Exact-WKB, complete resurgent structure, and mixed anomaly in quantum mechanics on S^1

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Collaboration with
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based on arXiv:2103.06586 [quant-th]
(See also JHEP **12** (2020), 114, [arXiv:2008.00379 [hep-th]])

String theory webinar at IST Lisbon – June 7th, 2021

Introduction

Path-integral

In this talk, we would begin with the asymptotic form of a path-integral in 1D (Euclidean) QM:

$$
\begin{aligned}
Z(\sim) &= \int Dx\, e^{\;\; S(x,\sim)} \\
&= \sum_{j=0}^{1} a_n \sim^j + \sum_{n,k=1}^{1} \sum_{j=0}^{1} b_{n,k,j} e^{\;\;\frac{nS_B}{\sim}} \sim^j (\log \sim)^k .
\end{aligned}
$$

The path-integral can be expressed by **transseries** generated by **transmonomials**, $(\sim, e^{\;\;\frac{S_B}{\sim}}, \log \sim)$.

- $\sim$ ... PT fluctuationg
- $e^{\;\;\frac{S_B}{\sim}}$ ... Instanton (Bion) energy
- $\log \sim$ ... Quasi-zero modes

Path-integral

Path-integral for 1D QM

$$Z(\hbar) = \int Dx\, e^{\;\;S(x,\hbar)}$$

- Perturbative expansion around a vacuum.

$$Z_{\rm p}(\hbar) \quad a_0 + a_1\hbar + a_2\hbar^2 + \quad .$$

- The PT expansion is a divergent series in general.

$$r_c := \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = 0.$$

- What does the PT expansion mean when $r_c = 0$?

Why a divergent series appears?

- This implies that there exists NPT sectors (bions).

  Typical examples: $V(x) = (x^2 \quad 1)^2$, $V(x) = 1 \quad \cos x$.

- Nonperturbarive information is available from the perturbative series via the Borel resummation (Borel transform + Laplace integral)

- PT sector , NPT sectors : **Resurgence relation**
  [J.Ecalle '81, A.Voros '81, D.Sauzin '14]

Borel resummation

Schematic figure of Borel resummation

$$\mathcal{B}[\hbar^k] := \frac{\xi^{k-1}}{\Gamma(k)}$$

Asymptotic form

$$\tilde{Z}(\hbar) \sim \sum_{k=0}^{\infty} a_k \hbar^k \quad \in \mathbb{R}[[\hbar]]$$

Borel transform

$$\mathcal{B}[\tilde{Z}](\xi) = \sum_{k=0}^{\infty} b_k \xi^k \quad \in \mathbb{R}[[\xi]]$$

Asymptotic limit

$0 < \hbar \ll 1$

$$\mathcal{L}[F](\hbar) := \int_0^{\infty} F(\xi) e^{-\xi/\hbar}\, d\xi$$

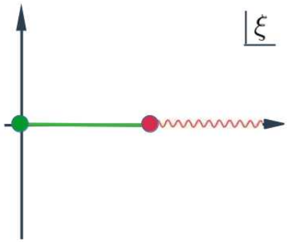
Borel resummation

$$\hat{Z}(\hbar) = \mathcal{L} \circ \mathcal{B}[\tilde{Z}](\hbar)$$

Borel summability

When acting the Laplace integration to $B[\tilde{Z}](\xi)$, ($S := L \circ B$)

- $\tilde{Z}(\hbar)$ is **Borel summable** if it is integrable.
- $\tilde{Z}(\hbar)$ is **Borel nonsummable** if it is not integrable due to a pole (branchcut).



Example:

$$\tilde{Z}(\hbar) \sim n! A^n \hbar^{n+1} \quad \text{as } n \to \infty$$

$$\Rightarrow \quad B[\tilde{Z}](\xi) = \sum_{n=0}^{\infty} (A\xi)^n = \frac{1}{1 - A\xi}$$

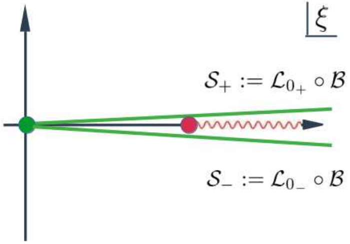
$$\Rightarrow \quad S[\tilde{Z}](\hbar) = \int_0^{\infty} d\xi \, \frac{e^{-\xi/\hbar}}{1 - A\xi}$$

Nonsummable if $A \in \mathbb{R}_+$.

Borel summability

To avoid from the singularity, we introduce the small complex phase to ~. However, the resulting function becomes complex and depends on the integration ray, S_{+} or S (**imaginary ambiguity**). By taking the Hankel contour, the NPT contribution is available from the PT sector (**Resurgence**):

$$(S_{+} \quad S \quad)[\tilde{Z}](\sim) \ / \ ie^{\ \frac{S_b}{\sim}}(1 + O(\sim))$$



- The singularity corresponds to the bion ($I\,\bar{I}$) energy.
- n-th sector $!$ $(n + k)$-sectors $(k\ 2\ \mathrm{N})$
- People expect the ambiguity should be cancelled by the NPT sectors in some way.

Questions and Problems

General questions and problems...

1. How to obtain the resurgence including all NP sectors?
2. How to get the mechanism of the imaginary ambiguity cancellation in full sectors?
3. To do it in the path-integral, all of coefficients are needed (but it is extremely difficult).

Schrödinger equation

Instead of beginning with the path-integral, we also have the Schöredinger equation.

$$\left[\quad \frac{\sim^2}{2} \frac{d^2}{dx^2} + V(x) \right] \psi(x, \sim) = E\psi(x, \sim).$$

- By putting an ansatz of asymptotic form for $\psi(x, \sim)$, its coefficients are easily calculable by the Schöredinger Eq.
- In general, a resurgence mechanism can be argued based on the structure of a given differential equation. [e.g. (non)linear, (non)autonomous, etc...]
- One has to consider Schöredinger eq) path-integral. It is possible through **the resolvent method**.

Resolvent method

- By using the Laplace transform, one can obtain the resolvent $G(E)$ which is a function of E from $Z(\beta)$, as

$$G(E) = \int_0^{\infty} Z(\beta) e^{\beta E} d\beta, \qquad Z(\beta) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} G(E) e^{-\beta E} dE.$$

- The resolvent $G(E)$ can be written by $D(E)$ called the Fredholm determinant,

$$G(E) = \mathrm{tr}\frac{1}{\hat{H} - E} = -\frac{\partial \log D_{\mathrm{FD}}(E)}{\partial E}, \qquad D_{\mathrm{FD}}(E) := \det\left(\hat{H} - E\right)$$

- $D_{\mathrm{FD}}(E) = 0$ gives **the spectral form**. Indeed, from the argument principle,

$$Z(\beta) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{D'_{\mathrm{FD}}(E)}{D_{\mathrm{FD}}(E)} e^{-\beta E} dE = \sum_{k=1}^{\infty} n_k e^{-\beta E_k} = \mathrm{tr}\left[e^{-\beta \hat{H}}\right],$$

where $D_{\mathrm{FD}}(E_k) = 0$ and n_k is the number of zero of $D_{\mathrm{FD}}(E_k)$.

Gutzwiller trace formula

– semiclassical construction of the resolvent $G(E)$.

Normally, it is defined for "Lorentzian" partition function.

$$Z(T) = \mathrm{tr} e^{\ \ i\hat{H}T} = \int_{\mathrm{periodic}} Dx\, e^{iS}$$

$$) \qquad G(E) = \quad i\,\mathrm{tr}\frac{1}{\hat{H} \quad E} = \int_0^{1} dT \int_{\mathrm{periodic}} Dx\, e^{\Gamma(\sim)},$$

where $\Gamma = S + ET = n \oint pdx \quad \int^T Hdt + ET$.

We evaluate it by the stationary phase approximation.

Gutzwiller trace formula (GTF)

By taking up to the sub-leading contribution, $G(E)$ can be expressed by

$$G(E) = i \sum_{\mathrm{p.p.o}} \sum_{n=1}^{\infty} T(E) e^{ni \oint_{\mathrm{p.p.o.}} p dx} (-1)^{n} |\det(\mathrm{Hess}(S))|^{-1/2},$$

where p.p.o. denotes a prime periodic orbit, $T(E)$ is the period of each cycle whose energy is E, and $(-1)^n = e^{\frac{\pi i}{4}\mathrm{sgn}(\mathrm{Hess}(S))}$ is **the Maslov index**.
(See Gutzwiller's book for the derivation
[M. Gutzwiller, Springer-Verlag New York '90])

In general, it is a tough problem to determine all p.p.o.

Strategy

We start with the Schrödinger equation and obtain the **quantization condition** by **the exact-WKB analysis**.

$$D_{\mathrm{FH}}(E) = 0 \qquad ! \qquad D_{\mathrm{WKB}}(E) = 0$$

- $D_{\mathrm{WKB}}(E)$ keeps all informations such as transseries, ambiguity cancellation, and resurgence structure for E. This gives **the genralized B-S quantization**.
- $D_{\mathrm{WKB}}(E)$ can be expressed by a kind of periodic orbits. This gives **the Gutzwiller trace formula** through the resolvent $G(E)$.
- As we saw, $D_{\mathrm{WKB}}(E)$ gives **the spectral form**. Furthermore, by taking integral by parts for $\partial_E \log D_{\mathrm{WKB}}(E)$, it gives **the path-integral**.

Strategy

We would take the following steps:

1. Obtain **the quantization condition** by using **the exact-WKB analysis**.
2. Then, consider **the resurgence relation** for the quantization condition.
3. Derive expressions such as GTF and path-integral from the quantization condition through **the resolvent method**.

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$
Schrödinger eq.

↓ exact-WKB

$$D(E) = 0$$
exact quantization condition

→ $$\frac{1}{2\pi i}\oint p dx \simeq (N + \tfrac{1}{2}) + O(e^{-\frac{S}{\hbar}})$$
generalized Bohr-Sommerfeld

↓

$$G(E) = -\partial_E \log D(E)$$

→ expand → $$G(E) \simeq \sum_n \sum_{p.p.o.} (-1)^n e^{in\oint pdx}$$
Gutzwiller trace formula

↓ inverse Laplace transform

$$Z(\beta) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} G(E) e^{-\beta E} dE$$

→ calculate the residues → $$Z(\beta) = \sum_n e^{-\beta E_n}$$
spectral summation form

↓ integral by parts and expand $\log D(E)$

$$Z(\beta) = \sum_n a_n \hbar^n + e^{-\frac{S_1}{\hbar}} \sum_n b_n \hbar^n + e^{-\frac{S_2}{\hbar}} \sum_n c_n \hbar^n + \ldots$$
path-integral(trans-series) form

Anomaly and the cosine model

(Mixed) 't Hooft anomaly

[G.'t Hooft '80, D.Gaiotto et al. '17, Y.Kikuchi et al. '17, etc.]

A obstruction to promoting the global symmetry to local gauge symmetry

- Global symmetry G) Gauging (background gauge A)
- Take the G-gauge transform. We say that the theory has an 't Hooft anomaly if it gives

$$Z[A + d\lambda] = Z[A] \exp\left(i\mathcal{A}[\lambda, A]\right).$$

  The phase can not be canceled by a local counter term.

- If $G = G_1 \quad G_2$, it is said to be a mixed 't Hooft anomaly.

Cosine model

$$L = \frac{\dot{x}^2}{2} + V(x) - \frac{i\theta}{2\pi}\dot{x},$$
$$V(x) = 1 - \cos(Nx), \quad x \sim x + 2\pi, \quad N \in \mathbb{N}.$$

Symmetry

$$\mathbb{Z}_N \text{ shift} \qquad \mathrm{U} : x(t) \mapsto x(t) + \frac{2\pi}{N} \qquad (\mathrm{U}^N = 1)$$
$$\text{Time reversal } (\mathbb{Z}_2) \qquad \mathrm{T} : (x(t), \dot{x}(t)) \mapsto (x(-t), -\dot{x}(-t)) \quad (\theta = 0, \pi)$$

Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\hat{p} - \frac{\theta}{2\pi}\right)^2 + V(\hat{x}), \qquad [\hat{x}, \hat{p}] = 1.$$

Anomaly and the cosine model

We consider when $\theta = 0$ or π.

$$\hat{H} = \frac{1}{2}\left(\hat{p} \quad \frac{\theta}{2\pi}\right)^2 + V(\hat{x}), \qquad [\hat{x}, \hat{p}] = 1.$$

$\mathrm{T}\hat{H}\mathrm{T}^{\ \ 1} = \hat{H}$ can be satisfied by

$$\mathrm{T}\hat{x}\mathrm{T}^{\ \ 1} = \hat{x}, \quad \mathrm{T}\hat{p}\mathrm{T}^{\ \ 1} = \begin{cases} \hat{p} & \theta = 0 \\ \hat{p} + 1 & \theta = \pi \end{cases}.$$

By using the coordinate basis, U and T can be expressed by

$$\mathrm{U} = \exp\left[\frac{2\pi i}{N}\partial_x\right], \quad \mathrm{T} = \begin{cases} K & \theta = 0 \\ \exp[ix]\,K & \theta = \pi \end{cases}, \quad (K\text{: c.c. operator})$$

$$) \quad \mathrm{TUT}^{\ \ 1} = \begin{cases} \mathrm{U} & \theta = 0 \\ \exp\left[\ \ \frac{2\pi i}{N}\right]\mathrm{U} & \theta = \pi \end{cases}.$$

Anomaly and the cosine model

Suppose that $\theta = \pi$. We redefine U as

$$\mathsf{U}' := \exp\left[-\frac{2\pi i}{N}k\right]\mathsf{U}, \qquad (k \in \mathbb{Z}_N)$$

$$\Rightarrow \quad \mathsf{T}\mathsf{U}'\mathsf{T}^{-1} = \exp\left[-\frac{2\pi i}{N}(2k-1)\right]\mathsf{U}'.$$

U' and T is commutative if there exists a solution satisfying $2k-1=0 \pmod N$ for $k \in \mathbb{Z}_N$.

- If $N \in 2\mathbb{N}$, no solution exists $\Rightarrow$ **Mixed 't Hooft anomaly**
  All the energy spectra is two-fold degenerate.
- If $N \in 2\mathbb{N}+1$, $k = \frac{N+1}{2}$ $\Rightarrow$ **Global incosistency**
  A energy singlet state at $\theta = 0$ is not continuously connected to a singlet state at $\theta = \pi$.

Anomaly and the cosine model

$\mathbb{Z}_N$ background gauge (A, B) $\qquad (NA = dB)$

$$S[x, A, B] = \int dt \left[\frac{1}{2}(\dot{x} + A_0)^2 + 1 \quad \cos(Nx + B)\right] \quad i\frac{\theta}{2\pi}\int (dx + A) + ik\int A,$$

with the Chern-Simons level $k\ 2\ \mathbb{Z}_N$.

$$N\int A = \int dB\ 2\ 2\pi\mathbb{Z}.$$

Gauge transform

$$x\ 7!\ x \quad \lambda, \quad A\ 7!\ A + d\lambda, \quad B\ 7!\ B + N\lambda.$$

Partition function

$$Z_{\theta,k}[(A, B)] = \int Dx \exp\left(\quad S[x, A, B]\right).$$

We take the gauge fixing condition as $B = 0 \pmod N$. Thus,

$$A = \sum_{\ell 2 Z} \frac{2\pi\ell}{N}\delta(t \quad t_\ell)dt, \qquad B = \sum_{\ell 2 Z} = \frac{2\pi\ell}{N}\Theta(t \quad t_\ell),$$

with the step function and the delta function, $\Theta(t)$ and $\delta(t)$, respectively. The partition function can be evaluated as

$$Z_{\theta,k}[(A,B)] = \left\langle \prod_{\ell 2 Z} \left(e^{\quad 2\pi i k/N} \mathrm{U}(t_\ell) \right)^{\ell} \right\rangle.$$

By acting T, one finds

$$Z_{\theta,k}[\mathrm{T}(A,B)] = \left\langle \prod_{\ell 2 Z} \left(e^{\quad 2\pi i k/N} \mathrm{T}\mathrm{U}(t_\ell)\mathrm{T}^{\quad 1} \right)^{\ell} \right\rangle = Z_{\theta,k}[(A,B)] e^{iA[k,A]}$$

$$e^{iA[k,A]} = \begin{cases} \prod_{\ell 2 Z} e^{2\pi\ell i(2k)/N} & = e^{2ki\int A} & \text{for } \theta = 0 \\ \prod_{\ell 2 Z} e^{2\pi\ell i(2k \quad 1)/N} & = e^{(2k \quad 1)i\int A} & \text{for } \theta = \pi \end{cases}.$$

No solution for k 2 Z_N such that $e^{iA[k,A]} = 1$ when $\theta = \pi$ and N 2 2N.

Exact-WKB analysis

Exact-WKB analysis

Consider the Schrödinger equation given by $(\sim, x\ 2\ \mathrm{C}, E\ 2\ \mathrm{R}_+)$
(See e.g.**[T.Kawai et al. AMS, c2005]** in technical details.)

$$\left[\quad \sim^2 \frac{d^2}{dx^2} + Q(x)\right]\psi(x) = 0, \quad Q(x) = 2(V(x) \quad E),$$

Put ansatz for a formal solution

$$\psi(x,\sim) = e^{\int^x S(x,\sim)dx},$$
$$S(x,\sim) = \sim\ \ ^1 S\ \ _1(x) + S_0(x) + \sim S_1(x) + \sim^2 S_2(x) +$$

where $S(x,\sim)$ satisfies the nonlinear Riccati equation.

$$S(x,\sim)^2 + \frac{\partial S(x,\sim)}{\partial x} = \sim\ \ ^2 Q(x), \qquad \boxed{S\ \ _1(x) = \quad \sqrt{Q(x)}}.$$

Exact-WKB analysis

Since the Sch eq is the 2nd order diff eq, there exists two independent solutions.

$$S_{\pm}(x,\hbar) = \pm S_{\rm odd}(x,\hbar) + S_{\rm even}(x,\hbar),$$
$$S_{\rm odd}(x,\hbar) = \sum_{n=0}^{\infty} S_{2n-1}(x)\hbar^{2n-1}, \quad S_{\rm even}(x,\hbar) = \sum_{n=0}^{\infty} S_{2n}(x)\hbar^{2n}.$$

By the Riccati eq, one finds

$$S_{\rm even}(x,\hbar) = -\frac{1}{2}\frac{\partial \log S_{\rm odd}(x,\hbar)}{\partial x},$$

hence, the formal solution can be expressed only by $S_{\rm odd}(x,\hbar)$.

$$\psi_a^{\pm}(x,\hbar) = \frac{e^{\pm\int_a^x S_{\rm odd}(x,\hbar)dx}}{\sqrt{S_{\rm odd}(x,\hbar)}} = e^{\pm\frac{\xi_0(x)}{\hbar}}\sum_{n=0}^{\infty}\psi_{a,n}^{\pm}(x)\hbar^{n+1/2}.$$

Exact-WKB analysis

Let us look at the Borel resummation of the wavefunction.

$$S_\theta[\psi_a(x)](\sim) = \int_{\xi_0(x)}^{1\, e^{i\theta}} e^{\ \frac{\xi}{\sim}} \mathrm{B}[\psi_a(x)](\xi) d\xi, \quad \theta = \arg(\sim),$$

$$\mathrm{B}[\psi_a(x)](\xi) = \sum_{n=0}^{1} \frac{\psi_{a,n}(x)}{\Gamma\left(n+\frac{1}{2}\right)}(\xi \quad \xi_0(x))^{n \ \ \frac{1}{2}}, \quad \xi_0(x) = \int_a^x dx\, S_{\mathrm{odd},\ \ 1}(x).$$

The Borel summablity is determined from

$$\frac{\xi_0(x)}{\sim} = \frac{1}{\sim}\int_a^x dx\, S_{\mathrm{odd},\ \ 1}(x) = \frac{1}{\sim}\int_a^x dx\sqrt{Q(x)}.$$

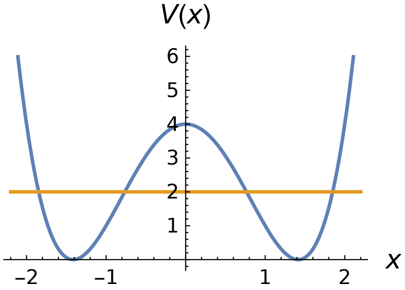
The Stokes phenomenon (in other words **Borel nonsummable**) happens when

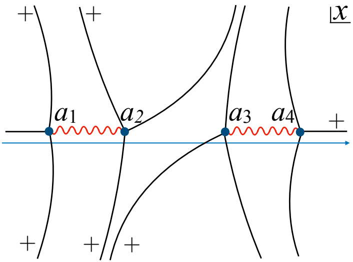
$$\mathrm{Im}\frac{\xi_0(x)}{\sim} = \quad \mathrm{Im}\frac{\xi_0(x)}{\sim}.$$

The Stokes graph

Since the Borel summability is relevant to $\frac{1}{\hbar}\int S_{\mathrm{odd},\ 1} : \mathbb{C} \to \mathbb{C}$, it is natural to see the Riemann surface defined by $\frac{1}{\hbar}\int S_{\mathrm{odd},\ 1}$, so called **the Stoke graph**.

Example: Double-well potential

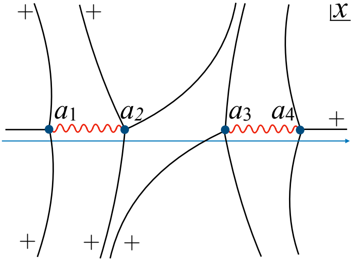




Since the Borel summability is relevant to $\frac{1}{\sim}\int S_{\mathrm{odd},\ \ 1}$: C ! C, it is natural to see the Riemann surface defined by $\frac{1}{\sim}\int S_{\mathrm{odd},\ \ 1}$, so called **the Stoke graph**.

Constitutive ingredients:

- Turning point $(a_1, a_2, \quad)$
  Def: $Q(x) = 0$ $(V(x) \quad E = 0)$
- Stokes line (black line)
  Def: $\mathrm{Im}\frac{1}{\sim}\int S_{\mathrm{odd},\ \ 1} = 0$
  labels $\int S_{\mathrm{odd},\ \ 1}$! 1
- Branch cut (red wave)
  $+S_{\mathrm{odd}}(x,\sim)$ \$ $S_{\mathrm{odd}}(x,\sim)$

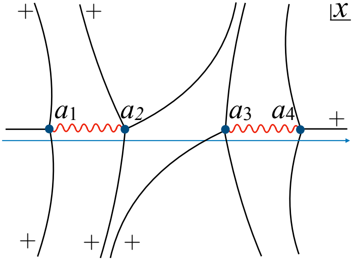


Example of the Stokes graph.
Double-well with arg(~) > 0.

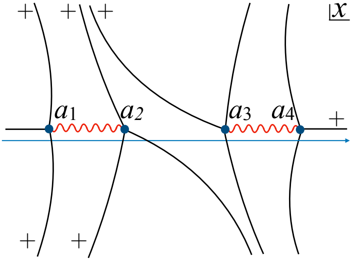
The Stokes graph

For getting the Q.C, we consider the analytic continuation of $\psi(x)$ for a given Stokes graph and a B.C. To do it, we have to know the effect of crossing Stokes line for $S[\psi](x)$.

$$S[\psi^{\mathrm{I}}](x) = \hat{M} S[\psi^{\mathrm{II}}](x).$$



arg(~) > 0



arg(~) < 0

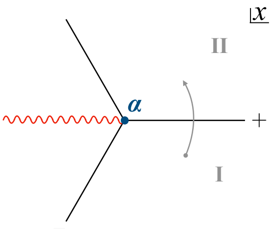
Connection formula for the Airy-type

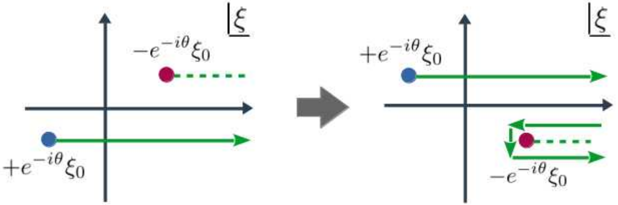
Consider crossing the Stoke line from I to II.

$$\psi = (\psi_+, \psi\ \ )^{>},$$

$$\psi_a^{\mathrm{I}} = M_+ \psi_a^{\mathrm{II}}, \qquad \psi_a^{\mathrm{I}} = M\ \ \psi_a^{\mathrm{II}},$$

$$M_+ = \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix}, \quad M\ \ = \begin{pmatrix} 1 & 0 \\ +i & 1 \end{pmatrix}.$$

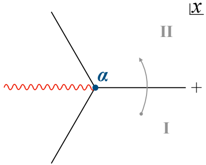




Connection formula for the Airy-type

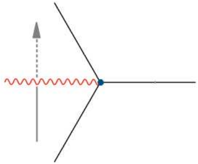
Connection matrix

$$M_{+} = \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix}, \quad M \;\; = \begin{pmatrix} 1 & 0 \\ +i & 1 \end{pmatrix},$$



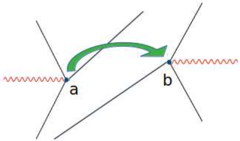
Branchcut matrix

$$T = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$



Normalization matrix (Voros multiplier)

$$N_{ba} = \begin{pmatrix} e^{+\int_a^b dx\, S_{\mathrm{odd}}(x,\sim)} & 0 \\ 0 & e^{\;\int_a^b dx\, S_{\mathrm{odd}}(x,\sim)} \end{pmatrix}.$$



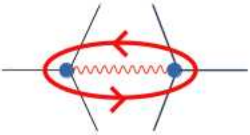
Cycle expression (Voros multiplier)

For the WKB analysis, it is convenient to introduce **cycle** expression, which is known as the Voros multipliers.

- A**-cycle (PT)**

$$A(\sim) := e^{\oint_A dx\, S_{\rm odd}(x,\sim)},$$

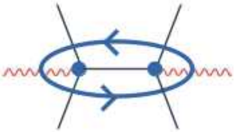
$$\oint_A dx \sqrt{2(V(x) \quad E)}\ 2\ i\mathbb{R}.$$



- B**-cycle (NPT)**

$$B(\sim) := e^{\oint_B dx\, S_{\rm odd}(x,\sim)},$$

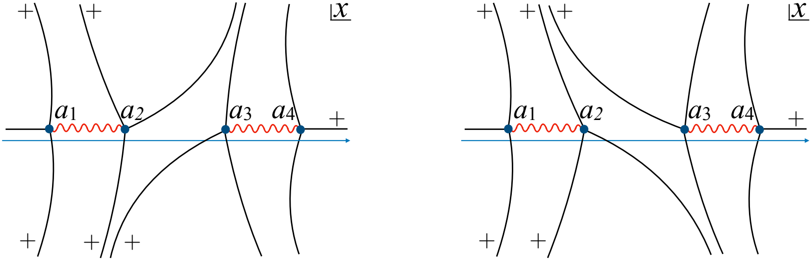
$$\oint_B dx \sqrt{2(V(x) \quad E)}\ 2\ \mathbb{R}.$$



Note: We would take the orientation such that $B(\sim)\ /\ e^{\ \frac{S_{\rm b}}{\sim}}$.

The Stokes graph

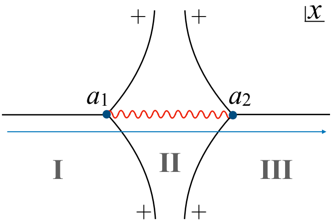
The Stokes graph generally depends on arg(~). In the below example, arg(~) = 0 gives a **Borel nonsummable** wavefunction. But it can be resolved, i.e. **Borel summable**, when 0 < jarg(~)j   1 (except exactly on the Stokes line).



arg(~) > 0   arg(~) < 0

Example 1: Harmonic oscillator $V(x)=\frac{1}{2}x^2$

$$
\begin{aligned}
\mathrm{I}\to\mathrm{II} &: \quad \psi^{\rm I}_{a_1} = M_+ \psi^{\rm II}_{a_1}\\
(1\to 2) &: \quad \psi^{\rm I}_{a_1} = N_{a_1,a_2} \psi^{\rm II}_{a_2}\\
\mathrm{II}\to\mathrm{III} &: \quad \psi^{\rm II}_{a_2} = M_+ \psi^{\rm III}_{a_2}\\
(2\to 1) &: \quad \psi^{\rm III}_{a_2} = N_{a_2,a_1} \psi^{\rm III}_{a_1}\\
&\Rightarrow \quad \psi^{\rm I}_{a_1} = \begin{pmatrix} 1 & i(1+A) \\ 0 & 1 \end{pmatrix} \psi^{\rm III}_{a_1}.
\end{aligned}
$$



Boundary condition for $\psi_a = (\psi_{+,a}, \psi_{-,a})^\top$:

$$\psi^{\rm I}_{a_1}(x,\hbar) \to 0 \quad \text{as} \quad x \to -\infty$$

$$\Rightarrow \quad \psi^{\rm I}_{-,a_1}(x,\hbar) = 0 \quad \text{and} \quad \boxed{D(\hbar) := (1 + A(\hbar)) = 0}$$

where $A(\hbar) := e^{2\int_{a_1}^{a_2} dx\, S_{\rm odd}(x,\hbar)} = e^{\oint_A dx\, S_{\rm odd}(x,\hbar)}$.

Example 1: Harmonic oscillator $V(x) = \frac{1}{2}x^2$

Since $Q(x) = x^2 - 2E$ with $E > 0$, the turning points are given by $a_1 = -\sqrt{2E}$, $a_2 = +\sqrt{2E}$. Hence,

$$2\int_{a_1}^{a_2} dx\, S_{\rm odd}(x,\hbar) = -\frac{2\pi i E}{\hbar}.$$

From the quantization condition, i.e. $D = (1 + A) = 0$,

$$1 + e^{-\frac{2\pi i E}{\hbar}} = 0 \quad \Rightarrow \quad E = \left(\frac{1}{2} + n\right)\hbar, \quad n \in \mathbb{Z}$$

From the positive energy condition,

$$\boxed{E = \left(\frac{1}{2} + n\right)\hbar, \quad n \in \mathbb{N}_0}$$

Application: the cosine model

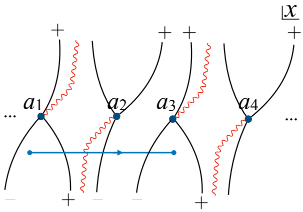
The cosine model

Let us consider the cosine model and the path from $x = 0$ to $x = 2\pi$ with $1 \quad \mathrm{Im}\, x < 0$ to obtain the Q.C.

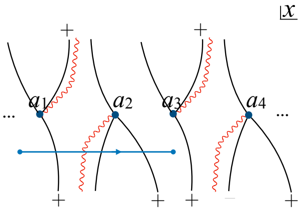
$$Q(x) = 2(V(x) \quad E), \qquad V(x) = 1 \quad \cos(Nx), \qquad (N \; 2 \; \mathrm{N})$$

$$\psi_{a_1}(x) = \hat{M}\psi_{a_1+2\pi \quad a_1}$$

$$\hat{M} = \begin{cases} [M_+ TN_{a_1a_2} M \quad N_{a_2a_3} M \quad]^N =: M^+ & \text{for} \quad \arg(\sim) > 0 \\ [M_+ TN_{a_1a_2} M \quad M_+ N_{a_2a_3}]^N =: M & \text{for} \quad \arg(\sim) < 0 \end{cases} .$$



$\arg(\sim) > 0$



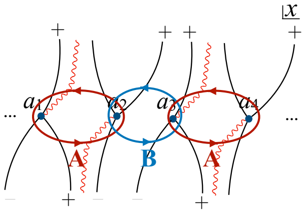
$\arg(\sim) < 0$

Set the twisted boundary condition, $\psi(x) = e^{i\theta}\psi(x + 2\pi)$. Thus,

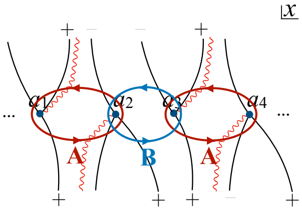
$$D \ := \det(M \quad \mathrm{I}e^{i\theta})/e^{i\theta}. \qquad (\theta\text{: boundary condition})$$

$$= \frac{1}{(A \ \ ^{1}B)^{N/2}} \prod_{p=0}^{N \quad 1} D_p \ = 0 \qquad \text{for} \ \ \mathrm{sign}(\mathrm{Im}(\ )) \quad 1$$

$$D_p \ := 1 + A \ \ ^{1}(1 + B) \quad 2 \ {}^{p}\overline{A \ \ ^{1}B}\cos\left(\frac{\theta + 2\pi p}{N}\right)$$



arg( ) > 0

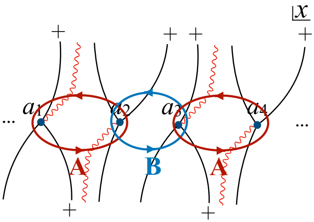


arg( ) < 0

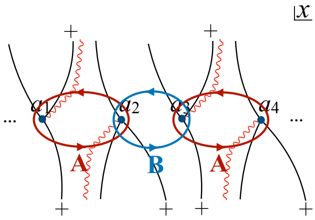
Delabaere-Dillinger-Pham (DDP) formula

Cycles have the resurgence relation called **the DDP formula**.

[E.Delabaere et al. '97, K.Iwaki et al. '14]



$\arg(\sim) > 0$



$\arg(\sim) < 0$

In the case of the cosine potential, it is given by

$$S_{+}[\sqrt{A}] = S\ [\sqrt{A}](1 + S[B]), \qquad S_{+}[B] = S\ [B] =: S[B].$$

(See DDP paper for details and generic cases.)

Resurgence for the quantization condition

Since $D\ \ (E)$ is a function of cycles, from the DDP formula given by

$$S_+[{}^{p}\overline{A}] = S\ \ [{}^{p}\overline{A}](1 + S[B]), \qquad S_+[B] = S\ \ [B] =: S[B],$$

one finds that

$$\boxed{S_+[(A\ \ {}^{1}B)\ \ {}^{1/2}D_p^+(A,B)] = S\ \ [(A^{+1}B)\ \ {}^{1/2}D_p\ \ (A,B)]}$$

Since $D\ \ = (A\ \ {}^{1}B)\ \ {}^{N/2}\prod_{p=0}^{N\ \ 1} D_p\ \ (E) = 0$, the energy spectrum is given by each of p-sectors. This fact means that the resurgence structure is closed on the fixed-p sector. Trivially,

$$\boxed{S_+[D^+(A,B)] = S\ \ [D\ \ (A,B)]}$$

Degeneracy of energy

When $N = 2K$ with $K \in \mathbb{N}$ and $\theta = \pi$ (APBC), one can immediately see the degeneracy of energy spectrum:

$$D_p = 1 + A^{-1}(1+B) - 2\sqrt{A^{-1}B}\cos\left(\frac{\theta + 2\pi p}{N}\right)$$

$$\Rightarrow \quad D_p = 1 + A^{-1}(1+B) - 2\sqrt{A^{-1}B}\cos\left[\frac{\pi(p+1/2)}{K}\right], \quad (p \in \mathbb{Z}_{2K})$$

Hence, one finds

$$\boxed{D_p = D_{2K-p-1}}$$

This degeneracy is a sign of an 't Hooft anomaly.

Gutzwiller trace formula

We derive the GTF through $G \quad = \quad \partial_E \log D \quad .$ ($N = 1$ for simplisity.)

$$G \quad (E) = G_{\rm pt}(E) + G_{\rm np}(E),$$

$$G_{\rm pt}(E) := \quad \partial_E A \quad ^1 \sum_{n=0}^{1} (\quad 1)^n A \quad ^n, \quad G_{\rm np}(E) := \quad \partial_E K \quad \sum_{n=0}^{1} (\quad 1)^n (K \quad)^n,$$

$$K \quad := B \sum_{n=0}^{1} (\quad 1)^n A \quad ^n \quad 2 \sqrt[p]{A \quad ^1 B} \sum_{n=0}^{1} (\quad 1)^n A \quad ^n \cos\theta.$$

We define "period" $T_{A,B}$ as

$$\partial_E A = \oint_A dx \left(\frac{1}{\sim\sqrt{2V(x) \quad E}} + O(\sim) \right) A =: \quad \frac{i}{\sim} T_A A,$$

$$\partial_E B = \oint_B dx \left(\frac{1}{\sim\sqrt{2V(x) \quad E}} + O(\sim) \right) B =: \quad \frac{i}{\sim} T_B B,$$

Notice that $\lim_{\sim\,!\,0} T_A$ 2 R and $\lim_{\sim\,!\,0} T_B$ 2 iR.

Gutzwiller trace formula

Maslov index

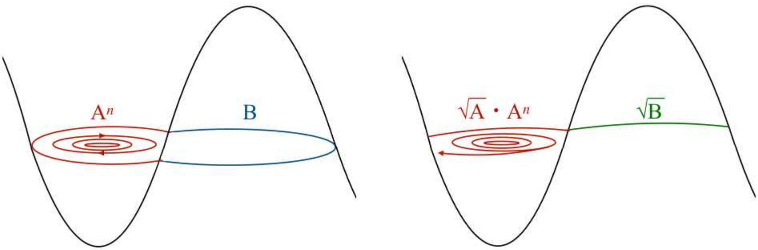
$$G_{\rm pt}(E) := \partial_E A^{1} \sum_{n=0}^{1} (\ 1)^n A^{\ n}, \quad G_{\rm np}(E) := \partial_E K \sum_{n=0}^{1} (\ 1)^n (K\ )^n,$$

$$K \ := B \sum_{n=0}^{1} (\ 1)^n A^{\ n} \quad 2\sqrt{A^{\ 1} B} \sum_{n=0}^{1} (\ 1)^n A^{\ n} \cos\theta.$$

- $G_{\rm pt}$ and $G_{\rm np}$ constitute of the periodic orbits, $A^{\ 1}$ and K , respectively. These give (1), which is the Maslov index.
- K is has fundamental nonpertubative orbits, $BA^{\ n}$ and $\sqrt{A^{\ 1}BA^{\ n}}$. $A^{\ 1}$ and B also give (1) there.
- The exact form of the GTF for the cosine model could be obtained.

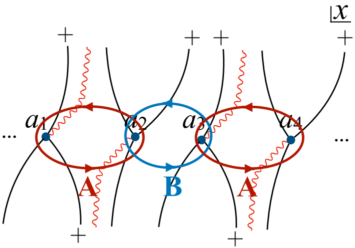
Gutzwiller trace formula

$$K \quad := B\sum_{n=0}^{1}(\quad 1)^n A \quad ^n \quad 2^{p}\overline{A \quad ^1B}\sum_{n=0}^{1}(\quad 1)^n A \quad ^n \cos\theta.$$

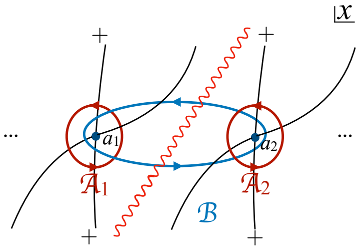


We consider the ground state energy and its nonpertubative correction. When considering the energy spectrum by solving the Q.C. and it gives $E(\hbar) = O(\hbar)$, it is useful to employ **the degenerate Weber (DW)-type** Stokes graph.

Two Airy-type ($E_0 > 0$) ! One DW-type ($E_0 = 0$)



$E_0 > 0$



$E_0 = 0$

Relation of cycles between Airy- and DW-types

- When considering the Q.C. based on the DW-type graph, one has to compute the DW connection formula.
- In principle, the result by the DW-type graph is available by replacing $E \ ! \ E\sim$ in the results by the Airy type. But, it is difficult in the generic cases.
- Instead the reduction from the Airy-type graph, one can make a **dictionary** between the cycles of Airy-type (A, B) and the ones of the DW-type (A, B).
- The cycles expression based on Voros multipliers is kept. (But, the functional forms are different from each others.)

$$\text{e.g.} \quad D(A, B) \quad ! \quad D(\mathsf{A}, \mathsf{B}).$$

Quantization condition by degenerate Weber-type

We replace (A, B) with (A, B) given by

$$A \;!\; \mathsf{A} \quad e^{\frac{2\pi i E}{N}}, \qquad B \;!\; \mathsf{B} \quad \frac{2\pi \mathsf{B}_0}{\Gamma(\frac{1}{2} + \frac{E}{N})^2} \left(\frac{N\tilde{}}{32}\right)^{\frac{2E}{N}}, \qquad \mathsf{B}_0 = e^{\frac{16}{N\tilde{}}}.$$

(Here, $E(\tilde{})$ $\quad E\tilde{}$) Thus, D $\quad$ becomes

$$D \qquad \prod_{p=0}^{N \quad 1} \Bigg[p\frac{1}{\overline{\mathsf{B}_0}\Gamma(\frac{1}{2} \quad \frac{E}{N})} \left(\frac{N\tilde{}}{32}\right)^{E/N} + \frac{p\overline{\mathsf{B}_0 e^{\quad \pi i E/N}}}{\Gamma(\frac{1}{2} + \frac{E}{N})} \left(\frac{N\tilde{}}{32}\right)^{\quad E/N} \quad \sqrt{\frac{2}{\pi}} \cos\left(\frac{\theta + 2\pi p}{N}\right) \Bigg] = 0.$$

Nonperturbative contribution to the energy

The ground state energy is obtained by the 1st. term of Q.C., $E/N = 1/2$. In order to consider the NPT contribution to the ground state energy, we substitute $E/N = 1/2 + \delta$ into Q.C. where $\delta = O(\sqrt{\mathrm{B}_0})$. The solution is given by

$$
\begin{aligned}
\delta_p &= \sqrt{\frac{64\,\mathrm{B}_0}{N\pi\hbar}} \cos\left(\frac{\theta + 2\pi p}{N}\right) \\
&\quad + \frac{64\,\mathrm{B}_0}{N\pi\hbar} \left[\cos^2\left(\frac{\theta + 2\pi p}{N}\right) \left(\gamma - \log\frac{N\hbar}{32}\right) - \frac{\pi i}{2}\right] + O(\mathrm{B}_0^{3/2}).
\end{aligned}
$$

Nonperturbative contribution to the energy

For $N = 1$ and 2,

$$\delta \;=\; \sqrt{\frac{64\,\mathrm{B}_0}{\pi\sim}}\cos\theta + \frac{64\,\mathrm{B}_0}{\pi\sim}\left[\cos^2\theta\left(\gamma \quad \log\frac{\sim}{32}\right) \quad \frac{\pi i}{2}\right] + O(\mathrm{B}_0^{3/2}),$$

$$\delta_p \;=\; (\quad 1)^p\sqrt{\frac{32\,\mathrm{B}_0}{\pi\sim}}\cos\frac{\theta}{2} + \frac{32\,\mathrm{B}_0}{\pi\sim}\left[\cos^2\frac{\theta}{2}\left(\gamma \quad \log\frac{\sim}{16}\right) \quad \frac{\pi i}{2}\right] + O(\mathrm{B}_0^{3/2}).$$

- $N = 1$
  $O(\mathrm{B}_0^{1/2})$ ... (Anti-)instanton $[I]$ $[\bar{I}]$
  $O(\mathrm{B}_0)$ ... (Anti-)instantons pair $[I\,I]$ $[\bar{I}\,\bar{I}]$
  Bion $[I\,\bar{I}]$.
- $N = 2$ (If $\theta = \pi$, instanton contributions disappear, and $\delta_0 \; = \delta_1$.)
  $O(\mathrm{B}_0^{1/2})$ ... (Anti-)instanton $[I_1][I_2][\bar{I}_1][\bar{I}_2]$
  $O(\mathrm{B}_0)$ ... (Anti-)instantons pair $[I_1 I_2][\bar{I}_1\bar{I}_2]$
  Bion $[I_1\bar{I}_1]$ $[I_2\bar{I}_2]$.

Partition function

We obtain the partition function via the resolvent method.

$$
\begin{aligned}
Z\ \ (\beta) \quad &= \quad \frac{1}{2\pi i}\int_{\epsilon+i\,1}^{\epsilon+i\,1} \frac{\partial \log D}{\partial E}\, e\ \ ^{\beta E} dE \\
&= \quad \frac{\beta}{2\pi i}\int_{\epsilon+i\,1}^{\epsilon+i\,1} \log D\ \ \ e\ \ ^{\beta E} dE.
\end{aligned}
$$

Since the DDP formula gives $S_{+}[D^{+}] = S\ \ [D\ \ ]$ and S is homomorphism for summation and multiplication, one can easily find

$$
\boxed{S_{+}[Z^{+}(\beta)] = S\ \ [Z\ \ (\beta)]}
$$

Since the path-integral is now expressed by cycles, the resurgenct relation for each sector can be traced.

Partition function

In order to see the details, we factorize the Q.C. as

$$D \quad / \quad \prod_{p=0}^{N\ \ 1}\left[1 + A\ \ ^{1}(1+B\ ) \quad 2\sqrt{A\ \ ^{1}B}\cos\left(\frac{\theta+2\pi p}{N}\right)\right]$$

$$= \quad (D_A\ )^N \prod_{p=0}^{N\ \ 1}\left[1+\frac{A\ \ ^{1}B}{D_A} \quad 2\frac{\sqrt{A\ \ ^{1}B}}{D_A}\cos\left(\frac{\theta+2\pi p}{N}\right)\right]$$

$$(D_A\ := 1 + A\ \ ^{1})$$

Each of the factors give

- $(D_A\ )^N$ ... (N copies of) pertubative sector,
- $[\quad]$ ... (p-th) nonpertubative sector,

by expanding $\log D_A$ and $\log[\quad]$ around 1.

Partition function

The partition function is given by

$$\begin{aligned} Z &= N Z_{\rm pt} + \sum_{p=0}^{N-1} \sum_{\substack{(Q_p,K_p)\in \mathbb{Z}\times\mathbb{N}_0 \\ |Q_p|+K_p>0}} Z_{\rm np}(p,Q_p,K_p) \\ &= N Z_{\rm pt} + N \sum_{\substack{(Q,K)\in \mathbb{Z}\times\mathbb{N}_0 \\ |Q|+K>0}} Z_{\rm np}(0,NQ,K), \end{aligned}$$

where

$$Z_{\rm pt} := \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \sum_{n=1}^{\infty} \frac{(-\mathcal{A}^{-1})^n}{n} e^{-\beta E} dE,$$

$$Z_{\rm np}(p,Q_p,K_p) := \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{2\pi i p Q_p/N}}{|Q_p|+K_p} \binom{|Q_p|+K_p}{K_p} \left(\frac{\mathcal{B}}{\mathcal{K}^2}\right)^{|Q_p|/2+K_p}$$
$$\times {}_2F_1(1-K_p, -K_p; |Q_p|+1; -\mathcal{A}^{-1})(-\mathcal{A}^{-K_p}) e^{-\beta E + iQ_p\theta/N} dE,$$

$$(\mathcal{K} := \sqrt{\mathcal{A}} + \sqrt{\mathcal{A}^{-1}})$$

Hilbert space , Partition function

Let us look at the resurgence structure of the Hilbert space and the partition function. We express the Q.C. using α and β :

$$D \;\; = \prod_{p=0}^{N \;\; 1} \alpha \;\; \left(1 \quad \beta \;\; e^{i(\theta+2\pi p)/N}\right)\left(1 \quad \beta \;\; e^{\;\; i(\theta+2\pi p)/N}\right) \;/\; \prod_{p=0}^{N \;\; 1} D_p \; ,$$

$$\alpha \;\; := \xi \;\; + \sqrt{(\xi \;\;)^2 \quad 1}, \quad \beta \;\; := \xi \qquad \sqrt{(\xi \;\;)^2 \quad 1}, \quad \xi \;\; := \frac{1 + \mathrm{A} \;\; {}^1 + \mathrm{B}}{p \, \overline{\mathrm{A} \;\; {}^1\mathrm{B}}},$$

$$S_+[\xi^+] = S \;\; [\xi \;\;] \qquad) \qquad S_+[\alpha^+] = S \;\; [\alpha \;\;] \quad \text{and} \quad S_+[\beta^+] = S \;\; [\beta \;\;].$$

E_p is given by $D_p \;\; = 0$, thus the resurgent relation of corresponding Hilbert space H_p is closed.

The partition function can be also expressed by α and β :

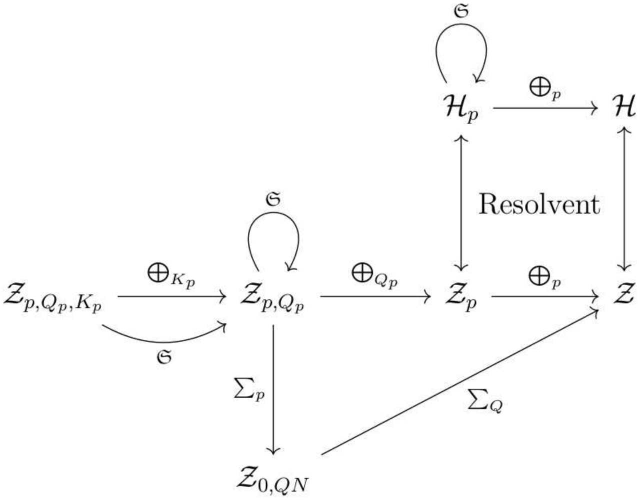
$$Z = \frac{\beta}{2\pi i}\int_{\epsilon \quad i 1}^{\epsilon+i 1} e^{\quad \beta E} dE$$

$$\sum_{p=0}^{N \quad 1}\left[\log\left(\sqrt[p]{A^{\quad 1}B\alpha}\quad\right) + \sum_{Q_p 2 Z n f 0 g} \frac{(\beta \quad)^{jQ_pj}}{jQ_pj} e^{i(\theta+2\pi i)Q_p/N}\right].$$

We can see that the resurgent structure is closed in [ ]. Not only that, each of the Q_p-sectors is also closed.
(The topological charge Q_p arises as powers by Taylor expansion.)

Thus, $\sum_{K_p=0}^{1} Z \ (p, Q_p, K_p)$ is irreducible for the Stokes automorphism S, and the resrugent structure is labeled by (p, Q_p).

Hilbert space , Partition function



Summary

- We considered QM of a particle on S^1 in the presence of a periodic potential with N-minima (N 2 N) by the exact-WKB method.
- We derived the energy spectrum, the Gutzwiller trace formula, and the partition function from the quantization condition.
- All orders perturbative/non-perturbative resurgent relation is shown.
- Our result obtained by the DW-type correctly reproduces the energy eigenvalues conjectured by Zinn-Justin**[J.Zinn-Justin et al. 04]**, and obtained earlier by using uniform WKB method for $N = 1$ **[G.Dunne et al. 14]**.
- The exact quantization condition naturally captures the mixed 't Hooft anomaly or global inconsistency.