Entanglement and Complexity from TQFT

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Based on work in collaboration with

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Quantum Entanglement

Spooky action at distance

- Quantum entanglement is the property of quantum correlations in a system

Consider a partition of the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$ and a state

$$|\Psi\rangle \in \mathcal{H}$$

We define the (pure state) density matrix as

$$\rho = |\Psi\rangle \otimes \langle\Psi|$$

Quantum Entanglement

For a partitioned system with a generic $\rho \in \mathcal{H} \otimes \mathcal{H}^*$ we also define the reduced density matrix

$$\rho_1 = \mathrm{Tr}_{\mathcal{H}_2, \mathcal{H}_3, \ldots}(\rho)$$

- ρ unentangled if $\forall\, \mathcal{H}_k,\ \rho_k = |\Psi_k\rangle \otimes \langle\Psi_k|$ for some $|\Psi_k\rangle$
- ρ entangled otherwise

Example: In the EPR (Bell) state, the spins are entangled

$$|\Psi_{12}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\right)$$

Quantum Entanglement

Measures of entanglement

- Entanglement (von Neumann) entropy

$$S(A) \;=\; -\operatorname{Tr}_A(\rho_A \log \rho_A), \qquad \rho_A = \operatorname{Tr}_{\bar{A}}(\rho)$$

- Relative entropy

$$S(A||B) \;=\; -\operatorname{Tr}_A(\rho_A \log \rho_B) - S(A)$$

- (Log) negativity (in terms of the eigenvalues of ρ^{Γ_A})

$$N(\rho^{\Gamma_A}) \;=\; \sum_{\lambda} \frac{|\lambda| - \lambda}{2}, \qquad E(\rho) \;=\; \log_2(2N(\rho) + 1)$$

Example

$$\text{EPR:} \qquad \rho_A \;=\; \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad S_A \;=\; \log 2$$

Quantum Entanglement

Classification (3-partite systems)

- Greenberger-Horne-Zeilinger states

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right)$$

  projector $|\uparrow\rangle\langle\uparrow| \otimes 1 \otimes 1$ (measurement of the first spin) makes the state unentangled

- W(olfgang Dür) states

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)$$

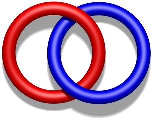
These are the only two classes of non-biseparable states, which cannot be connected by Local Operations and Classical Communication (LOCC)

Quantum Entanglement in TQFT

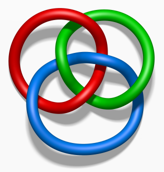
Quantum vs topological entanglement

Aravind's conjecture ('97): classifies types of entanglement using topology (linking)

- Bell state: $|\mathrm{B}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\right)$



- GHZ state: $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right)$

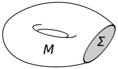


Recent elaborated map between link and QM states in the works of R. André and G. Quinta @Lisbon

Quantum Entanglement in TQFT

Formal definition of a TQFT [Atiyah]

- Map (functor) Z between (the category of) topological spaces and (the category of) linear spaces:
  1. d-dimensional $\Sigma \longrightarrow$ vector space $V = Z(\Sigma)$
  2. $d+1$ dimensional M, $\Sigma = \partial M \longrightarrow$ vector $v = Z(M) \in V$
  3. $\forall\ \Sigma_1, \Sigma_2$ and M, $\partial M = \Sigma_1 \cup \Sigma_2$, $\longrightarrow$ linear map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$



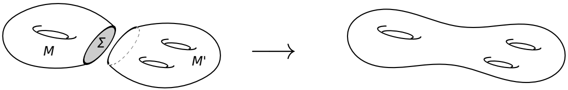
$$\equiv |\psi\rangle$$

Hilbert space is encoded by boundary Σ. Its elements are different ways of gluing-in manifolds M, up to homeomorphisms, and possible linear relations.

Quantum Entanglement in TQFT

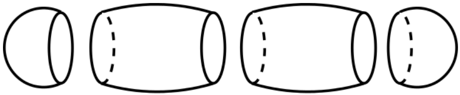
Properties of the TQFT functor

- Inner product



$$\longrightarrow \quad \equiv \langle \psi | \psi' \rangle$$

- For a disjoint union $Z(\Sigma_1 \sqcup \Sigma_2) \;=\; Z(\Sigma_1) \otimes Z(\Sigma_2)$
- Composition of maps



This provides a heuristic representation of a tensor multiplication and a path integral

Quantum Entanglement in TQFT

Functor as path integral in a QFT

State can be constructed from a path integral ($-\infty < t < 0$)

$$|\Psi(\Sigma)\rangle \;=\; \int \mathcal{D}A\Big|_{A(\Sigma)=A_\Sigma} \mathrm{e}^{iS_{\mathrm{CS}}[\mathcal{M}_3]}$$

with fixed configuration A_Σ at Cauchy surface Σ at $t = 0$

- partition function $\langle\Psi(\Sigma)|\Psi(\Sigma)\rangle = Z$
- density matrix $\hat{\rho} = |\Psi(\Sigma)\rangle \otimes \langle\Psi(\Sigma')|$
- reduced density matrix, $\Sigma = \Sigma_1 \cup \Sigma_2, \quad \rho_1(\Sigma_1, \Sigma_1') = \mathrm{Tr}_{\Sigma_2}\, \rho$

von Neumann entropy

$$S_{\mathrm{E}}(\Sigma_1) \;=\; -\,\mathrm{Tr}_{\Sigma_1}\, \rho_1 \log \rho_1$$

How does one compute S_{E}?

[Dong,Fradkin,Leigh,Nowling'08]

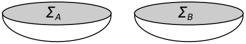
Quantum Entanglement in TQFT

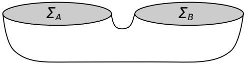
How is entanglement characterized by topology?

[DM,Mironov^2,Morozov^2'18]

Entangled vs. nonentangled

- Consider $\Sigma = \Sigma_A \cup \Sigma_B$. Two classes of states:

$|\Psi_1\rangle =$ 

$|\Psi_2\rangle =$ 

- We expect the left one to be unentangled

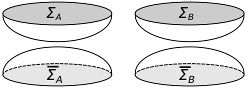
Quantum Entanglement in TQFT

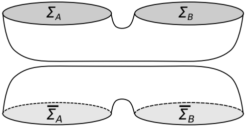
Replica trick

- compute $\mathrm{Tr}\ \rho_A^n$

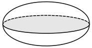
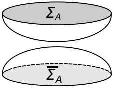
$$S_{\mathrm{E}} = -\frac{d}{dn}\ \mathrm{Tr}\ \rho_A^n\Big|_{n=1}$$

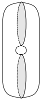
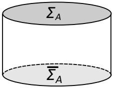
(Unnormalized) density matrices

$\widehat{\rho}_1 =$ 

$\widehat{\rho}_2 =$ 

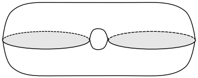
Normalized reduced density matrices

$\rho_1(A) = \Big[$  $\Big]^{-1}$ 

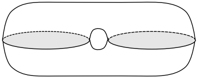
$\rho_2(A) = \Big[$  $\Big]^{-1}$ 

Quantum Entanglement in TQFT

Entanglement entropy

$$\mathrm{Tr}\,\left(\rho_1^A\right)^n \;=\; 1\,, \qquad \mathrm{Tr}\,\left(\rho_2^A\right)^n \;=\; \Big[$$

$$\Big]^{1-n}$$

Consequently,

$$S_{\mathrm{E}}(\rho_1) \;=\; 0\,, \qquad S_{\mathrm{E}}(\rho_2) \;=\; \log \Big[$$

$$\Big]$$

The donut is a top. invariant, $\mathrm{Tr}_{\mathcal{H}}\,\mathbf{1} \;=\; \dim \mathcal{H} = Z(\Sigma \times S^1)$

General observation: (Rényi) entropies are expressed in terms of topological invariants of closed 3D manifolds.

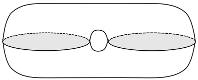
Quantum Entanglement in TQFT

Rényi entropies

$$S_n = \frac{1}{1-n} \operatorname{Tr} \rho^n$$

Relative entropy

$$S(\rho_1 || \rho_2) = \lim_{n \to 1} \frac{1}{n-1} \left(\operatorname{Tr} \rho_1^n - \operatorname{Tr} \rho_1 \rho_2^{n-1} \right) = \log \left[\quad \right]$$

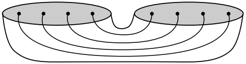
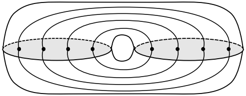


Quantum Entanglement in TQFT

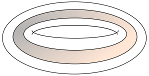
Examples in $SU(N)_k$ Chern-Simons

- $\Sigma = S^2$: $\quad Z(S^2 \times S^1) = 1 \Rightarrow S_{\mathrm{E}} = 0.$

- $\Sigma = S^2 \backslash \{P_i\}$: $\quad \dim \mathcal{H}_{ab} = \sum_c N_{abc}, \qquad \Phi_a \star \Phi_b = \oplus_c N_{abc} \Phi_c$

$|\Psi\rangle =$  $\qquad \dim \mathcal{H} =$ 

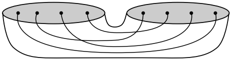
- $\Sigma = T^2$: $\quad Z(T^2 \times S^1) = \begin{pmatrix} k+N-1 \\ N-1 \end{pmatrix}, k \in \mathbb{Z}$

$|\Psi\rangle =$ 

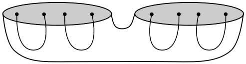
Quantum Entanglement in TQFT

Local operations and entanglers

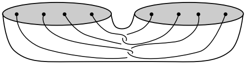
- Entanglement entropy is insensitive to local unitary operations



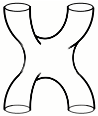
- Non-local operators can affect entanglement



or



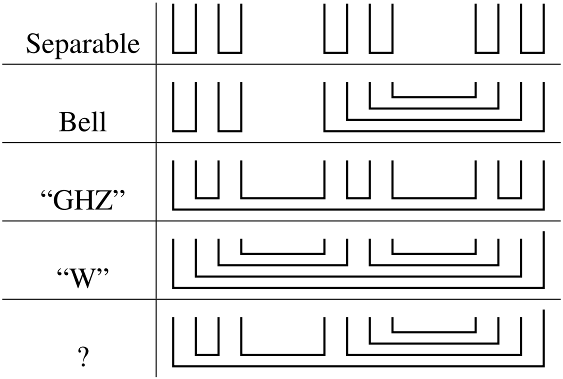
- Entangling operators are represented by manifolds of the form



Quantum Entanglement in TQFT

3-partite entanglement

- Count different ways to link Σ_1, Σ_2 and Σ_3



Knot and Knot Complement States

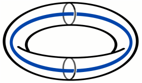
Quantum Chern-Simons theory on a torus

Hilbert Space

$$\dim \mathcal{H}_{T^2} = \begin{pmatrix} k+N-1 \\ N-1 \end{pmatrix}$$

Basis vectors

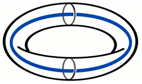
$$|\, R_i \,\rangle \;=\; \Big|$$



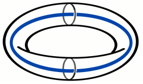
$$\Big\rangle, \qquad i = 1, \ldots, \dim \mathcal{H}_{T^2}$$

Scalar product

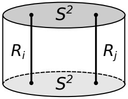
$$\langle R_i | R_j \rangle \;=\; \Big\langle$$



$$\Big|$$



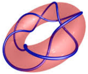
$$\Big\rangle \;=\; \mathrm{tr}$$



$$=\; Z(S^2 \times S^1; R_i, R_j) \;=\; \delta_{ij}$$

Knot and Knot Complement States

Torus knot states, $K_{m,n} = m\alpha + n\beta$ [Labastida,Llatas,Ramalho'91]

$\Big|$  $\Big\rangle \equiv |m,n;R\rangle \;=\; \sum_i W^{(m,n)}_{R,R_i}|R_i\rangle$

Knot operators

- $W^{(m,n)}_{R,R_i}$ must be the topological invariants
- Torus knots can be obtained via the $PSL(2,Z)$ action on T^2

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m & p \\ n & q \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  which produces (m,n)-knot from an unknot $(1,0)$
- The operator

$$|m,n;R\rangle \;=\; \hat{W}^{(m,n)}|1,0;R\rangle$$

  must realize $PSL(2,Z)$ representations

Knot and Knot Complement States

Unitary representations of $PSL(2,Z)$

- There exist unitary representations but they are not faithful: linear dependencies between torus knot states.
  - For $N=2$,

$$S_{R_j,R_l} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(2j+1)(2l+1)}{k+2}\right) , \quad T_{R_j,R_l} = e^{\frac{2\pi i j(j+1)}{k+2}} \delta_{j,l}$$

  - For $N=1$,

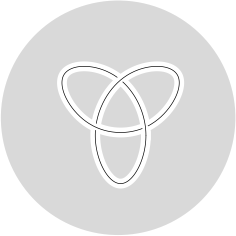
$$S_{q_j,q_l} = \frac{2}{\sqrt{k}} \exp\left(\frac{2\pi i}{k} q_j q_l\right) , \quad T_{q_j,q_l} = \exp\left(\frac{\pi i}{k} q_j q_l\right) \delta_{j,l}$$

- Number of linearly-independent states is equal to the index of the principal congruence subgroup $\Gamma(k+N)$ - finite, $|PSL(2,Z)/\Gamma_{k+N}|$

Knot and Knot Complement States

Knot complement states

[Balasubramanian et al'16]



- Choose a knot (link) in S^3 and cut its tubular neighborhood
- Then $\bar{\Sigma}$ is a disjoint union of tori $T^2 \sqcup T^2 \sqcup \cdots$

$$\mathcal{H} = \mathcal{H}_{T^2} \otimes \mathcal{H}_{T^2} \otimes \cdots$$

- The complement in S^3 corresponds to the state

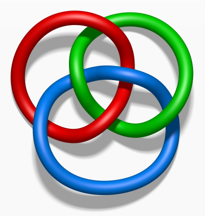
$$|\mathcal{L}\rangle = \sum_{R_1, R_2, \ldots, R_L} Z(S^3; R_1, R_2, \ldots, R_L) |R_1\rangle \otimes |R_2\rangle \otimes \cdots \otimes |R_L\rangle$$

where L is the number of link components

Knot and Knot Complement States

Entanglement of the knot complement states [Balasubramanian et al]

- Unlinked components have zero entanglement
- Hopf link has maximum entanglement, Borromean rings are not GHZ, ...



- All torus links (Lm, Ln) have GHZ type of entanglement

$$|\mathcal{L}_{(Lm,Ln)}\rangle = \sum_{R_1,\ldots,R_L} \sum_{R,Q} \frac{S_{R,R_1} S_{R,R_2} \cdots S_{R,R_L}}{(S_{0,R})^{L-1}} S_{R,Q} Z(\mathcal{K}_{(m,n)}, Q) |R_1, \ldots R_L\rangle$$

This state has simple coefficients in the basis $|\tilde{R}\rangle = S_{R,Q}|Q\rangle$

$$|\mathcal{L}_{(Lm,Ln)}\rangle = \sum_{R,Q} \frac{S_{R,Q}}{(S_{0,R})^{L-1}} Z(\mathcal{K}_{(m,n)}, Q) |\tilde{R}, \ldots \tilde{R}\rangle$$

Complexity of Knot States

Motivation

- Recent discussion of complexity in holography and QFT. Complexity=Volume [Susskind et al.]

$$\mathcal{C} = \frac{\mathrm{vol}(ERB)}{8\pi L G_N}$$

- Attempts to understand the holographic proposals of complexity in terms of circuit (network) constructions
- Path integral optimization [Takayanagi et al.]

$$\int D\varphi(x)\, \mathrm{e}^{-S_{M_\Sigma}[\varphi]} \delta\big(\varphi(x,t_0) - \varphi_\Sigma(x)\big) \;=\; \mathrm{e}^{C_M} \Psi_\Sigma$$

In the path integral formulation, the complexity prefactor appears similarly to the framing ambiguity of links

Complexity of Knot States

Circuit complexity

- $|\Psi_R\rangle$ – reference state
- $|\Psi_T\rangle$ – target state
- $\{U_n\}$ - set of "elementary" unitary operations (gates)

$$|\Psi_T\rangle = U_N|\Psi_R\rangle, \qquad U_N = U_{n_1}U_{n_2}\cdots U_{n_N}$$

What is the minimum number of gates necessary to generate the target state from the reference state?

$$\text{Complexity}: \quad \mathcal{C} = \min_{U_N} N$$

Geometric interpretation: let $\{U_n\}$ be generators of a Lie algebra

$$U[\gamma] = \mathrm{P}\exp\left(i\int_\gamma a_n U_n\right), \qquad \mathcal{C} = \min_\gamma \mathrm{Length}(\gamma)$$

Complexity of Knot States

Complexity of torus knot states [Camilo et al.]

Let the unknot be a reference state, while $|\Psi_T\rangle = |m, n; R\rangle$. What is the complexity of the (m, n) knot state?

- Define $PSL(2, Z)$ in terms of S and T generators - gates

$$\langle S, T | S^2 = (ST)^3 = 1\rangle$$

- write $W^{(m,n)} = T^{a_1} S T^{a_2} S \cdots S T^{a_r}$
- complexity can be defined as

$$\mathcal{C} = \min_{\{a_1, a_2, \ldots, a_n\}} \sum_{i=1}^{r} (|a_i| + 1)$$

What is the shortest ST word for a given $PSL(2, Z)$ element?

Complexity of Knot States

Continued fractions

$$\frac{m}{n} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_r}}} \equiv [a_1, a_2, \ldots, a_r] = b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_r}}}$$

We note that this is equivalent to

$$\frac{m}{n} \sim T^{a_1} S \circ T^{a_2} S \cdots \circ T^{a_r} S \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad T^a S : \; z \to a - \frac{1}{z}$$

Theorem (Camilo et al.'19)

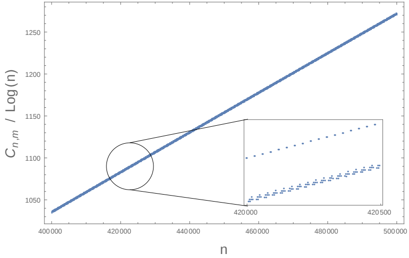
The continued fraction with all $b_i > 0$ and $b_r > 1$ gives a shortest word in terms of S and T generators. This presentation is unique.

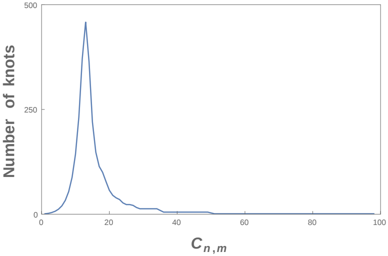
Complexity of Knot States

Classical vs quantum

- Due to linear dependences the actual quantum complexity may be lower, so $\mathcal{C} = \sum_i (b_i + 1)$ gives an upper bound
- In the semiclassical limit $k \to \infty$ the classical bound is saturated

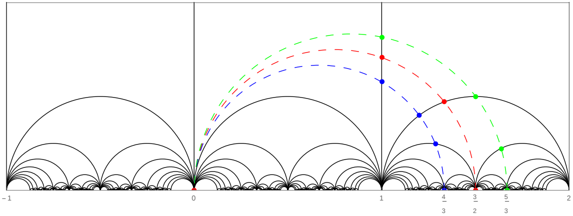
Asymptotics and distribution of the classical complexity





Complexity of Knot States

Geometric interpretation



Farey tesselation: arcs connect the Farey neighbors m/n and p/q,

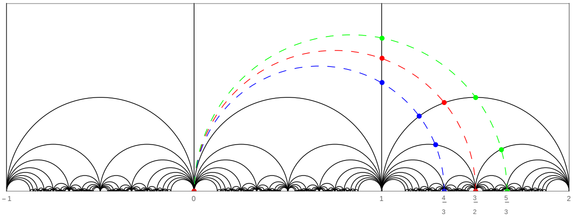
$$|mq - np| = 1$$

Each arc is an action of $T^a S$. In the hyperbolic geometry

- Curved triangles in the Farey tesselation have unit area
- Regularized area under the arc is proportional to the diameter

Complexity of Knot States

Geometric interpretation



- The complexity is a weighted length of the shortest path connecting ∞ and $\frac{m}{n}$, larger than #steps
- it asymptotically approaches the distance from the origin (the area under the arc connecting 0 and $\frac{m}{n}$)

Similar behavior of the holographic subregion complexity

$$\mathcal{C}(A) = \frac{\mathrm{vol}(\gamma_A)}{8\pi L G_N}$$

[Alishahiha'15]

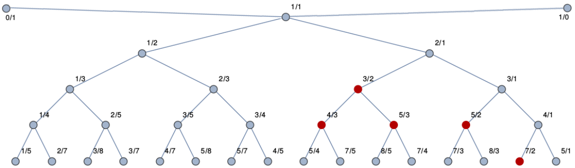
Complexity of Knot States

Geometric interpretation

Given a continued fraction $m/n = [b_1, \ldots, b_r]$

$$C_{m,n} \equiv d_{Caley} = \sum_{i=1}^{r}(b_i + 1) = d_{\mathrm{Farey}} + d_{\mathrm{Farey}^*}$$

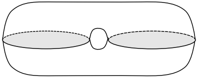
- r is the length of the shortest path
- The arc connecting m/n with 0 intersects Farey graph $\sum_i b_i$ times. This is the length of the path on the dual (Stern-Brocot) graph



Conclusions

- I introduced a TQFT interpretation of quantum entanglement

- Complicated measures of entanglement are easy to evaluate in TQFT

$S = \log$ []

- TQFT suggests an intuitive way to classify entaglement patterns

Conclusions

- Story of complexity was another investigation of TQFT states as quantum resources

$$\mathcal{C} = \min_{\{a_1, a_2, \ldots, a_n\}} \sum_{i=1}^{r} (|a_i| + 1)$$

- More recent work on complexity in TQFT in [Fliss'20,Leigh,Pai'20]
- Future directions: Quantum gravity looks like a possible field of application