# Exact-WKB, complete resurgent structure, and mixed anomaly in quantum mechanics on $S^1$

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Collaboration with
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based on arXiv:2103.06586 [quant-th]
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# Introduction

# Path-integral

In this talk, we would begin with the asymptotic form of a path-integral in 1D (Euclidean) QM:

$$\begin{aligned} Z(\hbar) &= \int \mathcal{D}x \, e^{-S(x,\hbar)} \\ &= \sum_{j=0}^{\infty} a_n \hbar^j + \sum_{n,k=1}^{\infty} \sum_{j=0}^{\infty} b_{n,k,j} e^{-\frac{nS_B}{\hbar}} \hbar^j (\log \hbar)^k. \end{aligned}$$

The path-integral can be expressed by **transseries** generated by **transmonomials**, $(\hbar, e^{-\frac{S_B}{\hbar}}, \log \hbar)$.

- $\hbar$ ... PT fluctuationg
- $e^{-\frac{S_B}{\hbar}}$ ... Instanton (Bion) energy
- $\log \hbar$ ... Quasi-zero modes

# Path-integral

**Path-integral for 1D QM**

$$Z(\hbar) = \int \mathcal{D}x \, e^{-S(x,\hbar)}$$

- Perturbative expansion around a vacuum.

$$Z_{\rm p}(\hbar) \sim a_0 + a_1\hbar + a_2\hbar^2 + \cdots .$$

- The PT expansion is a divergent series in general.

$$r_c := \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = 0.$$

- What does the PT expansion mean when $r_c = 0$?

# Path-integral

**Why a divergent series appears?**

- This implies that there exists NPT sectors (bions).

  Typical examples: $V(x) = (x^2 - 1)^2, \ V(x) = 1 - \cos x.$

- Nonperturbarive information is available from the perturbative series via the Borel resummation (Borel transform + Laplace integral)

- PT sector $\Leftrightarrow$ NPT sectors : **Resurgence relation**
  **[J.Ecalle '81, A.Voros '81, D.Sauzin '14]**

# Borel resummation

## Schematic figure of Borel resummation

$$\mathcal{B}[\hbar^k] := \frac{\xi^{k-1}}{\Gamma(k)}$$

**Asymptotic form**

$$\tilde{Z}(\hbar) \sim \sum_{k=0}^{\infty} a_k \hbar^k \ \in \mathbb{R}[[\hbar]]$$

**Borel transform**

$$\mathcal{B}[\tilde{Z}](\xi) = \sum_{k=0}^{\infty} b_k \xi^k \ \in \mathbb{R}[[\xi]]$$

$$\mathcal{L}[F](\hbar) := \int_0^{\infty} F(\xi) e^{-\xi/\hbar} \, d\xi$$

Asymptotic limit
$0 < \hbar \ll 1$

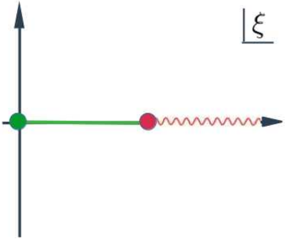
**Borel resummation**

$$\hat{Z}(\hbar) = \mathcal{L} \circ \mathcal{B}[\tilde{Z}](\hbar)$$

# Borel summability

When acting the Laplace integration to $\mathcal{B}[\tilde{Z}](\xi)$, ($\mathcal{S} := \mathcal{L} \circ \mathcal{B}$)

- $\tilde{Z}(\hbar)$ is **Borel summable** if it is integrable.
- $\tilde{Z}(\hbar)$ is **Borel nonsummable** if it is not integrable due to a pole (branchcut).




**Example:**

$$\tilde{Z}(\hbar) \sim n! A^n \hbar^{n+1} \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \mathcal{B}[\tilde{Z}](\xi) = \sum_{n=0}^{\infty} (A\xi)^n = \frac{1}{1 - A\xi}$$

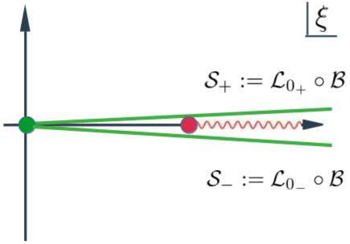
$$\Rightarrow \quad \mathcal{S}[\tilde{Z}](\hbar) = \int_0^{\infty} d\xi \, \frac{e^{-\xi/\hbar}}{1 - A\xi}$$

Nonsummable if $A \in \mathbb{R}_+$.

# Borel summability

To avoid from the singularity, we introduce the small complex phase to $\hbar$. However, the resulting function becomes complex and depends on the integration ray, $\mathcal{S}_+$ or $\mathcal{S}_-$ (**imaginary ambiguity**). By taking the Hankel contour, the NPT contribution is available from the PT sector (**Resurgence**):

$$(\mathcal{S}_+ - \mathcal{S}_-)[\tilde{Z}](\hbar) \propto i e^{-\frac{S_b}{\hbar}}(1 + O(\hbar))$$




- The singularity corresponds to the bion ($\mathcal{I}\bar{\mathcal{I}}$) energy.
- $n$-th sector $\rightarrow$ $(n+k)$-sectors ($k \in \mathbb{N}$)
- People expect the ambiguity should be cancelled by the NPT sectors in some way.

# Questions and Problems

**General questions and problems...**

1. How to obtain the resurgence including all NP sectors?
2. How to get the mechanism of the imaginary ambiguity cancellation in full sectors?
3. To do it in the path-integral, all of coefficients are needed (but it is extremely difficult).

# Schrödinger equation

Instead of beginning with the path-integral, we also have the Schöredinger equation.

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)\right]\psi(x,\hbar)=E\psi(x,\hbar).$$

- By putting an ansatz of asymptotic form for $\psi(x,\hbar)$, its coefficients are easily calculable by the Schöredinger Eq.
- In general, a resurgence mechanism can be argued based on the structure of a given differential equation. [e.g. (non)linear, (non)autonomous, etc...]
- One has to consider Schöredinger eq $\Rightarrow$ path-integral. It is possible through **the resolvent method**.

# Resolvent method

- By using the Laplace transform, one can obtain the resolvent $G(E)$ which is a function of $E$ from $Z(\beta)$, as

$$G(E) = \int_0^\infty Z(\beta) e^{\beta E} d\beta, \qquad Z(\beta) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} G(E) e^{-\beta E} dE.$$

- The resolvent $G(E)$ can be written by $D(E)$ called the Fredholm determinant,

$$G(E) = \mathrm{tr} \frac{1}{\hat{H} - E} = -\frac{\partial \log D_{\mathrm{FD}}(E)}{\partial E}, \qquad D_{\mathrm{FD}}(E) := \det \left( \hat{H} - E \right)$$

- $D_{\mathrm{FD}}(E) = 0$ gives **the spectral form**. Indeed, from the argument principle,

$$Z(\beta) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{D'_{\mathrm{FD}}(E)}{D_{\mathrm{FD}}(E)} e^{-\beta E} dE = \sum_{k=1}^{\infty} n_k e^{-\beta E_k} = \mathrm{tr} \left[ e^{-\beta \hat{H}} \right],$$

where $D_{\mathrm{FD}}(E_k) = 0$ and $n_k$ is the number of zero of $D_{\mathrm{FD}}(E_k)$.

# Gutzwiller trace formula (GTF)

**Gutzwiller trace formula**

– semiclassical construction of the resolvent $G(E)$.

Normally, it is defined for "Lorentzian" partition function.

$$Z(T) = \mathrm{tr} e^{-i\hat{H}T} = \int_{\text{periodic}} \mathcal{D}x\, e^{iS}$$

$$\Rightarrow \qquad G(E) = -i\mathrm{tr}\frac{1}{\hat{H}-E} = \int_0^\infty dT \int_{\text{periodic}} \mathcal{D}x\, e^{\Gamma(\hbar)},$$

where $\Gamma = S + ET = n\oint pdx - \int^T Hdt + ET$.

We evaluate it by the stationary phase approximation.

# Gutzwiller trace formula (GTF)

By taking up to the sub-leading contribution, $G(E)$ can be expressed by

$$G(E) = i \sum_{\mathrm{p.p.o}} \sum_{n=1}^{\infty} T(E) e^{ni \oint_{\mathrm{p.p.o.}} pdx} (-1)^{n} \left|\det(\mathrm{Hess}(S))\right|^{-1/2},$$

where p.p.o. denotes a prime periodic orbit, $T(E)$ is the period of each cycle whose energy is $E$, and $(-1)^n = e^{\frac{\pi i}{4}\mathrm{sgn}(\mathrm{Hess}(S))}$ is **the Maslov index**.
(See Gutzwiller's book for the derivation
[M. Gutzwiller, Springer-Verlag New York '90])

In general, it is a tough problem to determine all p.p.o.

# Strategy

We start with the Schrödinger equation and obtain the **quantization condition** by **the exact-WKB analysis**.

$$D_{\mathrm{FH}}(E) = 0 \quad \to \quad D_{\mathrm{WKB}}(E) = 0$$

- $D_{\mathrm{WKB}}(E)$ keeps all informations such as transseries, ambiguity cancellation, and resurgence structure for $E$. This gives **the genralized B-S quantization**.
- $D_{\mathrm{WKB}}(E)$ can be expressed by a kind of periodic orbits. This gives **the Gutzwiller trace formula** through the resolvent $G(E)$.
- As we saw, $D_{\mathrm{WKB}}(E)$ gives **the spectral form**. Furthermore, by taking integral by parts for $\partial_E \log D_{\mathrm{WKB}}(E)$, it gives **the path-integral**.

# Strategy

We would take the following steps:

1. Obtain **the quantization condition** by using **the exact-WKB analysis**.
2. Then, consider **the resurgence relation** for the quantization condition.
3. Derive expressions such as GTF and path-integral from the quantization condition through **the resolvent method**.

# Outline

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$

Schrödinger eq.

↓ exact-WKB

$$D(E) = 0$$

exact quantization condition

→ $$\frac{1}{2\pi i}\oint p dx \simeq \left(N + \frac{1}{2}\right) + O\left(e^{-\frac{S}{\hbar}}\right)$$

generalized Bohr-Sommerfeld

↓

$$G(E) = -\partial_E \log D(E)$$

→ expand → $$G(E) \simeq \sum_n \sum_{p.p.o.} (-1)^n e^{in \oint pdx}$$

Gutzwiller trace formula

↓ inverse Laplace transform

$$Z(\beta) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} G(E) e^{-\beta E} dE$$

→ calculate the residues → $$Z(\beta) = \sum_n e^{-\beta E_n}$$

spectral summation form

↓ integral by parts and expand $\log D(E)$

$$Z(\beta) = \sum_n a_n \hbar^n + e^{-\frac{S_1}{\hbar}} \sum_n b_n \hbar^n + e^{-\frac{S_2}{\hbar}} \sum_n c_n \hbar^n + \dots$$

path-integral(trans-series) form

# Anomaly and the cosine model

**(Mixed) 't Hooft anomaly**
**[G.'t Hooft '80, D.Gaiotto et al. '17, Y.Kikuchi et al. '17, etc.]**

A obstruction to promoting the global symmetry to local gauge symmetry

- Global symmetry $G \Rightarrow$ Gauging (background gauge $A$)
- Take the $G$-gauge transform. We say that the theory has an 't Hooft anomaly if it gives

$$Z[A + d\lambda] = Z[A] \exp\left(i\mathcal{A}[\lambda, A]\right).$$

  The phase can not be canceled by a local counter term.
- If $G = G_1 \times G_2$, it is said to be a mixed 't Hooft anomaly.

# Anomaly and the cosine model

**Cosine model**

$$L = \frac{\dot{x}^2}{2} + V(x) - \frac{i\theta}{2\pi}\dot{x},$$
$$V(x) = 1 - \cos(Nx), \quad x \sim x + 2\pi, \quad N \in \mathbb{N}.$$

**Symmetry**

| | | |
|---|---|---|
| $\mathbb{Z}_N$ shift | $\mathrm{U} : x(t) \mapsto x(t) + \frac{2\pi}{N}$ | $(\mathrm{U}^N = 1)$ |
| Time reversal ($\mathbb{Z}_2$) | $\mathrm{T} : (x(t), \dot{x}(t)) \mapsto (x(-t), -\dot{x}(-t))$ | $(\theta = 0, \pi)$ |

**Hamiltonian**

$$\hat{H} = \frac{1}{2}\left(\hat{p} - \frac{\theta}{2\pi}\right)^2 + V(\hat{x}), \qquad [\hat{x}, \hat{p}] = 1.$$

# Anomaly and the cosine model

We consider when $\theta = 0$ or $\pi$.

$$\hat{H} = \frac{1}{2}\left(\hat{p} - \frac{\theta}{2\pi}\right)^2 + V(\hat{x}), \qquad [\hat{x}, \hat{p}] = 1.$$

$\mathsf{T}\hat{H}\mathsf{T}^{-1} = \hat{H}$ can be satisfied by

$$\mathsf{T}\hat{x}\mathsf{T}^{-1} = \hat{x}, \quad \mathsf{T}\hat{p}\mathsf{T}^{-1} = \begin{cases} -\hat{p} & \theta = 0 \\ -\hat{p} + 1 & \theta = \pi \end{cases}.$$

By using the coordinate basis, U and T can be expressed by

$$\mathsf{U} = \exp\left[\frac{2\pi i}{N}\partial_x\right], \quad \mathsf{T} = \begin{cases} \mathcal{K} & \theta = 0 \\ \exp[ix]\,\mathcal{K} & \theta = \pi \end{cases}, \quad (\mathcal{K}\text{: c.c. operator})$$

$$\Rightarrow \quad \mathsf{TUT}^{-1} = \begin{cases} \mathsf{U} & \theta = 0 \\ \exp\left[-\frac{2\pi i}{N}\right]\mathsf{U} & \theta = \pi \end{cases}.$$

# Anomaly and the cosine model

Suppose that $\theta = \pi$. We redefine U as

$$\mathrm{U}' := \exp\left[-\frac{2\pi i}{N}k\right]\mathrm{U}, \qquad (k \in \mathbb{Z}_N)$$
$$\Rightarrow \quad \mathrm{T}\mathrm{U}'\mathrm{T}^{-1} = \exp\left[-\frac{2\pi i}{N}(2k-1)\right]\mathrm{U}'.$$

U′ and T is commutative if there exists a solution satisfying $2k - 1 = 0 \pmod N$ for $k \in \mathbb{Z}_N$.

- If $N \in 2\mathbb{N}$, no solution exists ⇒ **Mixed 't Hooft anomaly**
  **All the energy spectra is two-fold degenerate.**
- If $N \in 2\mathbb{N}+1$, $k = \frac{N+1}{2}$ ⇒ **Global incosistency**
  **A energy singlet state at $\theta = 0$ is not continuously connected to a singlet state at $\theta = \pi$.**

# Anomaly and the cosine model

$\mathbb{Z}_N$ **background gauge** $(A, B)$ $\qquad (NA = dB)$

$$S[x, A, B] = \int dt \left[\frac{1}{2}(\dot{x} + A_0)^2 + 1 - \cos(Nx + B)\right] - i\frac{\theta}{2\pi}\int (dx + A) + ik\int A,$$

with the Chern-Simons level $k \in \mathbb{Z}_N$.

$$N\int A = \int dB \in 2\pi\mathbb{Z}.$$

**Gauge transform**

$$x \mapsto x - \lambda, \quad A \mapsto A + d\lambda, \quad B \mapsto B + N\lambda.$$

**Partition function**

$$Z_{\theta,k}[(A, B)] = \int \mathcal{D}x \exp\left(-S[x, A, B]\right).$$

# Anomaly and the cosine model

We take the gauge fixing condition as $B = 0 \pmod N$. Thus,

$$A = \sum_{\ell \in \mathbb{Z}} \frac{2\pi\ell}{N}\delta(t - t_\ell)dt, \qquad B = \sum_{\ell \in \mathbb{Z}} = \frac{2\pi\ell}{N}\Theta(t - t_\ell),$$

with the step function and the delta function, $\Theta(t)$ and $\delta(t)$, respectively. The partition function can be evaluated as

$$Z_{\theta,k}[(A, B)] = \left\langle \prod_{\ell \in \mathbb{Z}} \left(e^{-2\pi i k/N}\mathrm{U}(t_\ell)\right)^{\ell} \right\rangle.$$

By acting T, one finds

$$Z_{\theta,k}[\mathrm{T}(A, B)] = \left\langle \prod_{\ell \in \mathbb{Z}} \left(e^{-2\pi i k/N}\mathrm{T}\mathrm{U}(t_\ell)\mathrm{T}^{-1}\right)^{\ell} \right\rangle = Z_{\theta,k}[(A, B)]e^{i\mathcal{A}[k,A]}$$

$$e^{i\mathcal{A}[k,A]} = \begin{cases} \prod_{\ell \in \mathbb{Z}} e^{2\pi\ell i(2k)/N} = e^{2ki\int A} & \text{for } \theta = 0 \\ \prod_{\ell \in \mathbb{Z}} e^{2\pi\ell i(2k-1)/N} = e^{(2k-1)i\int A} & \text{for } \theta = \pi \end{cases}.$$

**No solution for $k \in \mathbb{Z}_N$ such that $e^{i\mathcal{A}[k,A]} = 1$ when $\theta = \pi$ and $N \in 2\mathbb{N}$.**

# Exact-WKB analysis

# Exact-WKB analysis

Consider the Schrödinger equation given by $(\hbar, x \in \mathbb{C}, E \in \mathbb{R}_+)$
(See e.g.**[T.Kawai et al. AMS, c2005]** in technical details.)

$$\left[-\hbar^2 \frac{d^2}{dx^2} + Q(x)\right] \psi(x) = 0, \quad Q(x) = 2(V(x) - E),$$

Put ansatz for a formal solution

$$\psi(x, \hbar) = e^{\int^x S(x,\hbar) dx},$$
$$S(x, \hbar) = \hbar^{-1} S_{-1}(x) + S_0(x) + \hbar S_1(x) + \hbar^2 S_2(x) + \cdots$$

where $S(x, \hbar)$ satisfies the nonlinear Riccati equation.

$$S(x, \hbar)^2 + \frac{\partial S(x, \hbar)}{\partial x} = \hbar^{-2} Q(x), \qquad \boxed{S_{-1}(x) = \pm\sqrt{Q(x)}}.$$

# Exact-WKB analysis

Since the Sch eq is the 2nd order diff eq, there exists two independent solutions.

$$S^{\pm}(x,\hbar) = \pm S_{\rm odd}(x,\hbar) + S_{\rm even}(x,\hbar)\,,$$
$$S_{\rm odd}(x,\hbar) = \sum_{n=0}^{\infty} S_{2n-1}(x)\hbar^{2n-1}, \quad S_{\rm even}(x,\hbar) = \sum_{n=0}^{\infty} S_{2n}(x)\hbar^{2n}.$$

By the Riccati eq, one finds

$$S_{\rm even}(x,\hbar) = -\frac{1}{2}\frac{\partial \log S_{\rm odd}(x,\hbar)}{\partial x}\,,$$

hence, the formal solution can be expressed only by $S_{\rm odd}(x,\hbar)$.

$$\psi_a^{\pm}(x,\hbar) = \frac{e^{\pm\int_a^x S_{\rm odd}(x,\hbar)dx}}{\sqrt{S_{\rm odd}(x,\hbar)}} = e^{\pm\frac{\xi_0(x)}{\hbar}}\sum_{n=0}^{\infty}\psi_{a,n}^{\pm}(x)\hbar^{n+1/2}.$$

# Exact-WKB analysis

Let us look at the Borel resummation of the wavefunction.

$$\mathcal{S}_\theta[\psi_a^\pm(x)](\hbar) = \int_{\mp\xi_0(x)}^{\infty e^{i\theta}} e^{-\frac{\xi}{\hbar}} \mathfrak{B}[\psi_a^\pm(x)](\xi) d\xi, \quad \theta = \arg(\hbar),$$

$$\mathfrak{B}[\psi_a^\pm(x)](\xi) = \sum_{n=0}^{\infty} \frac{\psi_{a,n}^\pm(x)}{\Gamma\left(n+\frac{1}{2}\right)} (\xi \pm \xi_0(x))^{n-\frac{1}{2}}, \quad \xi_0(x) = \int_a^x dx\, S_{\mathrm{odd},-1}(x).$$

The Borel summablity is determined from

$$\frac{\xi_0(x)}{\hbar} = \frac{1}{\hbar}\int_a^x dx\, S_{\mathrm{odd},-1}(x) = \frac{1}{\hbar}\int_a^x dx\, \sqrt{Q(x)}.$$

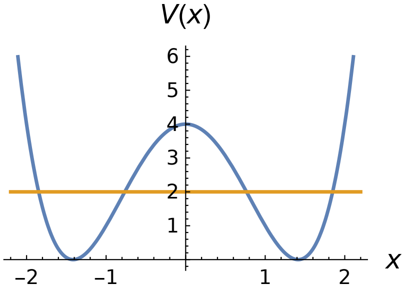
**The Stokes phenomenon** (in other words **Borel nonsummable**) happens when

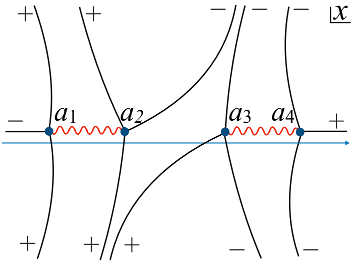
$$\mathrm{Im}\frac{\xi_0(x)}{\hbar} = -\mathrm{Im}\frac{\xi_0(x)}{\hbar}.$$

# The Stokes graph

Since the Borel summability is relevant to $\frac{1}{\hbar}\int S_{\mathrm{odd},-1} : \mathbb{C} \to \mathbb{C}$, it is natural to see the Riemann surface defined by $\frac{1}{\hbar}\int S_{\mathrm{odd},-1}$, so called **the Stoke graph**.

**Example: Double-well potential**



# The Stokes graph

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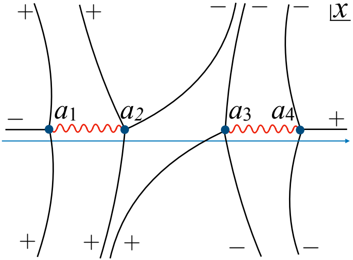
**Constitutive ingredients:**

- Turning point $(a_1, a_2, \cdots)$
  Def: $Q(x) = 0$ $(V(x) - E = 0)$
- Stokes line (black line)
  Def: $\mathrm{Im}\frac{1}{\hbar}\int S_{\mathrm{odd},-1} = 0$
  $\pm$ labels $\int S_{\mathrm{odd},-1} \to \pm\infty$
- Branch cut (red wave)
  $+S_{\mathrm{odd}}(x,\hbar) \leftrightarrow -S_{\mathrm{odd}}(x,\hbar)$

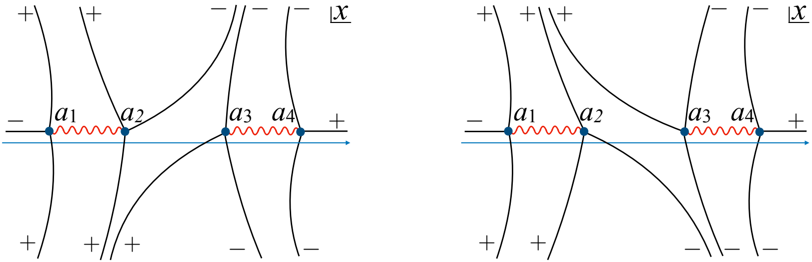



Example of the Stokes graph.
Double-well with $\arg(\hbar) > 0$.

# The Stokes graph

For getting the Q.C, we consider the analytic continuation of $\psi(x)$ for a given Stokes graph and a B.C. To do it, we have to know the effect of crossing Stokes line for $\mathcal{S}[\psi](x)$.

$$\mathcal{S}[\psi^{\mathrm{I}}](x) = \hat{M}\mathcal{S}[\psi^{\mathrm{II}}](x).$$




arg($\hbar$) $> 0$      arg($\hbar$) $< 0$

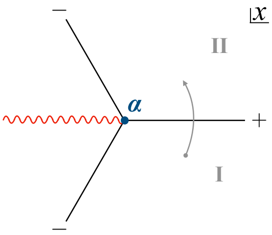
# Connection formula for the Airy-type

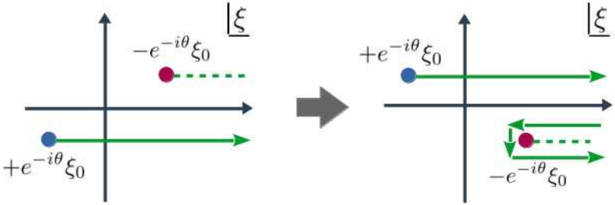
Consider crossing the Stoke line from I to II.

$$\psi = (\psi_+, \psi_-)^\top,$$

$$\psi_a^{\rm I} = M_+ \psi_a^{\rm II}, \qquad \psi_a^{\rm I} = M_- \psi_a^{\rm II},$$

$$M_+ = \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix}, \quad M_- = \begin{pmatrix} 1 & 0 \\ +i & 1 \end{pmatrix}.$$

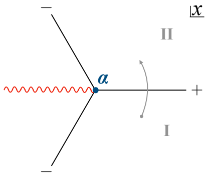


# Connection formula for the Airy-type

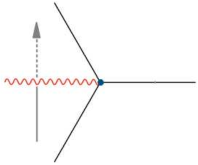
Connection matrix

$$M_+ = \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix}, \quad M_- = \begin{pmatrix} 1 & 0 \\ +i & 1 \end{pmatrix},$$



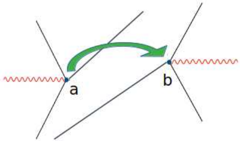

Branchcut matrix

$$T = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$



Normalization matrix (Voros multiplier)

$$N_{ba} = \begin{pmatrix} e^{+\int_a^b dx\, S_{\mathrm{odd}}(x,\hbar)} & 0 \\ 0 & e^{-\int_a^b dx\, S_{\mathrm{odd}}(x,\hbar)} \end{pmatrix}.$$

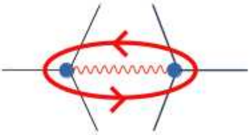


# Cycle expression (Voros multiplier)

For the WKB analysis, it is convenient to introduce **cycle** expression, which is known as the Voros multipliers.

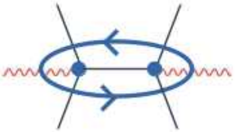
- *A*-**cycle (PT)**

$$A(\hbar) := e^{\oint_A dx\, S_{\rm odd}(x,\hbar)},$$
$$\oint_A dx\,\sqrt{2(V(x)-E)} \in i\mathbb{R}.$$



- *B*-**cycle (NPT)**

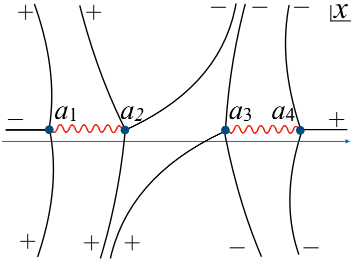
$$B(\hbar) := e^{\oint_B dx\, S_{\rm odd}(x,\hbar)},$$
$$\oint_B dx\,\sqrt{2(V(x)-E)} \in \mathbb{R}.$$



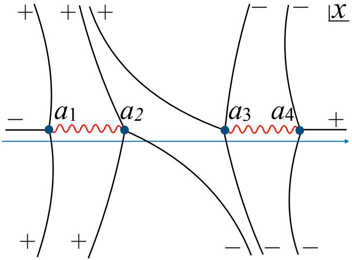
Note: We would take the orientation such that $B(\hbar) \propto e^{-\frac{S_{\rm b}}{\hbar}}$.

# The Stokes graph

The Stokes graph generally depends on $\arg(\hbar)$. In the below example, $\arg(\hbar) = 0$ gives a **Borel nonsummable** wavefunction. But it can be resolved, i.e. **Borel summable**, when $0 < |\arg(\hbar)| \ll 1$ (except exactly on the Stokes line).



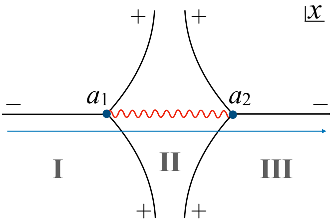

$\arg(\hbar) > 0$




$\arg(\hbar) < 0$

# Example 1: Harmonic oscillator $V(x) = \frac{1}{2}x^2$

$$
\begin{aligned}
\mathrm{I} \to \mathrm{II} &: \quad \psi^{\mathrm{I}}_{a_1} = M_+ \psi^{\mathrm{II}}_{a_1} \\
(1 \to 2) &: \quad \psi^{\mathrm{I}}_{a_1} = N_{a_1,a_2} \psi^{\mathrm{II}}_{a_2} \\
\mathrm{II} \to \mathrm{III} &: \quad \psi^{\mathrm{II}}_{a_2} = M_+ \psi^{\mathrm{III}}_{a_2} \\
(2 \to 1) &: \quad \psi^{\mathrm{III}}_{a_2} = N_{a_2,a_1} \psi^{\mathrm{III}}_{a_1} \\
&\Rightarrow \quad \psi^{\mathrm{I}}_{a_1} = \begin{pmatrix} 1 & i(1+A) \\ 0 & 1 \end{pmatrix} \psi^{\mathrm{III}}_{a_1}.
\end{aligned}
$$




Boundary condition for $\psi_a = (\psi_{+,a}, \psi_{-,a})^\top$:

$$
\begin{aligned}
&\psi^{\mathrm{I}}_{a_1}(x,\hbar) \to 0 \quad \text{as} \quad x \to \pm\infty \\
\Rightarrow \quad &\psi^{\mathrm{I}}_{-,a_1}(x,\hbar) = 0 \quad \text{and} \quad \boxed{D(\hbar) := (1 + A(\hbar)) = 0}
\end{aligned}
$$

where $A(\hbar) := e^{2\int_{a_1}^{a_2} dx\, S_{\mathrm{odd}}(x,\hbar)} = e^{\oint_A dx\, S_{\mathrm{odd}}(x,\hbar)}$.

# Example 1: Harmonic oscillator $V(x) = \frac{1}{2}x^2$

Since $Q(x) = x^2 - 2E$ with $E > 0$, the turning points are given by $a_1 = -\sqrt{2E}, a_2 = +\sqrt{2E}$. Hence,

$$2\int_{a_1}^{a_2} dx\, S_{\rm odd}(x,\hbar) = -\frac{2\pi i E}{\hbar}.$$

From the quantization condition, i.e. $D = (1 + A) = 0$,

$$1 + e^{-\frac{2\pi i E}{\hbar}} = 0 \quad \Rightarrow \quad E = \left(\frac{1}{2} + n\right)\hbar, \quad n \in \mathbb{Z}$$

From the positive energy condition,

$$\boxed{E = \left(\frac{1}{2} + n\right)\hbar, \quad n \in \mathbb{N}_0}$$

# Application: the cosine model

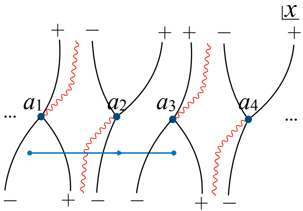
# The cosine model

Let us consider the cosine model and the path from $x = 0$ to $x = 2\pi$ with $-1 \ll \operatorname{Im} x < 0$ to obtain the Q.C.

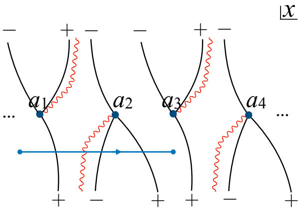
$$Q(x) = 2(V(x) - E), \qquad V(x) = 1 - \cos(Nx), \qquad (N \in \mathbb{N})$$

$$\psi_{a_1}(x) = \hat{M}\psi_{a_1+2\pi\sim a_1}$$

$$\hat{M} = \begin{cases} [M_+ T N_{a_1 a_2} M_- N_{a_2 a_3} M_-]^N =: \mathcal{M}^+ & \text{for} \quad \arg(\hbar) > 0 \\ [M_+ T N_{a_1 a_2} M_- M_+ N_{a_2 a_3}]^N =: \mathcal{M}^- & \text{for} \quad \arg(\hbar) < 0 \end{cases}.$$




$\arg(\hbar) > 0$




$\arg(\hbar) < 0$

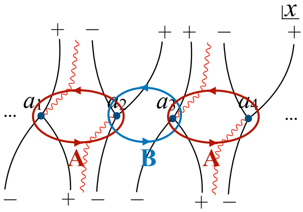
# The cosine model

Set the twisted boundary condition, $\psi(x) = e^{i\theta}\psi(x + 2\pi)$. Thus,

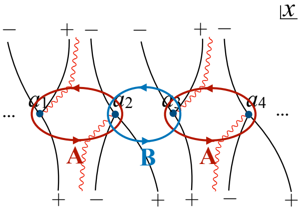
$$D^{\pm} := \det(\mathcal{M}^{\pm} - \mathbb{I}e^{i\theta})/e^{i\theta}. \qquad (\theta\text{: boundary condition})$$

$$= \frac{1}{(A^{\mp 1}B)^{N/2}} \prod_{p=0}^{N-1} D_p^{\pm} = 0 \qquad \text{for} \quad \mathrm{sign}(\mathrm{Im}(\hbar)) \pm 1$$

$$D_p^{\pm} := 1 + A^{\mp 1}(1 + B) - 2\sqrt{A^{\mp 1}B} \cos\left(\frac{\theta + 2\pi p}{N}\right)$$




$\arg(\hbar) > 0$

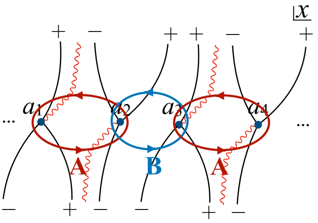



$\arg(\hbar) < 0$

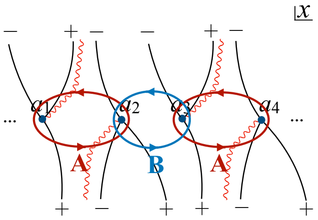
# Delabaere-Dillinger-Pham (DDP) formula

Cycles have the resurgence relation called **the DDP formula**.

**[E.Delabaere et al. '97, K.Iwaki et al. '14]**




$\arg(\hbar) > 0$




$\arg(\hbar) < 0$

In the case of the cosine potential, it is given by

$$\mathcal{S}_+[\sqrt{A}] = \mathcal{S}_-[\sqrt{A}](1 + \mathcal{S}[B]), \qquad \mathcal{S}_+[B] = \mathcal{S}_-[B] =: \mathcal{S}[B].$$

(See DDP paper for details and generic cases.)

# Resurgence for the quantization condition

Since $D^{\pm}(E)$ is a function of cycles, from the DDP formula given by

$$\mathcal{S}_+[\sqrt{A}] = \mathcal{S}_-[\sqrt{A}](1+\mathcal{S}[B]), \qquad \mathcal{S}_+[B] = \mathcal{S}_-[B] =: \mathcal{S}[B],$$

one finds that

$$\boxed{\mathcal{S}_+[(A^{-1}B)^{-1/2}D_p^+(A,B)] = \mathcal{S}_-[(A^{+1}B)^{-1/2}D_p^-(A,B)]}$$

Since $D^{\pm} = (A^{\mp 1}B)^{-N/2}\prod_{p=0}^{N-1} D_p^{\pm}(E) = 0$, the energy spectrum is given by each of $p$-sectors. This fact means that the resurgence structure is closed on the fixed-$p$ sector. Trivially,

$$\boxed{\mathcal{S}_+[D^+(A,B)] = \mathcal{S}_-[D^-(A,B)]}$$

# Degeneracy of energy

When $N = 2K$ with $K \in \mathbb{N}$ and $\theta = \pi$ (APBC), one can immediately see the degeneracy of energy spectrum:

$$
\begin{aligned}
& D_p^{\pm} = 1 + A^{\mp 1}(1+B) - 2\sqrt{A^{\mp 1}B}\cos\left(\frac{\theta + 2\pi p}{N}\right) \\
\Rightarrow \quad & D_p^{\pm} = 1 + A^{\mp 1}(1+B) - 2\sqrt{A^{\mp 1}B}\cos\left[\frac{\pi(p+1/2)}{K}\right], \quad (p \in \mathbb{Z}_{2K})
\end{aligned}
$$

Hence, one finds

$$
\boxed{D_p^{\pm} = D_{2K-p-1}^{\pm}}
$$

This degeneracy is a sign of an 't Hooft anomaly.

# Gutzwiller trace formula

We derive the GTF through $G^{\pm} = -\partial_E \log D^{\pm}$. ($N = 1$ for simplisity.)

$$G^{\pm}(E) = G^{\pm}_{\rm pt}(E) + G^{\pm}_{\rm np}(E),$$

$$G^{\pm}_{\rm pt}(E) := -\partial_E A^{\mp 1} \cdot \sum_{n=0}^{\infty} (-1)^n A^{\mp n}, \quad G^{\pm}_{\rm np}(E) := -\partial_E K^{\pm} \cdot \sum_{n=0}^{\infty} (-1)^n (K^{\pm})^n,$$

$$K^{\pm} := B \sum_{n=0}^{\infty} (-1)^n A^{\pm n} - 2\sqrt{A^{\pm 1} B} \sum_{n=0}^{\infty} (-1)^n A^{\pm n} \cos\theta.$$

We define "period" $T_{A,B}$ as

$$\partial_E A = \oint_A dx \left( \frac{-1}{\hbar\sqrt{2V(x) - E}} + O(\hbar) \right) \cdot A =: -\frac{i}{\hbar} T_A A,$$

$$\partial_E B = \oint_B dx \left( \frac{-1}{\hbar\sqrt{2V(x) - E}} + O(\hbar) \right) \cdot B =: -\frac{i}{\hbar} T_B B,$$

Notice that $\lim_{\hbar\to 0} T_A \in \mathbb{R}$ and $\lim_{\hbar\to 0} T_B \in i\mathbb{R}$.

# Gutzwiller trace formula

**Maslov index**

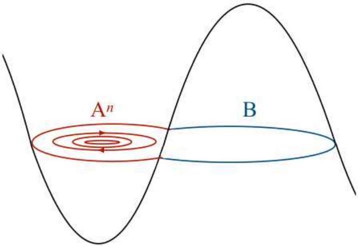
$$G^{\pm}_{\rm pt}(E) := -\partial_E A^{\mp 1} \cdot \sum_{n=0}^{\infty} (-1)^n A^{\mp n}, \quad G^{\pm}_{\rm np}(E) := -\partial_E K^{\pm} \cdot \sum_{n=0}^{\infty} (-1)^n (K^{\pm})^n,$$

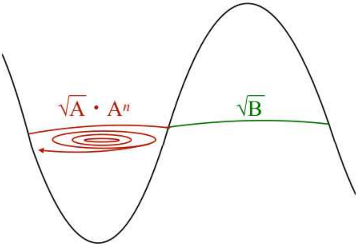
$$K^{\pm} := B \sum_{n=0}^{\infty} (-1)^n A^{\pm n} - 2\sqrt{A^{\pm 1} B} \sum_{n=0}^{\infty} (-1)^n A^{\pm n} \cos\theta.$$

- $G^{\pm}_{\rm pt}$ and $G^{\pm}_{\rm np}$ constitute of the periodic orbits, $A^{\pm 1}$ and $K^{\pm}$, respectively. These give $(-1)$, which is the Maslov index.
- $K^{\pm}$ is has fundamental nonpertubative orbits, $BA^{\pm n}$ and $\sqrt{A^{\pm 1} B} A^{\pm n}$. $A^{\pm 1}$ and $B$ also give $(-1)$ there.
- The exact form of the GTF for the cosine model could be obtained.

# Gutzwiller trace formula

$$K^{\pm} := B\sum_{n=0}^{\infty}(-1)^n A^{\pm n} - 2\sqrt{A^{\pm 1}B}\sum_{n=0}^{\infty}(-1)^n A^{\pm n}\cos\theta.$$

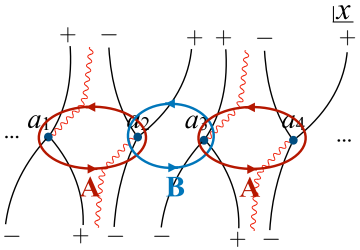


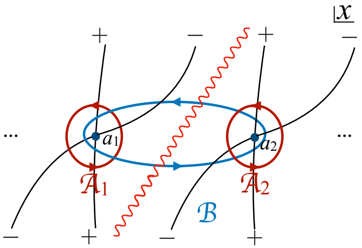
# Relation of cycles between Airy- and DW-types

We consider the ground state energy and its nonpertubative correction. When considering the energy spectrum by solving the Q.C. and it gives $E(\hbar) = O(\hbar)$, it is useful to employ **the degenerate Weber (DW)-type** Stokes graph.

Two Airy-type ($E_0 > 0$) $\rightarrow$ One DW-type ($E_0 = 0$)




$E_0 > 0$




$E_0 = 0$

# Relation of cycles between Airy- and DW-types

- When considering the Q.C. based on the DW-type graph, one has to compute the DW connection formula.
- In principle, the result by the DW-type graph is available by replacing $E \to E\hbar$ in the results by the Airy type. But, it is difficult in the generic cases.
- Instead the reduction from the Airy-type graph, one can make a **dictionary** between the cycles of Airy-type $(A, B)$ and the ones of the DW-type $(\mathfrak{A}, \mathfrak{B})$.
- The cycles expression based on Voros multipliers is kept. (But, the functional forms are different from each others.)

$$\text{e.g.} \quad D(A, B) \;\to\; D(\mathfrak{A}, \mathfrak{B}).$$

# Quantization condition by degenerate Weber-type

We replace $(A, B)$ with $(\mathfrak{A}, \mathfrak{B})$ given by

$$A \to \mathfrak{A} \approx e^{-\frac{2\pi i E}{N}}, \qquad B \to \mathfrak{B} \approx \frac{2\pi \mathfrak{B}_0}{\Gamma(\frac{1}{2} + \frac{E}{N})^2} \left(\frac{N\hbar}{32}\right)^{-\frac{2E}{N}}, \qquad \mathfrak{B}_0 = e^{-\frac{16}{N\hbar}}.$$

(Here, $E(\hbar) \approx E\hbar$) Thus, $D^{\pm}$ becomes

$$D^{\pm} \approx \prod_{p=0}^{N-1} \left[ \frac{1}{\sqrt{\mathfrak{B}_0}\Gamma(\frac{1}{2} - \frac{E}{N})} \left(\frac{N\hbar}{32}\right)^{E/N} + \frac{\sqrt{\mathfrak{B}_0} e^{\pm \pi i E/N}}{\Gamma(\frac{1}{2} + \frac{E}{N})} \left(\frac{N\hbar}{32}\right)^{-E/N} - \sqrt{\frac{2}{\pi}} \cos\left(\frac{\theta + 2\pi p}{N}\right) \right] = 0.$$

# Nonperturbative contribution to the energy

The ground state energy is obtained by the 1st. term of Q.C., $E/N = 1/2$. In order to consider the NPT contribution to the ground state energy, we substitute $E/N = 1/2 + \delta$ into Q.C. where $\delta = O(\sqrt{\mathfrak{B}_0})$. The solution is given by

$$\begin{aligned}\delta_p^{\pm} &= -\sqrt{\frac{64\mathfrak{B}_0}{N\pi\hbar}}\cos\left(\frac{\theta+2\pi p}{N}\right)\\ &\quad +\frac{64\mathfrak{B}_0}{N\pi\hbar}\cdot\left[\cos^2\left(\frac{\theta+2\pi p}{N}\right)\cdot\left(\gamma-\log\frac{N\hbar}{32}\right)\pm\frac{\pi i}{2}\right]+O(\mathfrak{B}_0^{3/2}).\end{aligned}$$

# Nonperturbative contribution to the energy

For $N = 1$ and 2,

$$\delta^{\pm} = -\sqrt{\frac{64\mathfrak{B}_0}{\pi\hbar}}\cos\theta + \frac{64\mathfrak{B}_0}{\pi\hbar}\cdot\left[\cos^2\theta\cdot\left(\gamma - \log\frac{\hbar}{32}\right) \pm \frac{\pi i}{2}\right] + O(\mathfrak{B}_0^{3/2}),$$

$$\delta_p^{\pm} = -(-1)^p\sqrt{\frac{32\mathfrak{B}_0}{\pi\hbar}}\cos\frac{\theta}{2} + \frac{32\mathfrak{B}_0}{\pi\hbar}\cdot\left[\cos^2\frac{\theta}{2}\cdot\left(\gamma - \log\frac{\hbar}{16}\right) \pm \frac{\pi i}{2}\right] + O(\mathfrak{B}_0^{3/2}).$$

- $N = 1$
  $O(\mathfrak{B}_0^{1/2})$ ... (Anti-)instanton $[\mathcal{I}]$ $[\bar{\mathcal{I}}]$
  $O(\mathfrak{B}_0)$ ... (Anti-)instantons pair $[\mathcal{I}\mathcal{I}]$ $[\bar{\mathcal{I}}\bar{\mathcal{I}}]$
  Bion $[\mathcal{I}\bar{\mathcal{I}}]_{\pm}$.
- $N = 2$ (If $\theta = \pi$, instanton contributions disappear, and $\delta_0^{\pm} = \delta_1^{\pm}$.)
  $O(\mathfrak{B}_0^{1/2})$ ... (Anti-)instanton $[\mathcal{I}_1][\mathcal{I}_2][\bar{\mathcal{I}}_1][\bar{\mathcal{I}}_2]$
  $O(\mathfrak{B}_0)$ ... (Anti-)instantons pair $[\mathcal{I}_1\mathcal{I}_2][\bar{\mathcal{I}}_1\bar{\mathcal{I}}_2]$
  Bion $[\mathcal{I}_1\bar{\mathcal{I}}_1]_{\pm}$ $[\mathcal{I}_2\bar{\mathcal{I}}_2]_{\pm}$.

# Partition function

We obtain the partition function via the resolvent method.

$$
\begin{aligned}
Z^{\pm}(\beta) &= -\frac{1}{2\pi i}\int_{\epsilon+i\infty}^{\epsilon+i\infty} \frac{\partial \log D^{\pm}}{\partial E} e^{-\beta E} dE \\
&= -\frac{\beta}{2\pi i}\int_{\epsilon+i\infty}^{\epsilon+i\infty} \log D^{\pm} e^{-\beta E} dE.
\end{aligned}
$$

Since the DDP formula gives $\mathcal{S}_{+}[D^{+}] = \mathcal{S}_{-}[D^{-}]$ and $\mathcal{S}_{\pm}$ is homomorphism for summation and multiplication, one can easily find

$$
\boxed{\mathcal{S}_{+}[Z^{+}(\beta)] = \mathcal{S}_{-}[Z^{-}(\beta)]}
$$

Since the path-integral is now expressed by cycles, the resurgenct relation for each sector can be traced.

# Partition function

In order to see the details, we factorize the Q.C. as

$$
\begin{aligned}
D^{\pm} &\propto \prod_{p=0}^{N-1}\left[1+\mathfrak{A}^{\mp 1}(1+\mathfrak{B})-2\sqrt{\mathfrak{A}^{\mp 1}\mathfrak{B}}\cos\left(\frac{\theta+2\pi p}{N}\right)\right] \\
&= (D_A^{\pm})^N \cdot \prod_{p=0}^{N-1}\left[1+\frac{\mathfrak{A}^{\mp 1}\mathfrak{B}}{D_A^{\pm}}-2\frac{\sqrt{\mathfrak{A}^{\mp 1}\mathfrak{B}}}{D_A^{\pm}}\cos\left(\frac{\theta+2\pi p}{N}\right)\right] \\
&\quad (D_A^{\pm} := 1+\mathfrak{A}^{\pm 1})
\end{aligned}
$$

Each of the factors give

- $(D_A^{\pm})^N$ ... (N copies of) pertubative sector,
- $[\cdots]$ ... ($p$-th) nonpertubative sector,

by expanding $\log D_A^{\pm}$ and $\log[\cdots]$ around 1.

# Partition function

The partition function is given by

$$
\begin{aligned}
Z^{\pm} &= N Z^{\pm}_{\mathrm{pt}} + \sum_{p=0}^{N-1} \sum_{\substack{(Q_p,K_p)\in\mathbb{Z}\otimes\mathbb{N}_0 \\ |Q_p|+K_p>0}} Z^{\pm}_{\mathrm{np}}(p,Q_p,K_p) \\
&= N Z^{\pm}_{\mathrm{pt}} + N \sum_{\substack{(Q,K)\in\mathbb{Z}\otimes\mathbb{N}_0 \\ |Q|+K>0}} Z^{\pm}_{\mathrm{np}}(0,NQ,K),
\end{aligned}
$$

where

$$
Z^{\pm}_{\mathrm{pt}} := \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \sum_{n=1}^{\infty} \frac{(-\mathfrak{A}^{\pm 1})^n}{n} e^{-\beta E} dE,
$$

$$
\begin{aligned}
Z^{\pm}_{\mathrm{np}}(p,Q_p,K_p) &:= \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{2\pi i p Q_p/N}}{|Q_p|+K_p} \binom{|Q_p|+K_p}{K_p} \left(\frac{\mathfrak{B}}{\mathfrak{K}^2}\right)^{|Q_p|/2+K_p} \\
&\quad \cdot {}_2F_1(1-K_p,-K_p;|Q_p|+1;-\mathfrak{A}^{\pm 1})(-\mathfrak{A}^{\mp K_p}) e^{-\beta E + i Q_p \theta/N} dE,
\end{aligned}
$$

$$
(\mathfrak{K} := \sqrt{\mathfrak{A}} + \sqrt{\mathfrak{A}}^{-1})
$$

Let us look at the resurgence structure of the Hilbert space and the partition function. We express the Q.C. using $\alpha^{\pm}$ and $\beta^{\pm}$:

$$D^{\pm} = \prod_{p=0}^{N-1} \alpha^{\pm} \left(1 - \beta^{\pm} e^{i(\theta+2\pi p)/N}\right) \left(1 - \beta^{\pm} e^{-i(\theta+2\pi p)/N}\right) \propto \prod_{p=0}^{N-1} D_p^{\pm},$$

$$\alpha^{\pm} := \xi^{\pm} + \sqrt{(\xi^{\pm})^2 - 1}, \quad \beta^{\pm} := \xi^{\pm} - \sqrt{(\xi^{\pm})^2 - 1}, \quad \xi^{\pm} := \frac{1 + \mathfrak{A}^{\pm 1} + \mathfrak{B}}{\sqrt{\mathfrak{A}^{\pm 1}\mathfrak{B}}},$$

$$\mathcal{S}_{+}[\xi^{+}] = \mathcal{S}_{-}[\xi^{-}] \quad \Rightarrow \quad \mathcal{S}_{+}[\alpha^{+}] = \mathcal{S}_{-}[\alpha^{-}] \quad \text{and} \quad \mathcal{S}_{+}[\beta^{+}] = \mathcal{S}_{-}[\beta^{-}].$$

$E_p$ is given by $D_p^{\pm} = 0$, thus the resurgent relation of corresponding Hilbert space $\mathcal{H}_p$ is closed.

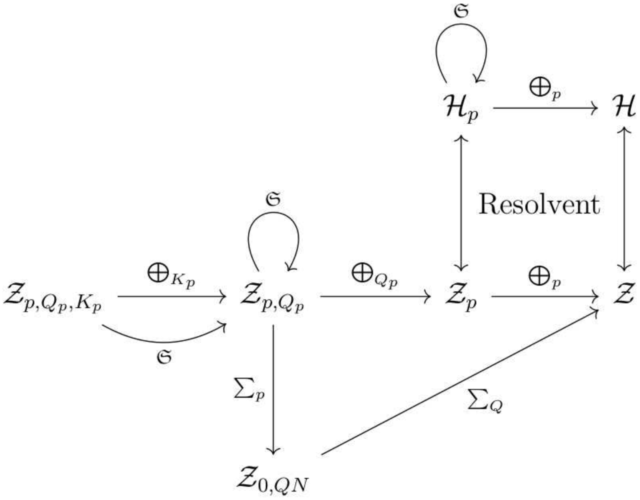
The partition function can be also expressed by $\alpha^{\pm}$ and $\beta^{\pm}$:

$$
\begin{aligned}
Z^{\pm} &= \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{-\beta E} dE \\
&\cdot \sum_{p=0}^{N-1}\left[-\log\left(\sqrt{\mathfrak{A}^{\mp 1}\mathfrak{B}}\alpha^{\pm}\right) + \sum_{Q_p\in\mathbb{Z}\setminus\{0\}} \frac{(\beta^{\pm})^{|Q_p|}}{|Q_p|} e^{i(\theta+2\pi i)Q_p/N}\right].
\end{aligned}
$$

We can see that the resurgent structure is closed in $[\cdots]$. Not only that, each of the $Q_p$-sectors is also closed.
(The topological charge $Q_p$ arises as powers by Taylor expansion.)

Thus, $\sum_{K_p=0}^{\infty} Z^{\pm}(p, Q_p, K_p)$ is irreducible for the Stokes automorphism $\mathfrak{S}$, and the resrugent structure is labeled by $(p, Q_p)$.

# Hilbert space $\Leftrightarrow$ Partition function



# Summary

- We considered QM of a particle on $S^1$ in the presence of a periodic potential with $N$-minima ($N \in \mathbb{N}$) by the exact-WKB method.
- We derived the energy spectrum, the Gutzwiller trace formula, and the partition function from the quantization condition.
- All orders perturbative/non-perturbative resurgent relation is shown.
- Our result obtained by the DW-type correctly reproduces the energy eigenvalues conjectured by Zinn-Justin**[J.Zinn-Justin et al. 04]**, and obtained earlier by using uniform WKB method for $N = 1$ **[G.Dunne et al. 14]**.
- The exact quantization condition naturally captures the mixed 't Hooft anomaly or global inconsistency.