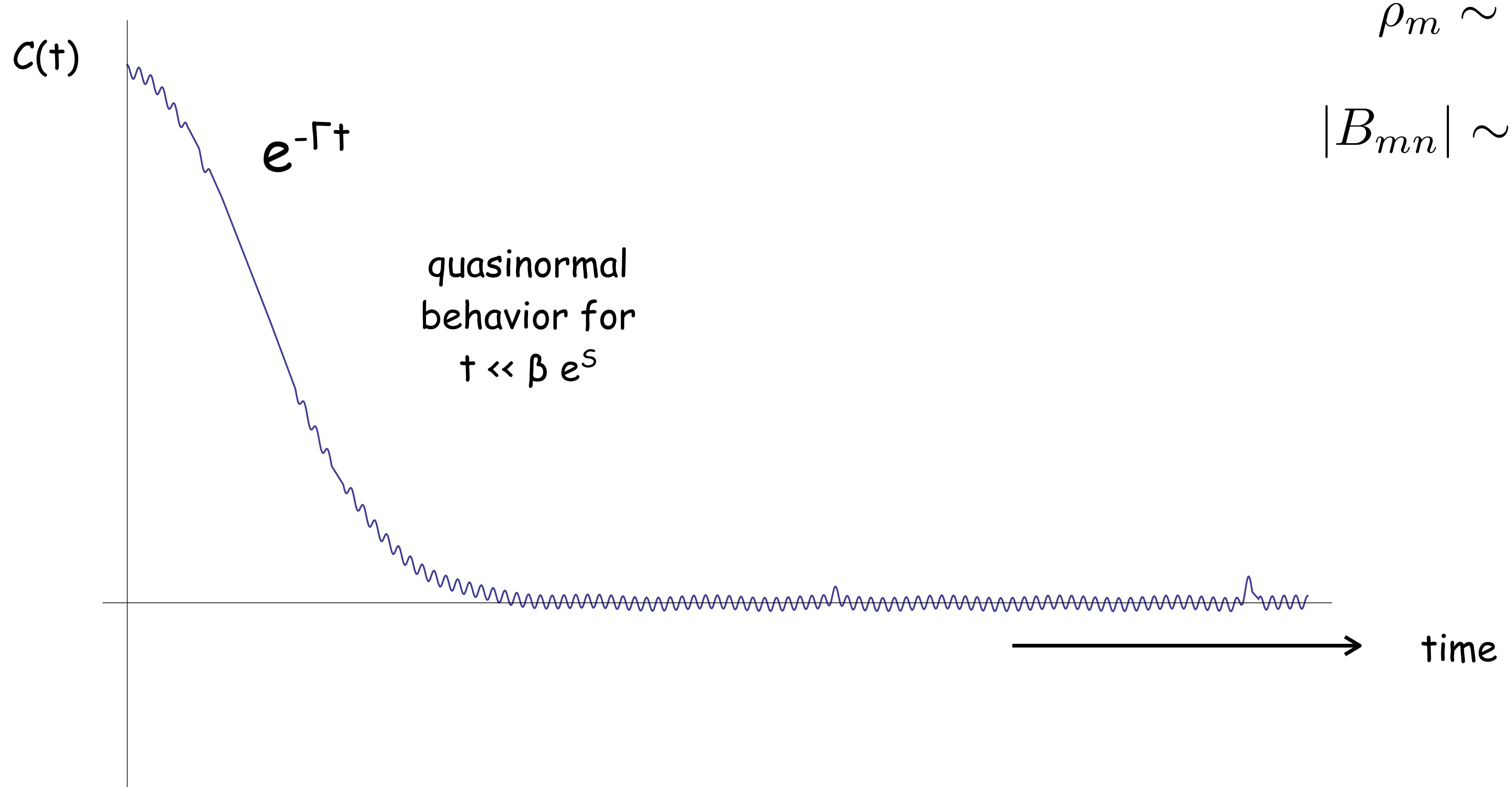
$$C(t) = \mathrm{Tr}\left[\rho\, B(t)\, B(0)\right] = \mathrm{Tr}\left[\rho\, B\, e^{itH}\, B\, e^{-itH}\right]$$

$$C(t) = \sum_{mn}^{e^{2S}} \rho_m \, |B_{mn}|^2 \, e^{i(E_m - E_n)t}$$

$$C(0) = 1$$

$$\rho_m \sim e^{-S}$$

$$|B_{mn}| \sim e^{-S/2}$$



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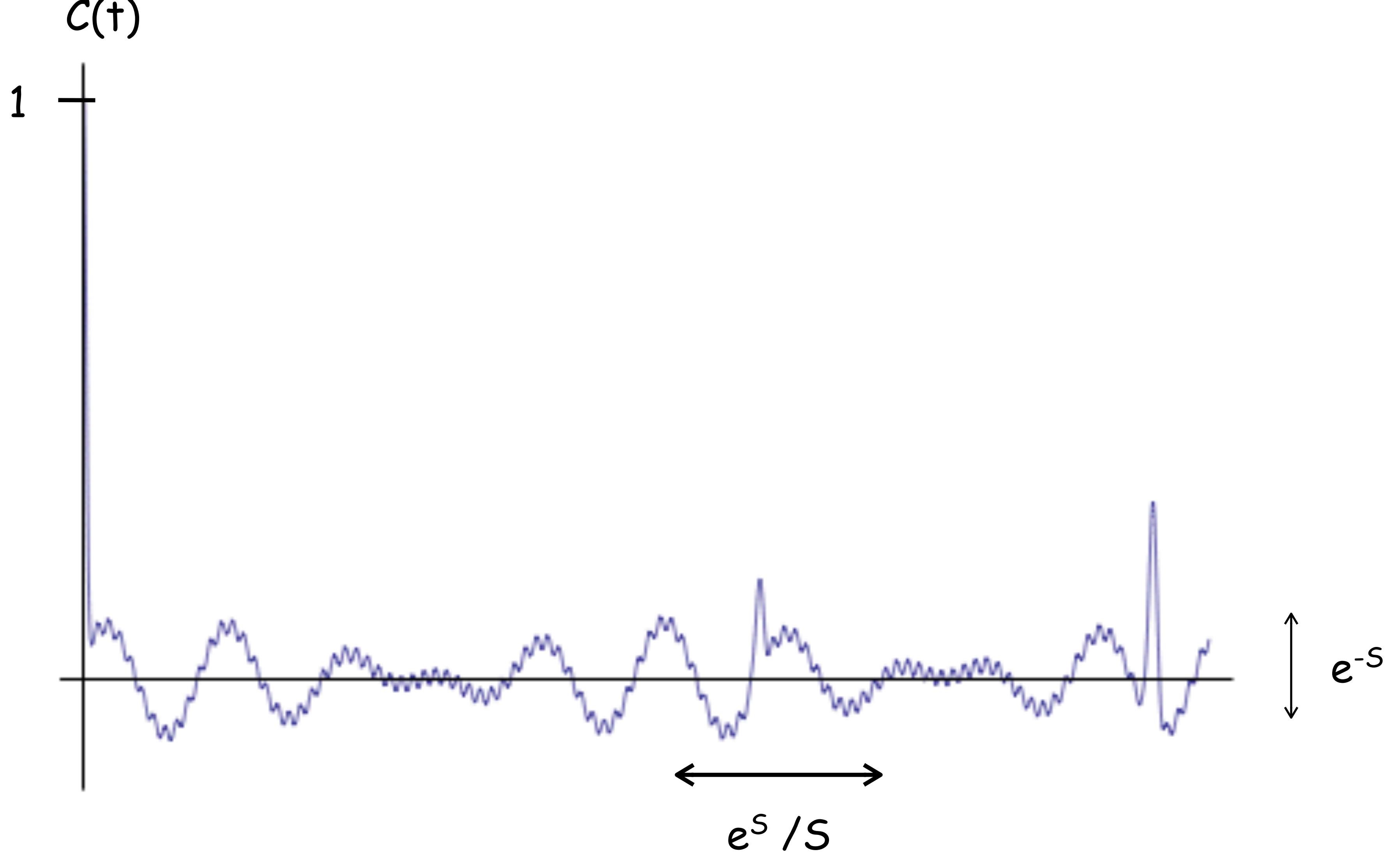

Heisenberg time scale

$$C(t) = \sum_{mn}^{e^{2S}} \rho_m \, |B_{mn}|^2 \, e^{i(E_m - E_n)t}$$

$$C(0) = 1$$

$$\rho_m \sim e^{-S}$$

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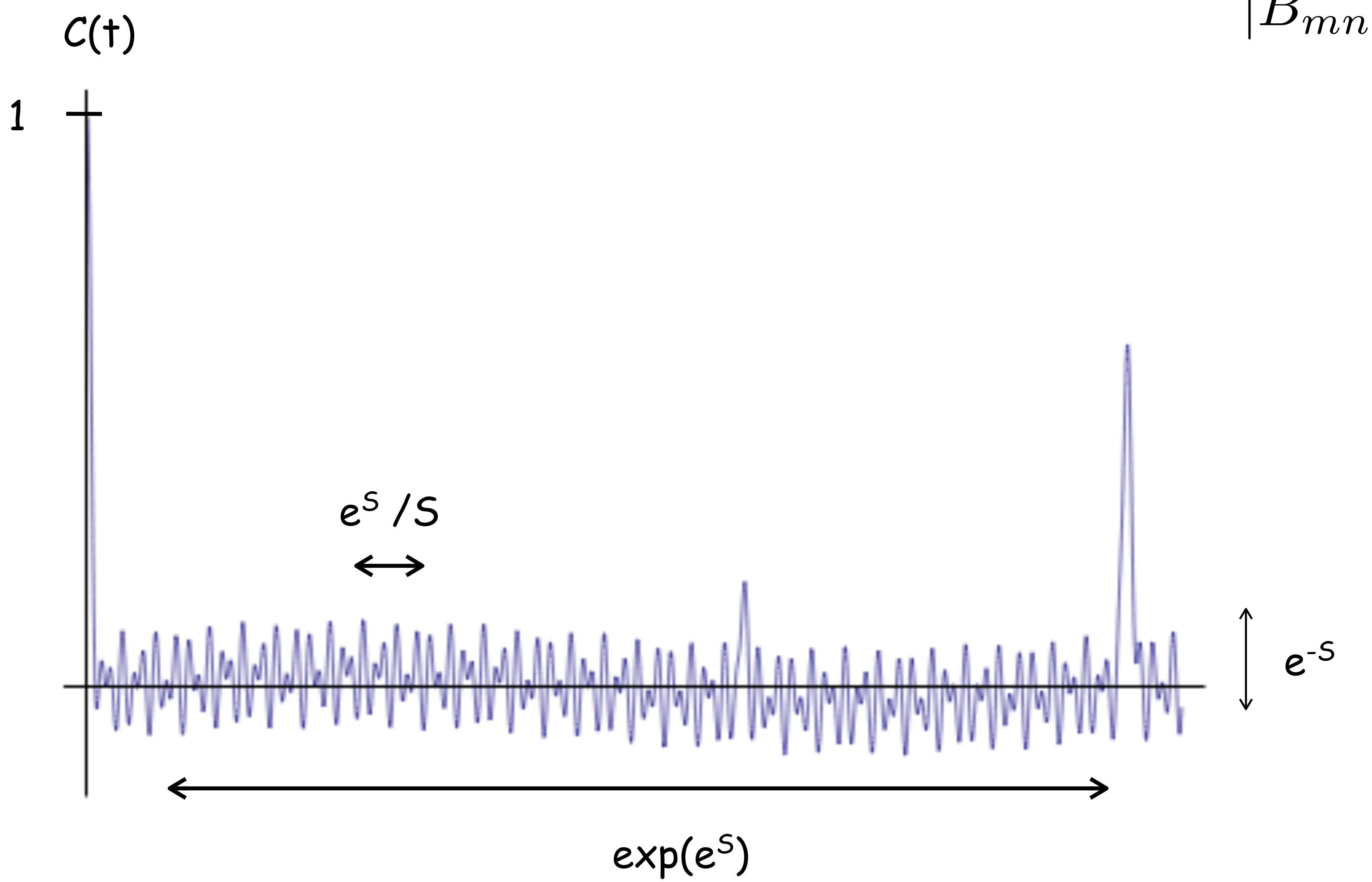


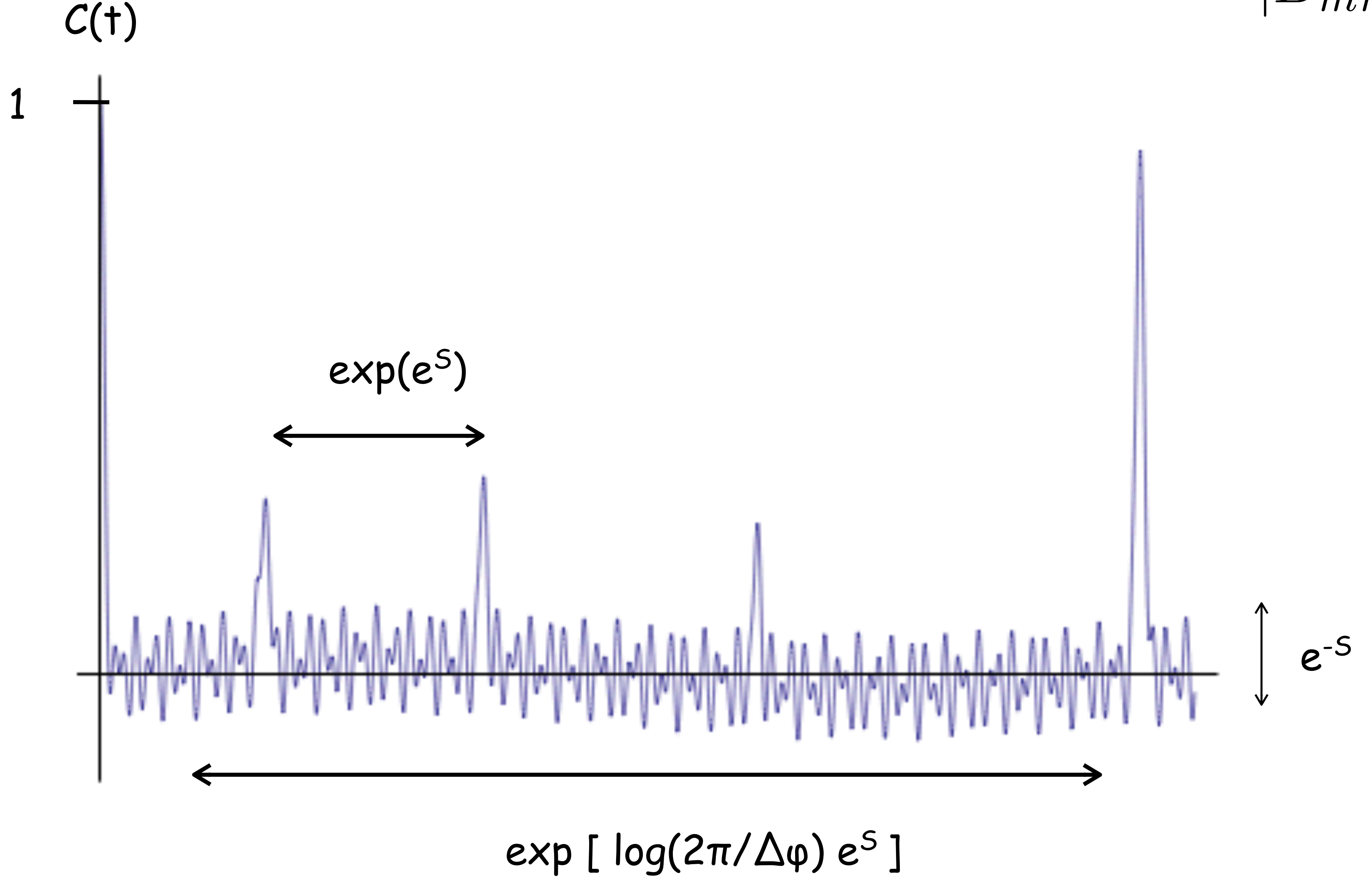
Poincaré time scale

$$C(t) = \sum_{mn}^{e^{2S}} \rho_m \, |B_{mn}|^2 \, e^{i(E_m - E_n)t}$$

$$C(0) = 1$$

$$\rho_m \sim e^{-S}$$

$$|B_{mn}| \sim e^{-S/2}$$



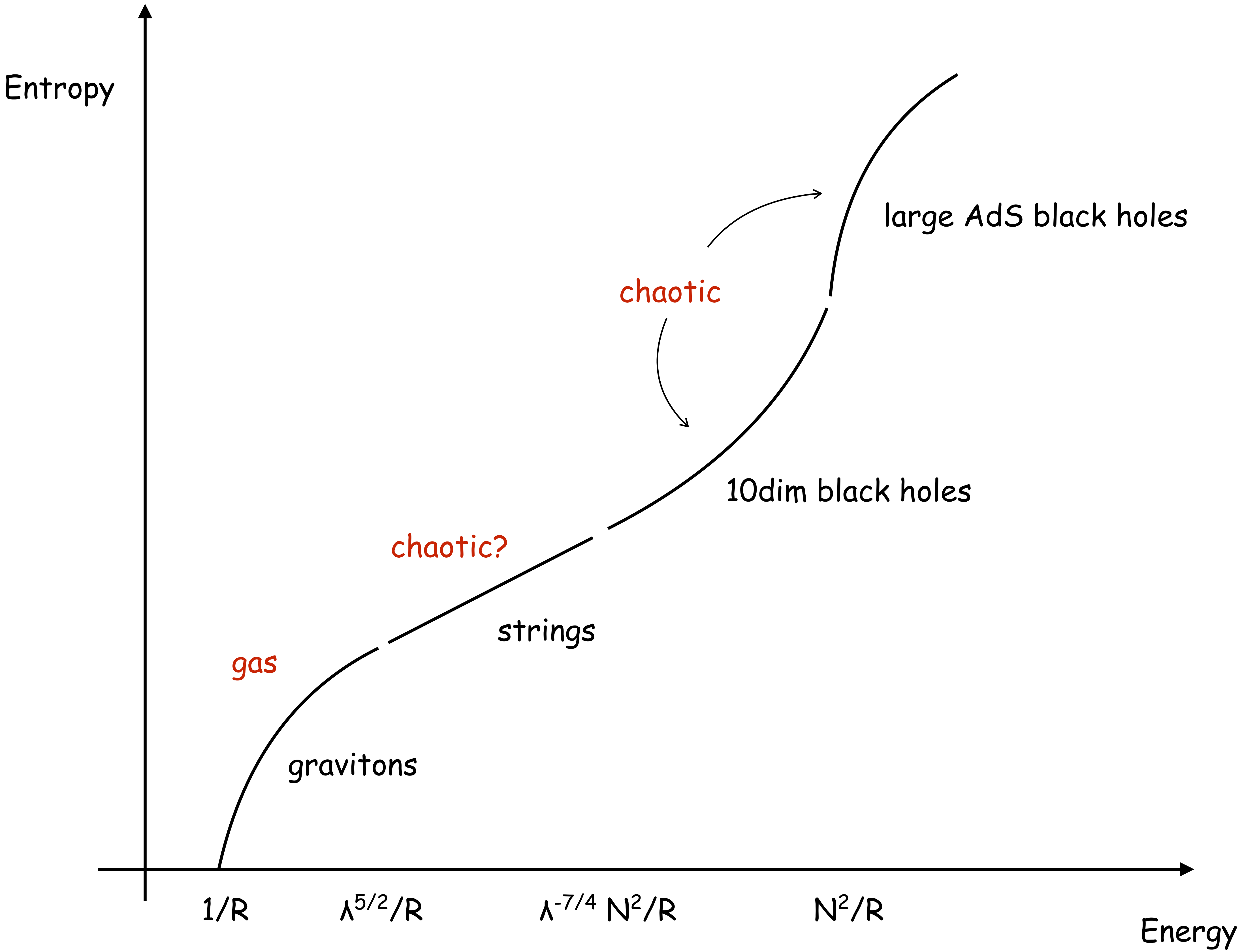
detailed Poincaré time scale

An interesting property of systems with AdS duals is the possibility of doing some amount of "quantum noise tomography"

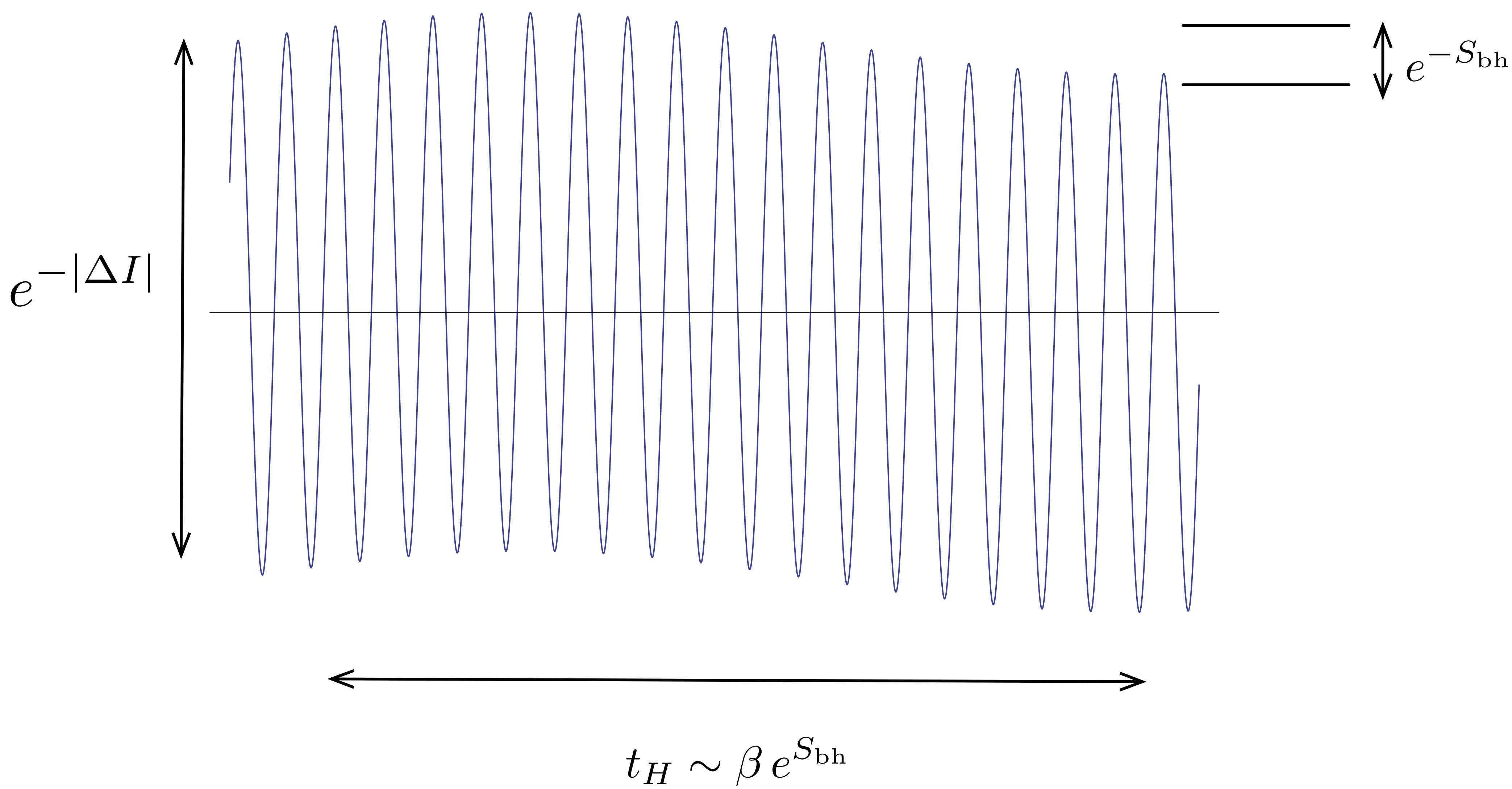
If we have a chaotic high-energy band and a non-chaotic low energy band, the ETH ansatz has to be modified as

$$B_{ETH} = \left(\begin{array}{c|c} \begin{array}{c}\text{Random}\\ O(e^{-S/2})\\ \text{entries}\end{array} & 0 \\ \hline 0 & \begin{array}{c}\text{sparse}\\ O(1)\\ \text{entries}\end{array} \end{array} \right)$$

Such coexistence is an expected property of AdS/CFT models



In such systems the noise tells of the existence of topological sectors



Barbón & Rabinovici

COMPLEXITY

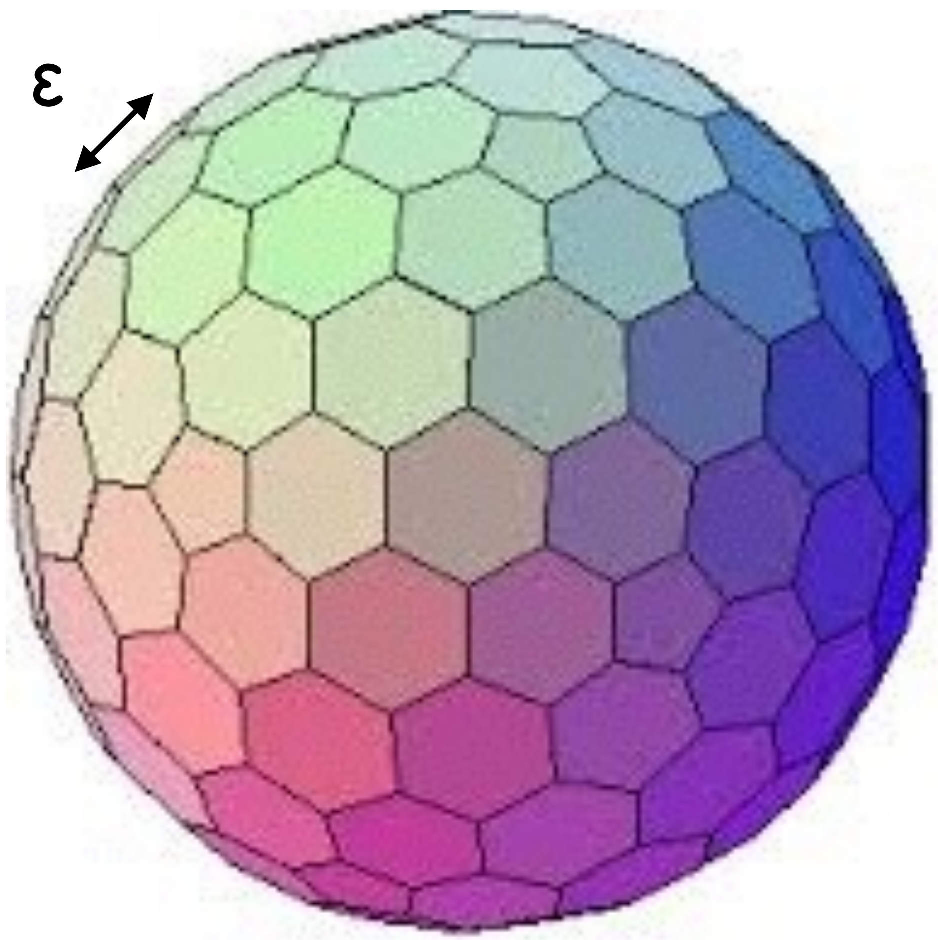
shortest exit
path
from here?

and
from here?

How difficult is to "prepare" specific butterfly effects?

In the quantum realm, look at the Hilbert space as a (huge) set

For a Hilbert space of complex dimension e^S



$$\frac{U(e^S)}{U(e^S - 1) \times U(1)}$$

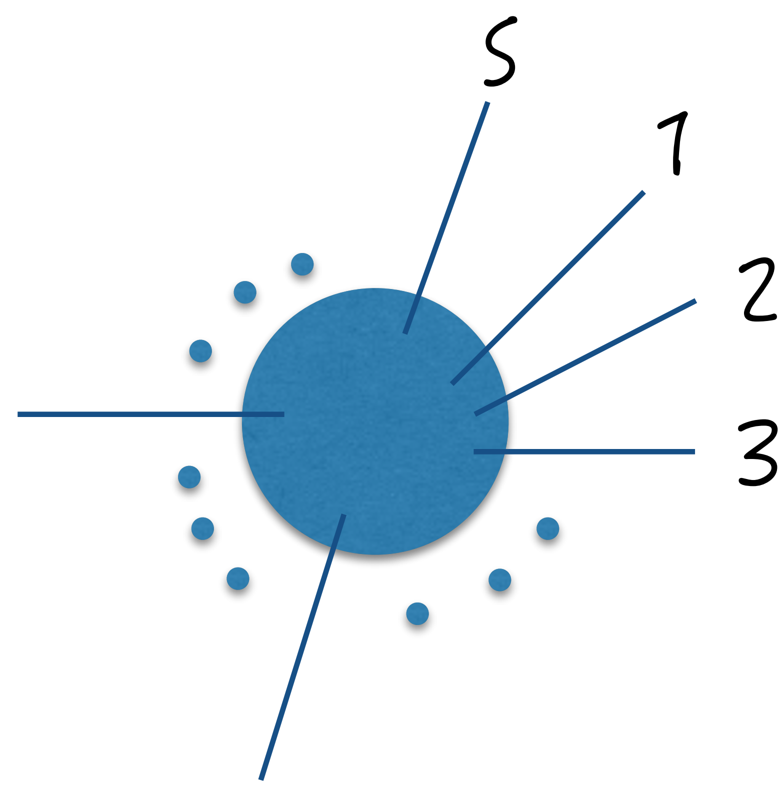
$$\# \text{ cells} \sim \exp\left(e^S \log(1/\epsilon)\right)$$

Exponentially larger than classical phase space

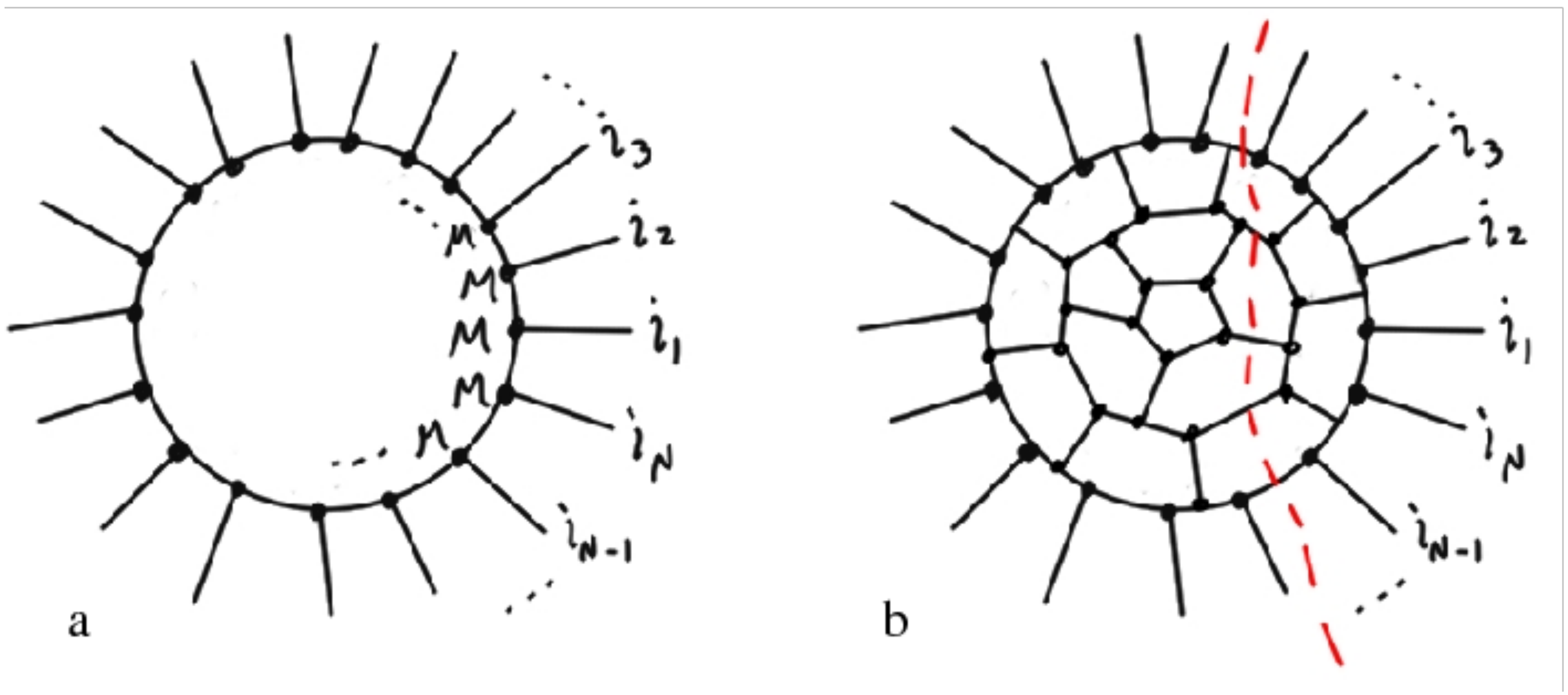
Assuming a notion of tensor factor locality in Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_S$$

A tensor of **S** indices gives a generic state



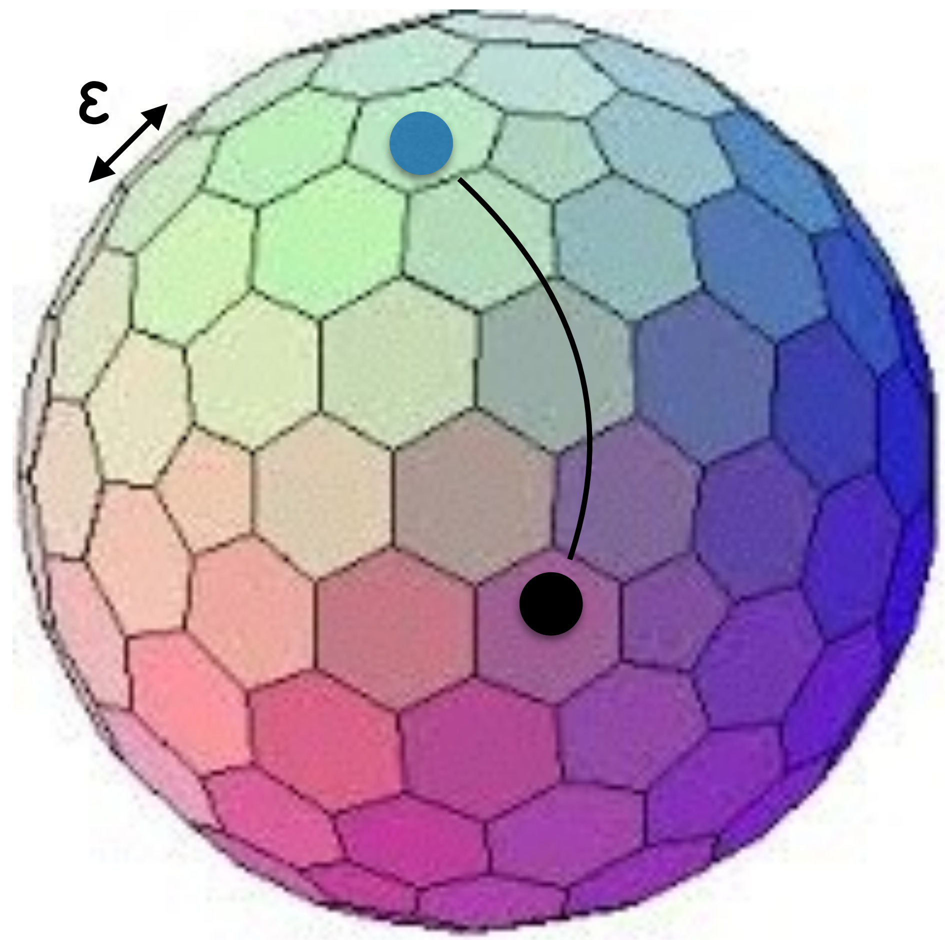
The tensor decomposition in **small** building blocks gives a notion of "complexity"



simple complex

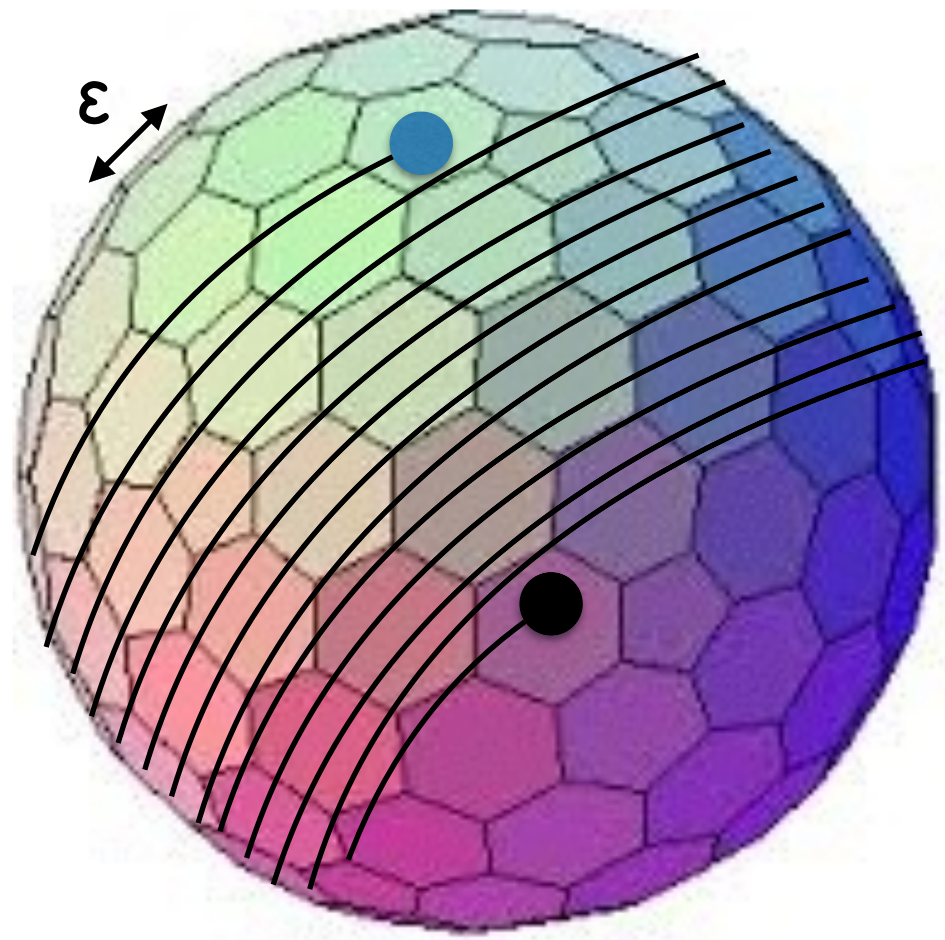
cf. M van Raamsdonk

Paths



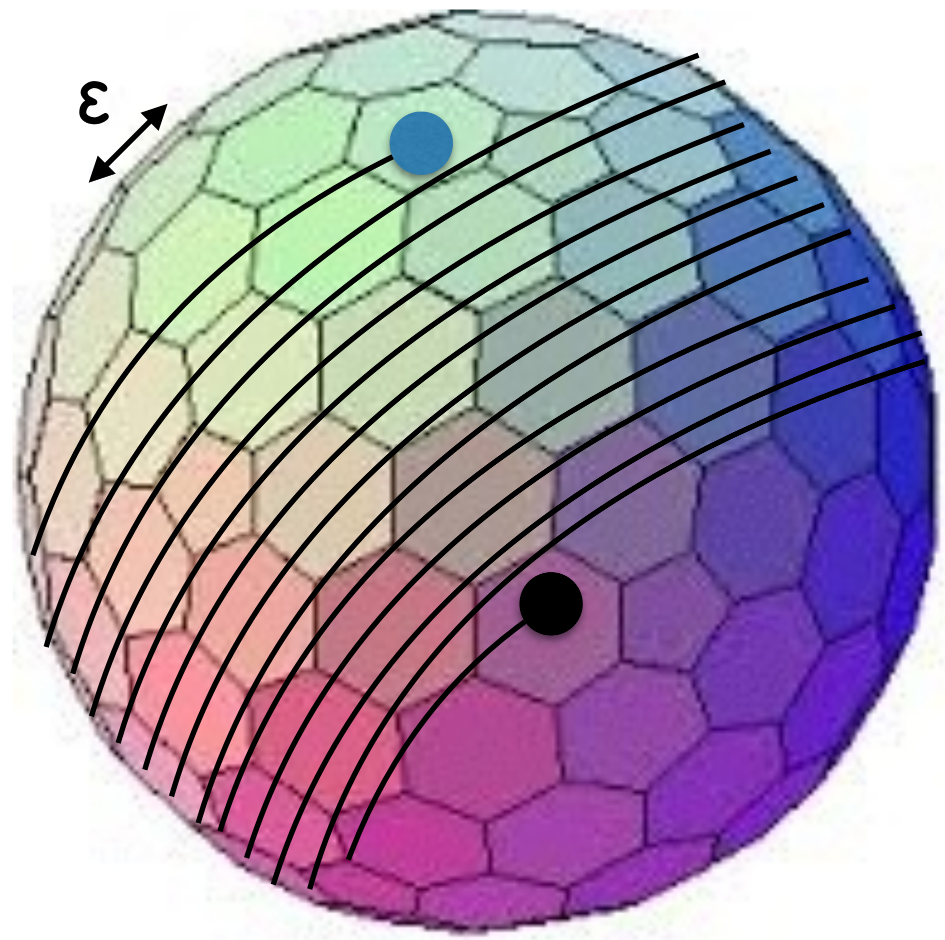
In general, shortest paths in the Hilbert space correspond to very non-local Hamiltonians

Paths



Restricting to "local Hamiltonians", involving few qbuits at a time leads to ergodically-long paths

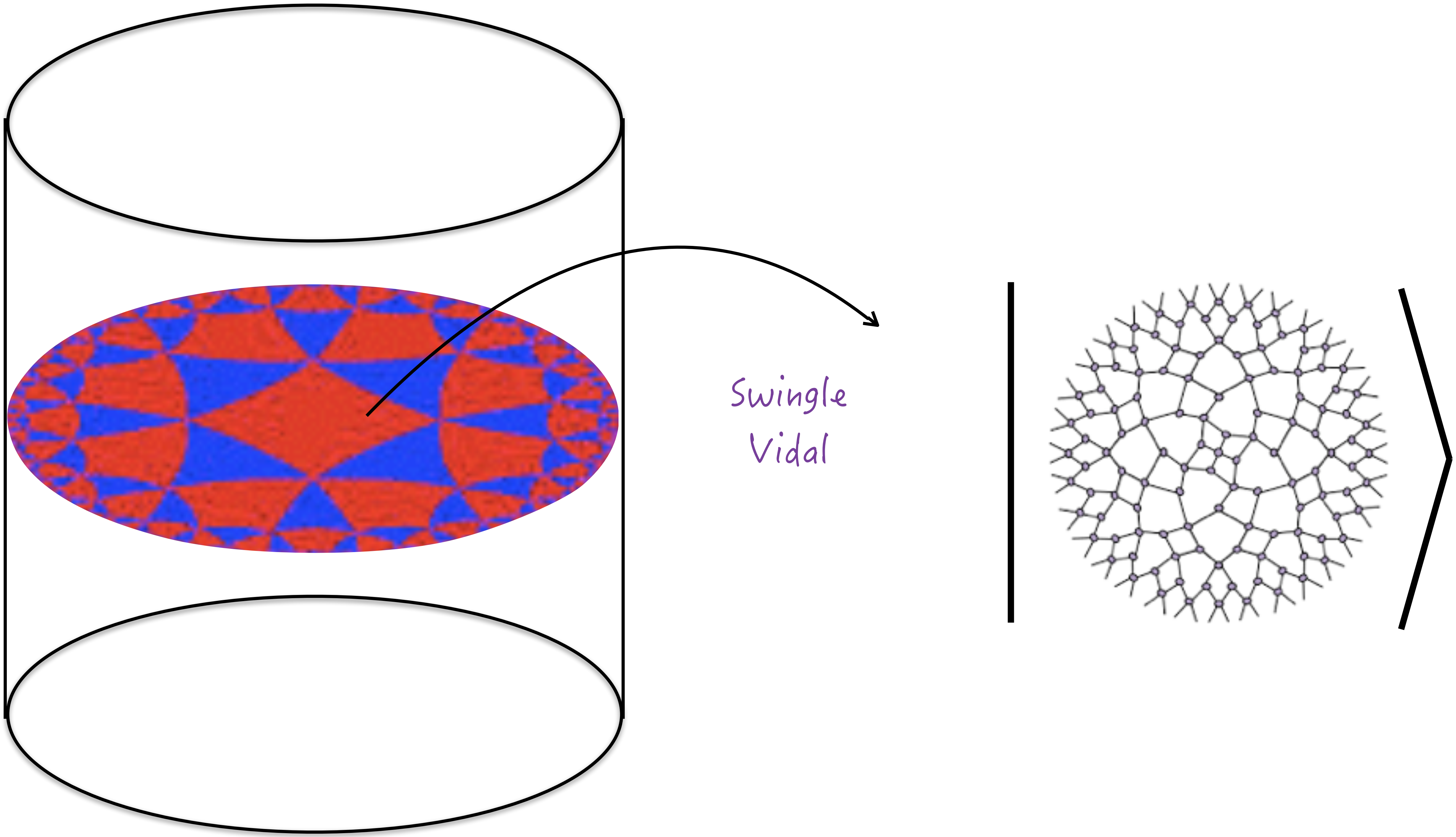
Complexity can be roughly defined as the size of the minimal tensor network building the state with ε-accuracy



$$\text{Complexity}_\epsilon \leq \log[\#_{\text{cells}}] \sim e^S \log(1/\epsilon)$$

What is the connection to black hole physics?

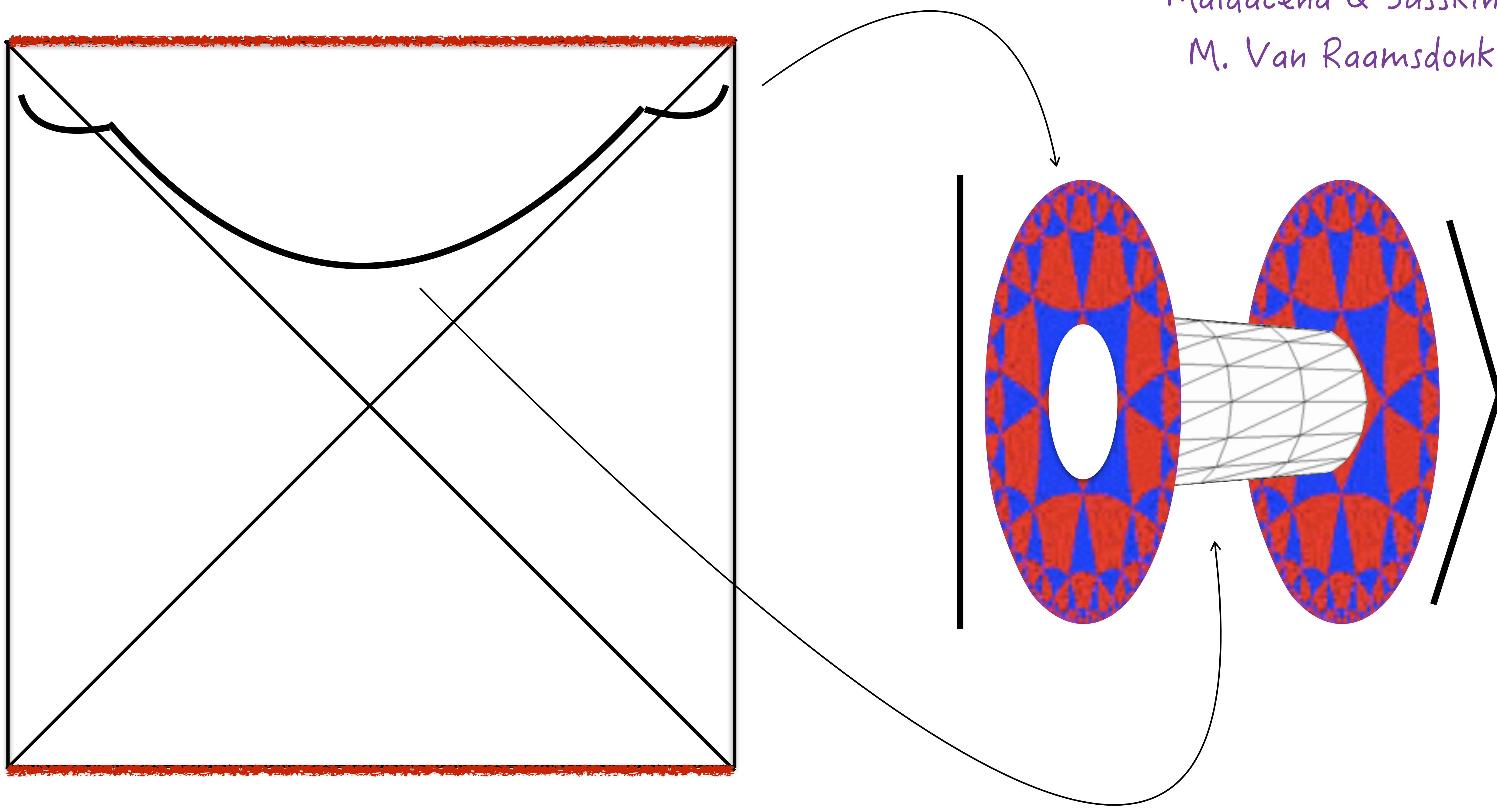
Tensor networks as discrete building blocks of space



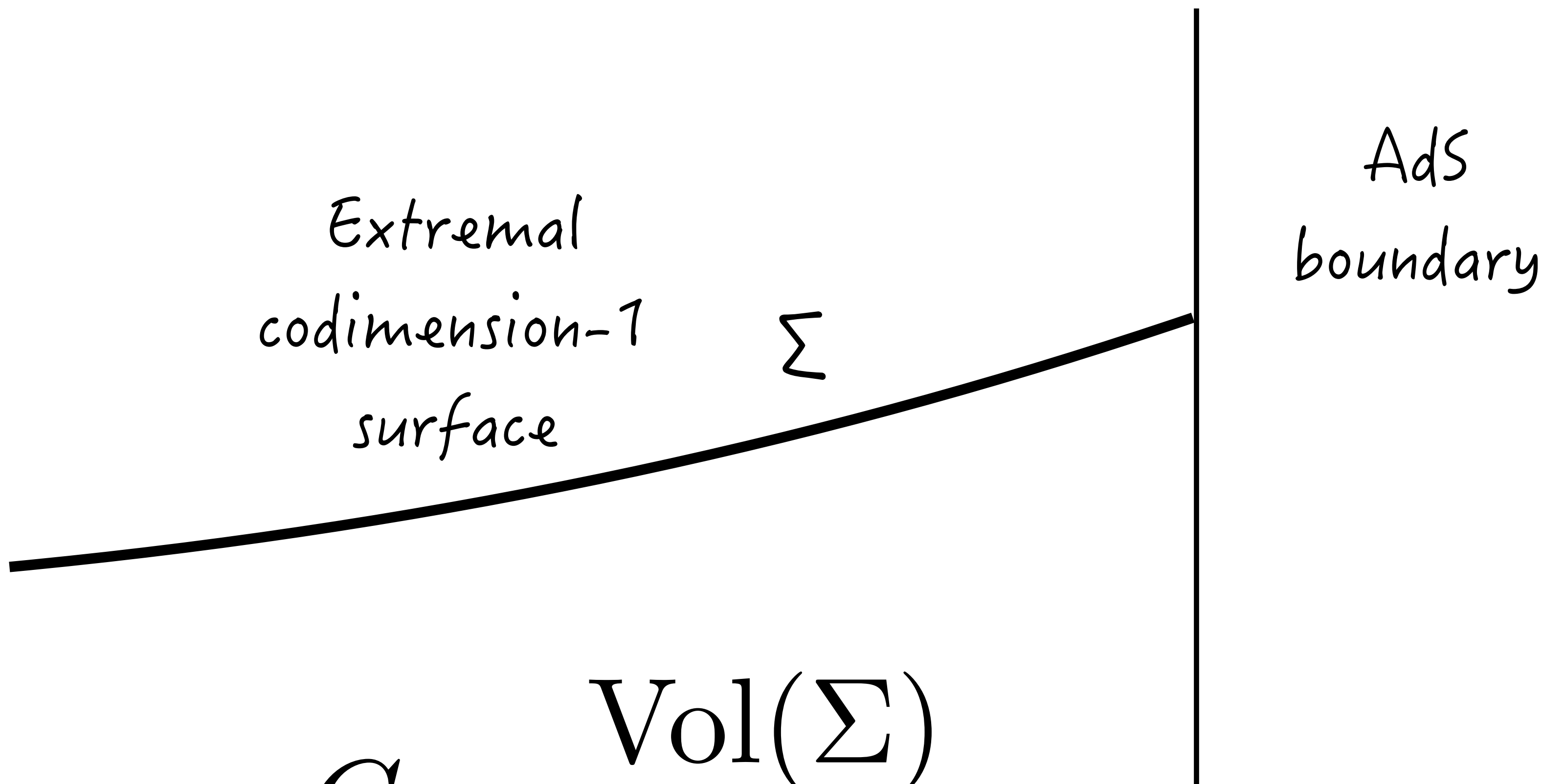
What is the connection to black hole physics?

Tensor networks as discrete building blocks of space

Hartman & Maldacena
Maldacena & Susskind
M. Van Raamsdonk



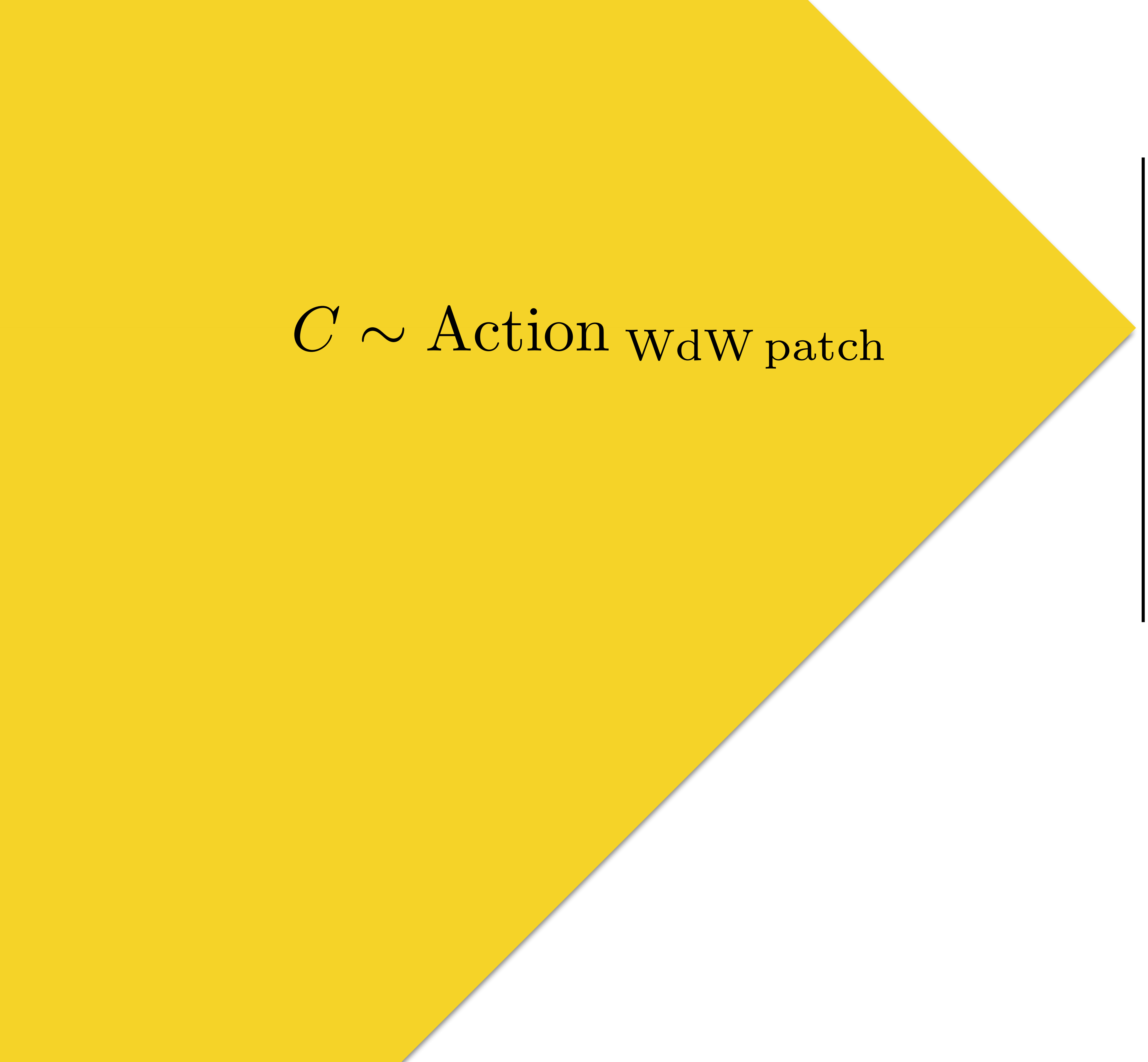
VOLUME / COMPLEXITY duality



$$C \sim \frac{\mathrm{Vol}(\Sigma)}{G R_{\mathrm{AdS}}}$$

A codimension-one generalisation of Ryu-Takayanagi   Susskind et al

ACTION / COMPLEXITY duality



$$C \sim \text{Action}_{\text{WdW patch}}$$

Susskind et al

Things being done … explorations …

Sample qualitative tests of holographic complexity ansatze in assorted black hole states

Stanford group, Kyoto group, Perimeter group
…

→ next talk by J. Martín García

Growing wormholes ⟷ linear growth of complexity

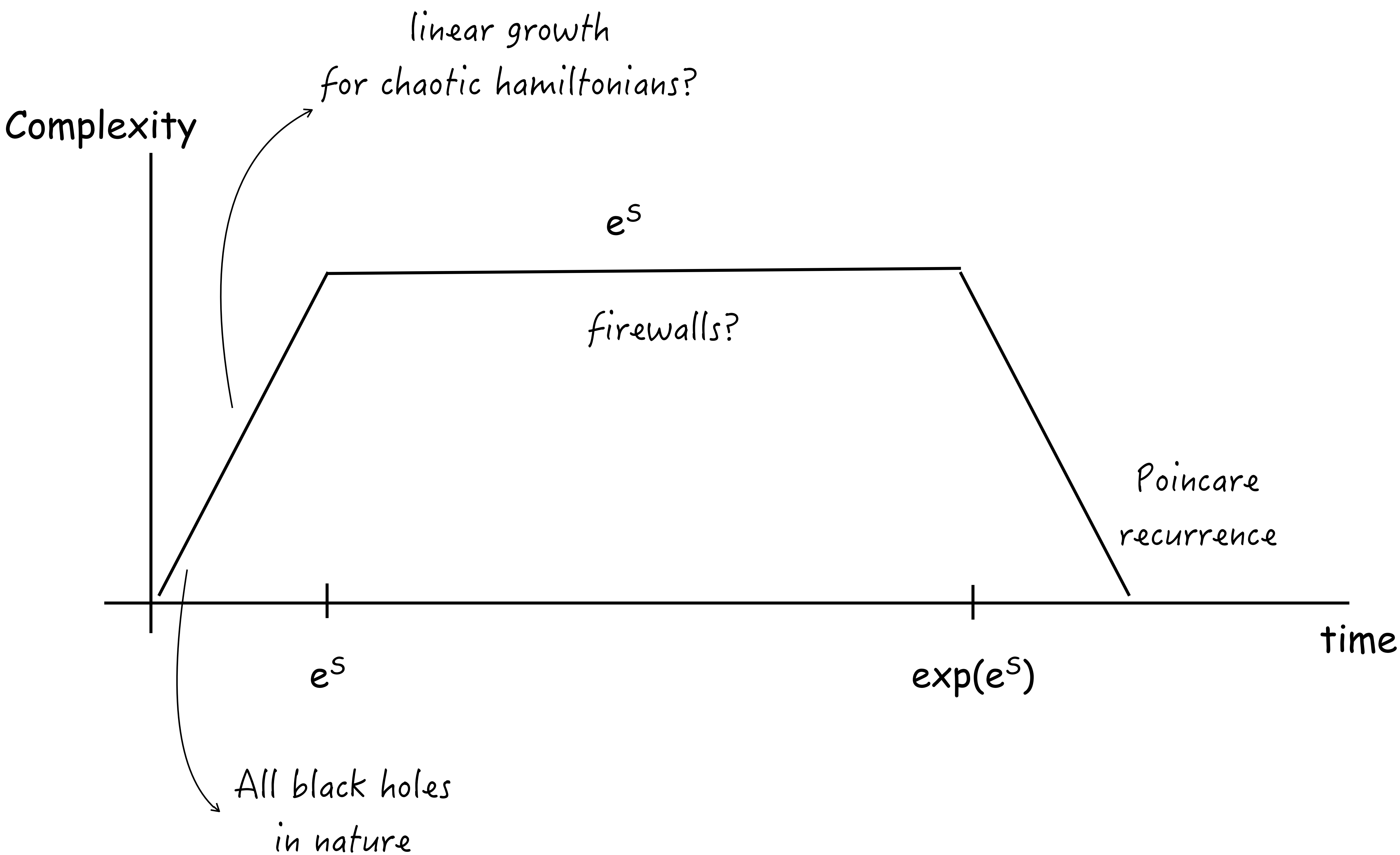
A first look at complexity of space-time singularities

Barbón & Rabinovici

…

"Second law" of quantum complexity theory ?

Susskind



CONCLUSIONS

- Saturating quantum Lyapunov exponents are now part of the check-list for "good" models of holography.
- ETH emerges as a natural assumption of generic black hole dynamics
- Holographic complexity remains mysterious, especially its definition on the CFT side.
- SYK type models a good playground for these questions
- What can be said beyond AdS/CFT ?

MUITO OBRIGADO