Asymptotics of the classical and quantum 6j symbols

Bruce Bartlett
(j/w Hosana Ranaivomanana)

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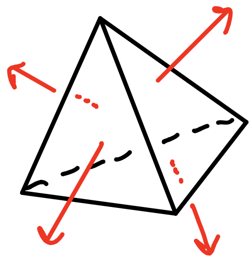

TQFT club seminar, Instituto Superior Técnico, 21 May 2021

1. The 6j symbols

Recall that up to isomorphism, there is a unique irreducible rep V_j of $SU(2)$ of dimension $2j+1$ for each $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$.

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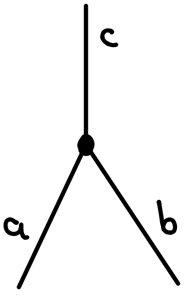
$$\mathrm{Hom}_{\mathrm{Rep}\,SU(2)}(V_c, V_a \otimes V_b) = \begin{cases} \mathbb{C} \cdot f^c_{ab} & \text{if } |a-b| \leq c \leq a+b \\ 0 & \text{otherwise} \end{cases}$$

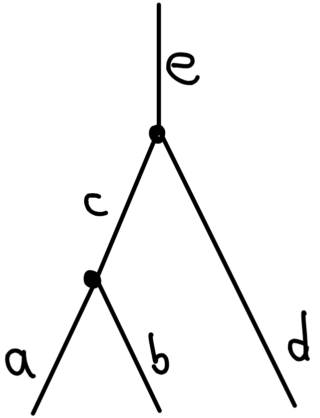
$$V_c \xrightarrow{f^c_{ab}} V_a \otimes V_b \quad \overset{\text{graphical calculus}}{\rightsquigarrow}$$



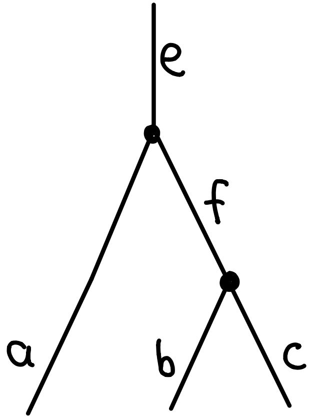
So there are two natural bases for

$$\mathrm{Hom}_{\mathrm{Rep}\,SU(2)}\left(V_e\,,\ V_a \otimes V_b \otimes V_d\right)$$

and the classical <u>6j symbols</u> are defined as the <u>change-of-basis coefficients</u>:



$$= \sum_f \underbrace{\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}}_{\in\ \mathbb{R}}$$



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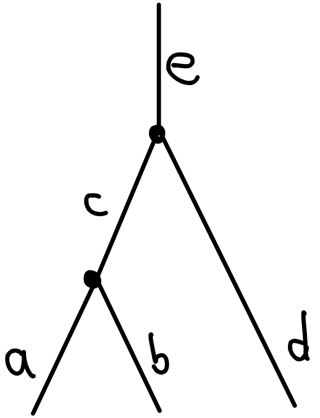

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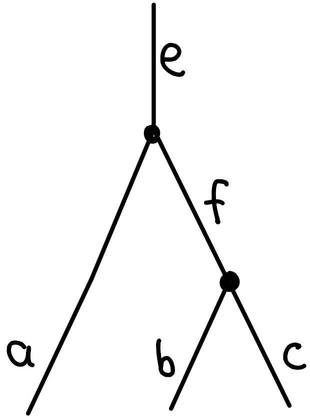
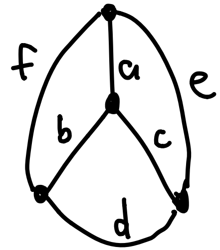


$$\boxed{\text{Mathematically, the 6j symbols encode the \underline{associator} on } \mathrm{Rep}\,SU(2).}$$

From this definition, it is not hard to see that the 6j symbol can be computed via string diagrams as the "Mercedes graph":

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{pmatrix} \text{normalization} \\ \text{factors} \end{pmatrix} \times$$



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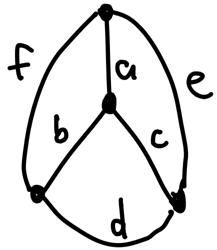

(Racah, 1942)

$$= \Delta(a,b,c)\,\Delta(c,d,e)\,\Delta(a,e,f)\,\Delta(b,d,f) \sum_n \frac{(-1)^n (n+1)!}{(n-a-b-c)!\,(n-c-d-e)! \cdots (b+c+e+f-n)!}$$

$$\Delta(a,b,c) = \sqrt{\frac{(a+b-c)!\,(a-b+c)!\,(-a+b+c)!}{(a+b+c+1)!}}$$

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$$|j;m\rangle, \qquad m = j, j-1, \dots, -j$$

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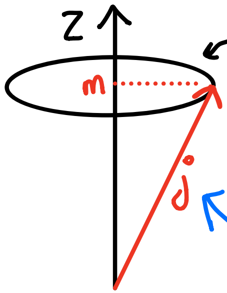
Each basis vector $|j;m\rangle$ is to be thought of as the quantum-mechanical avatar of an unknown vector $\vec{J}$ in $\mathbb{R}^3$ with magnitude j and z-component m.



$\vec{J}$ "is" somewhere on this circle.

actually, the magnitude is $\sqrt{j(j+1)}$

This gives a <u>physical interpretation</u> of the 6j symbol

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{Bmatrix}$$

as the <u>amplitude</u> for three quantum-mechanical vectors with magnitude j_1, j_2 and j_3 to combine to give vectors with magnitude j_{12}, j_{23}, j_{123}:

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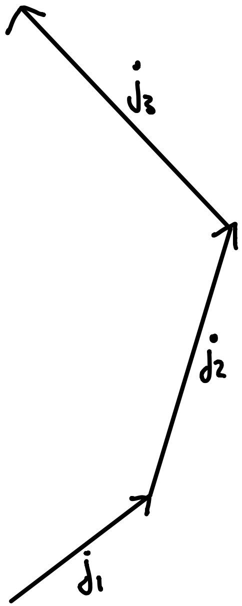

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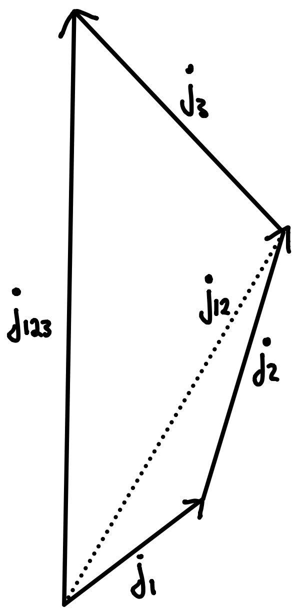

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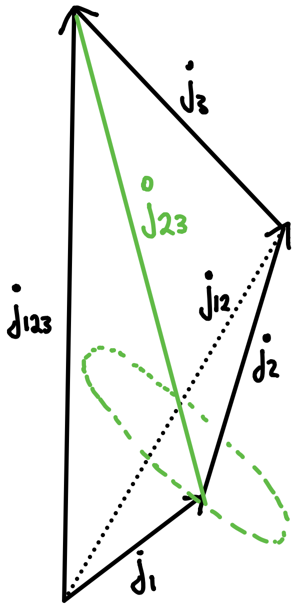

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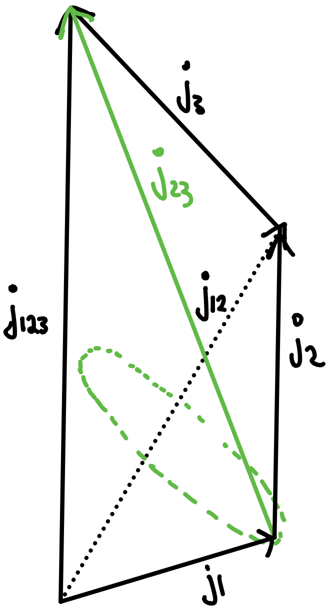

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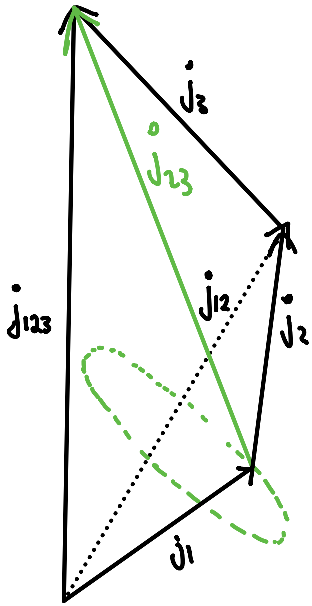

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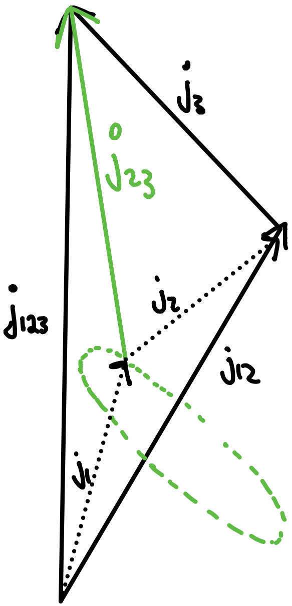

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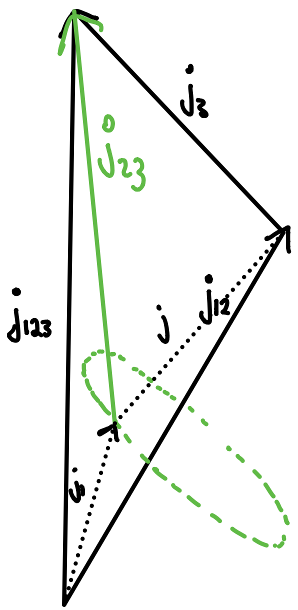

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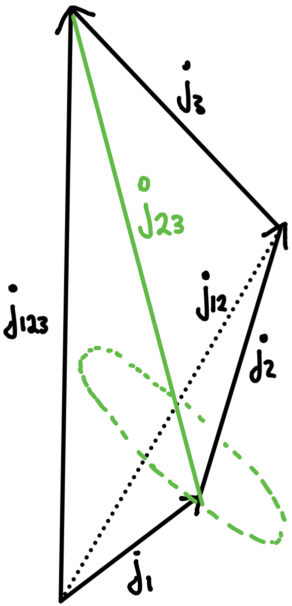

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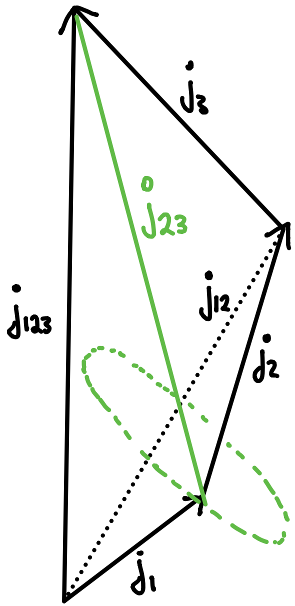

<u>Wigner (1959)</u> All vectors $\vec{J}_{23}$ on the green circle should be uniformly probable.

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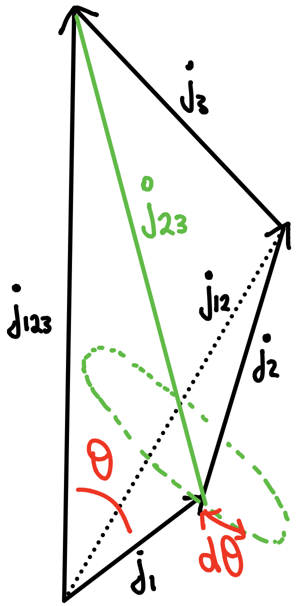

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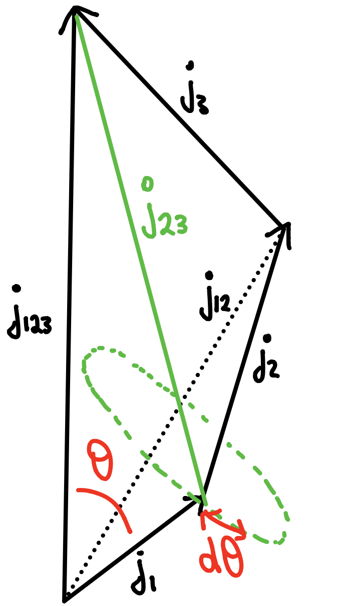

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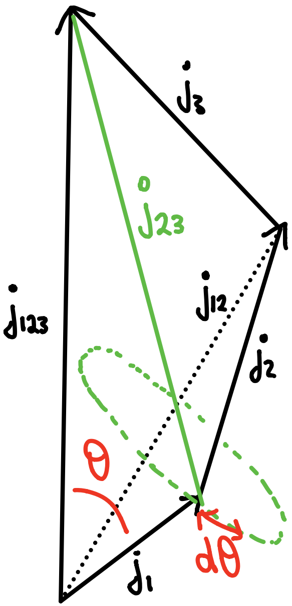

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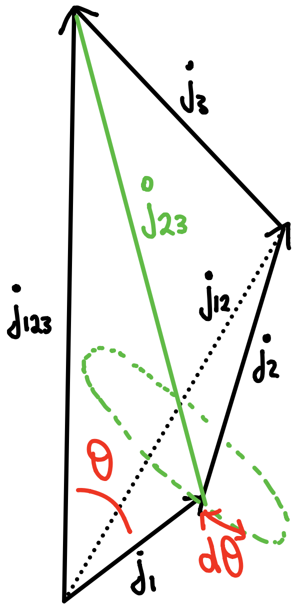

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Wigner derivative

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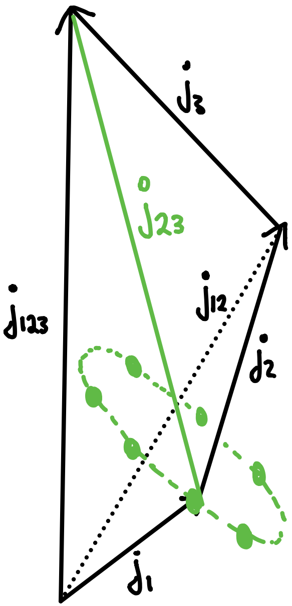

<u>Wigner (1959)</u> For large j's,

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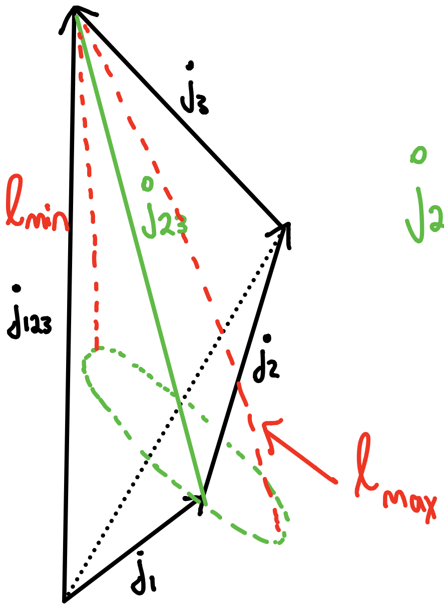

$$j_{23} \in \left(\mathbb{Z} \text{ or } \mathbb{Z}+\tfrac{1}{2}\right) \cap [\ell_{min}, \ell_{max}]$$

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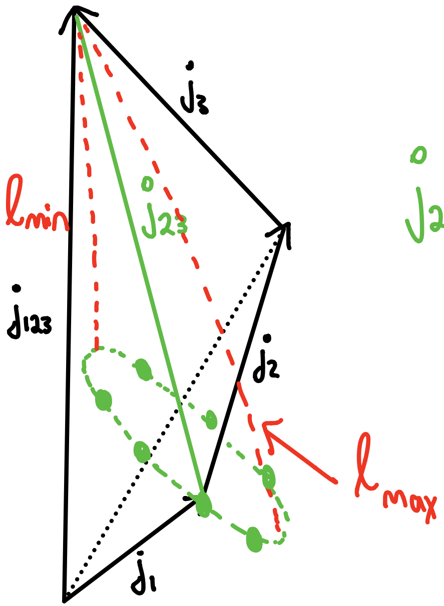

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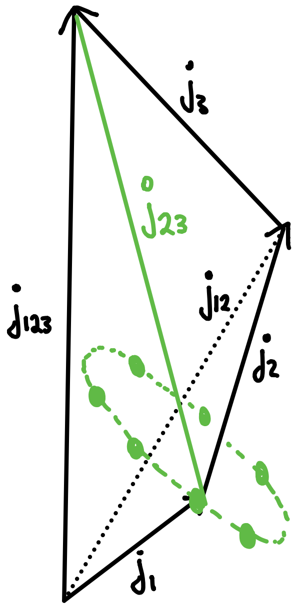

<u>Ponzano-Regge (1968)</u> For large $\mathring{j}$'s,

$$\begin{Bmatrix} \mathring{j}_1 & \mathring{j}_2 & \mathring{j}_{12} \\ \mathring{j}_3 & \mathring{j}_{123} & \mathring{j}_{23} \end{Bmatrix}^2 \sim \frac{1}{12\pi V} \cos^2\left(\sum_e \mathring{j}_e \theta_e + \frac{\pi}{4} \right)$$

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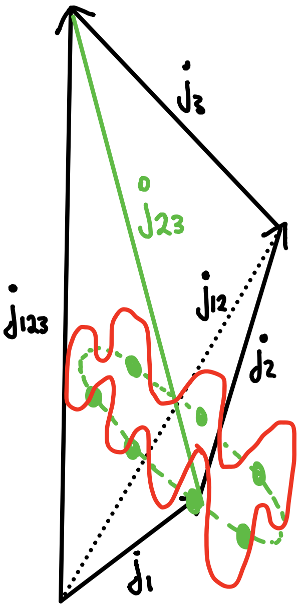

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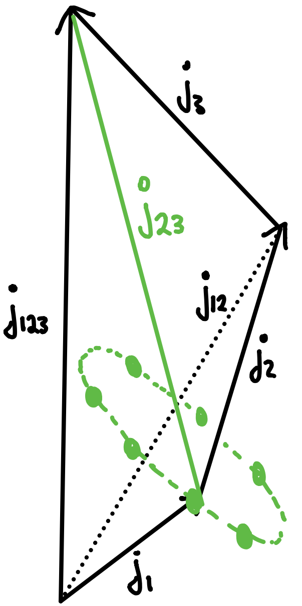

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edges of the tetrahedron

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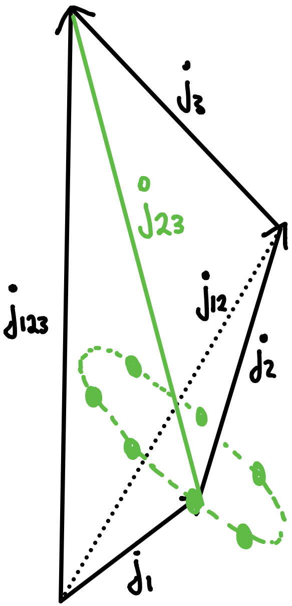

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length of edge

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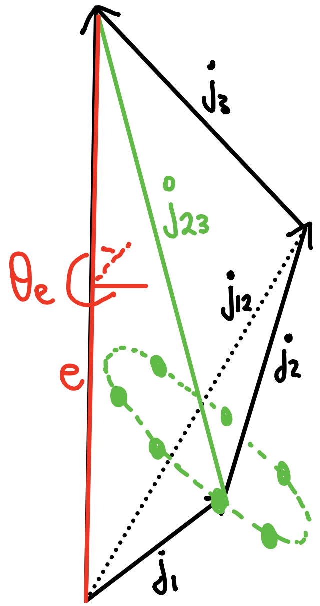

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exterior dihedral angle at edge

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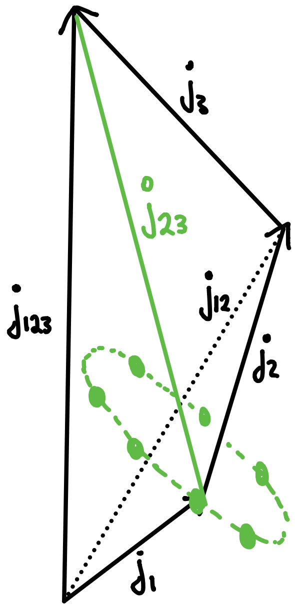


<u>Ponzano-Regge (1968)</u> For large j's,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{Bmatrix}^2 \sim \frac{1}{12\pi V} \cos^2\left(\sum_e j_e \theta_e + \frac{\pi}{4}\right)$$

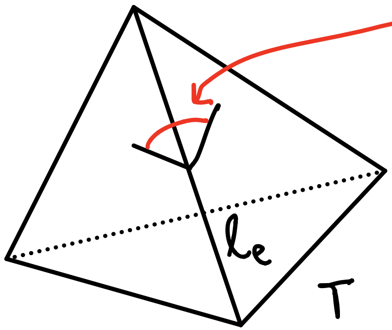
Proved by Roberts (1999) via <u>geometric quantization</u>.

2. The Ponzano-Regge model

Wait!

Regge (1961) In a Riemannian 3-manifold (M,g) constructed by gluing tetrahedra together along their faces,

$$\int_M R\,\mathrm{vol} = 2\sum_{\text{edges } e} \ell_e \sum_{\substack{\text{tetrahedra } T\\ \text{incident to } e}} \underbrace{\left(\pi-\theta_{T,e}\right)}_{=\ \text{interior dihedral angle at edge } e \text{ in } T}$$



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Ponzano-Regge model

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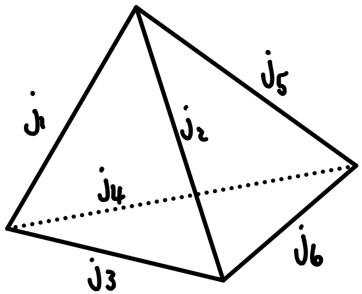

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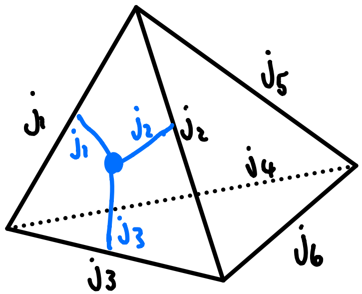

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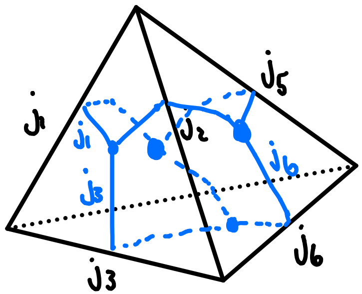

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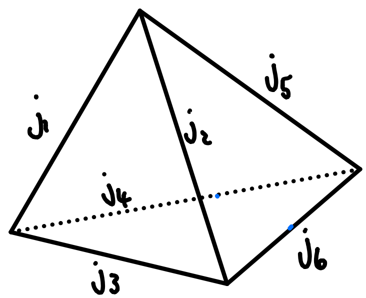

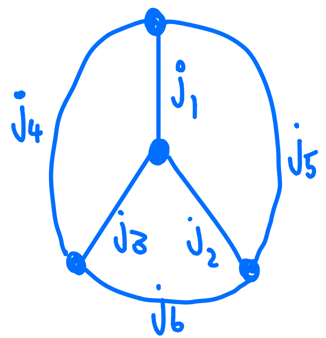

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a sum over all assignments of spins (i.e. irreps of SU(2)) to the edges of the triangulation

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$$= \text{“} \int_{\substack{\text{metrics } g \\ \text{on } M}} \mathcal{D}g \; e^{i \int_M R \, \mathrm{vol}_g} \text{”}$$

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$$\boxed{\text{Problem: for many triangulated 3-manifolds, this sum diverges.}}$$

This discrete sum over irreps of $SU(2)$ assigned to the edges of the polyhedral decomposition Δ can be rewritten as an integral over all connections (group elements assigned to the edges) on the <u>dual</u> polyhedral decomposition Δ^*!

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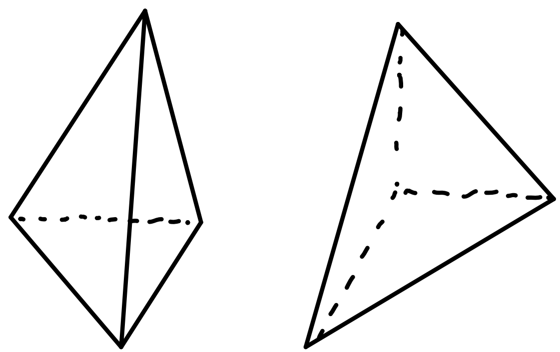

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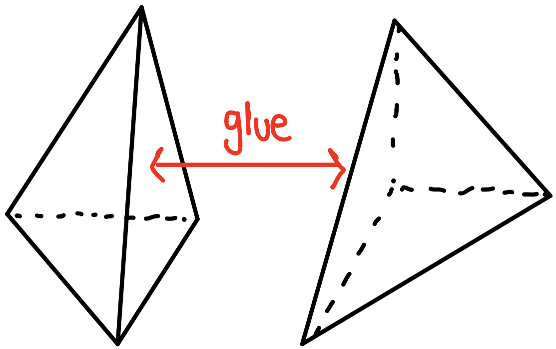

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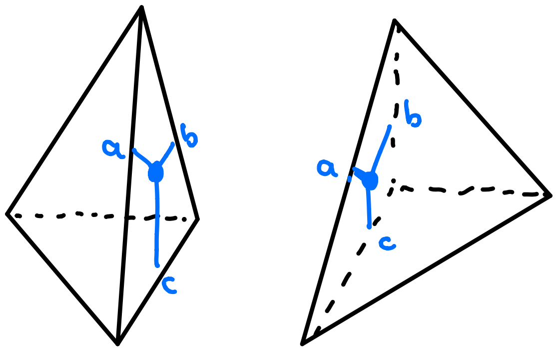

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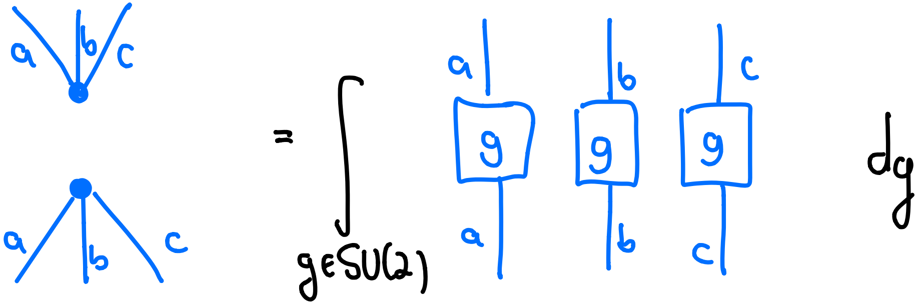

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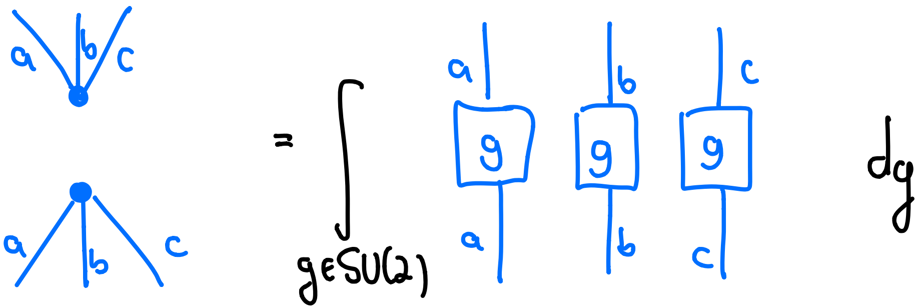

This is simply the well-known representation theory identity that the projection $P : V \longrightarrow V$ of a representation onto its trivial subspace is given by:

$$= \frac{1}{|G|} \sum_{g \in G} \rho_V(g)$$

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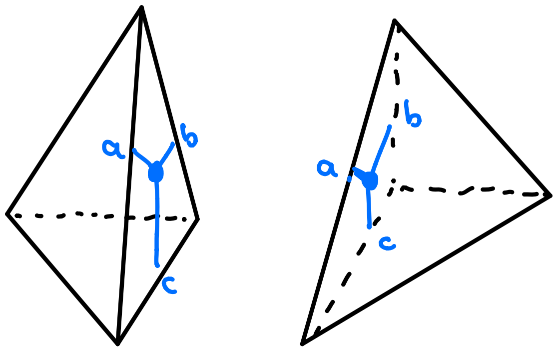

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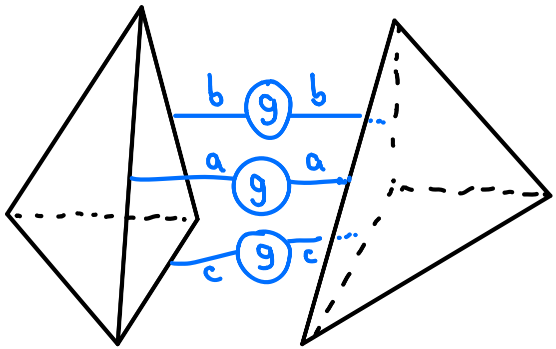

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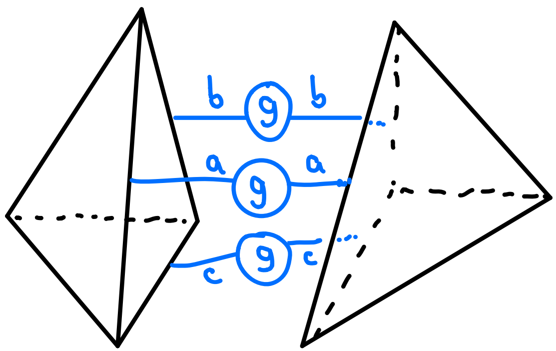

Use:

$$\sum_a \dim(V_a)\, \chi_a(g)$$

$$= \delta(g)$$

We obtain:

$$Z_{\text{Ponzano-Regge}} = \int_{\{g_e,\ e\in \text{edges}(\Delta^*)\}} \prod_{f\in \text{faces}(\Delta^*)} \delta(\text{holonomy around } f)\ d\{g_e\}$$

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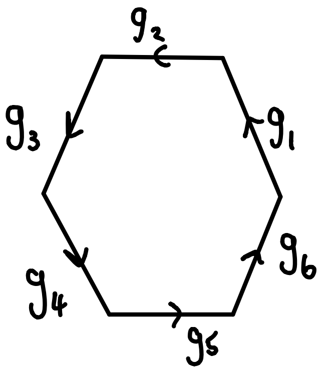

$$\text{holonomy} = g_6 g_5 g_4 g_3 g_2 g_1$$

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(Barrett and Naish-Guzman, 2008)

$$= \int_{[\rho] \in \text{Hom}(\pi_1 M, G)/G} R_{[\rho]}$$

Reidemeister torsion

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$$\left(\begin{array}{l}\text{Barrett and}\\ \text{Naish-Guzman, 2008}\end{array}\right) = \int_{[\rho]\in \mathrm{Hom}(\pi_1 M,\, G)/G} R_{[\rho]}$$

Reidemeister torsion

providing $H^2(\Delta^*, [\rho])$ vanishes for all $[\rho]$.

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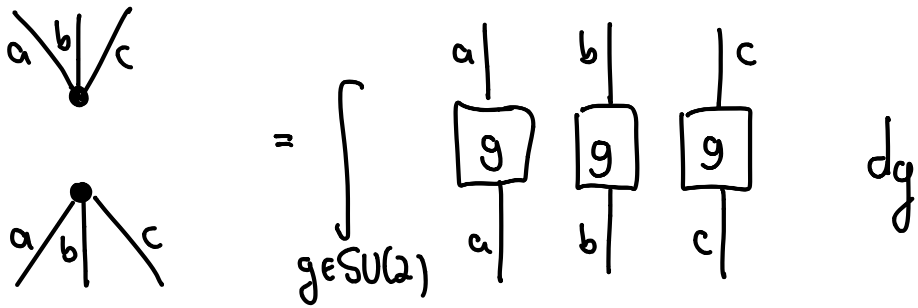

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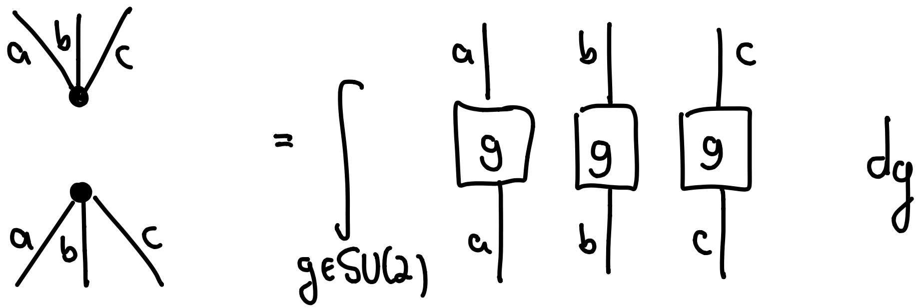

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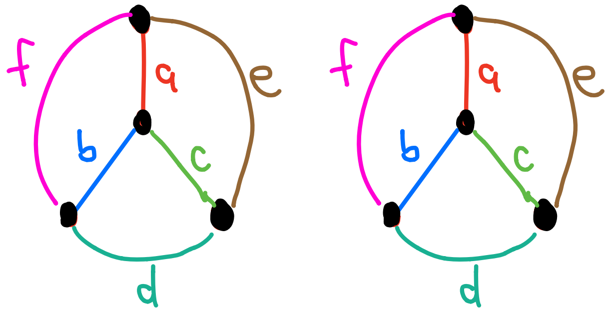

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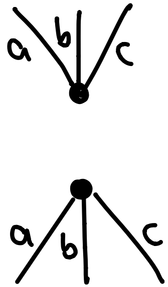

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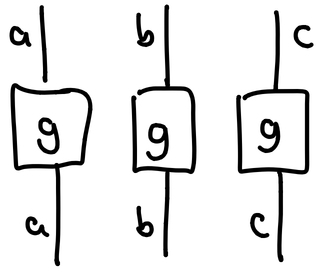

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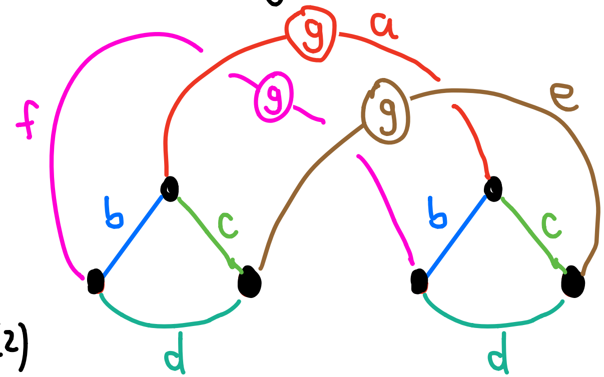

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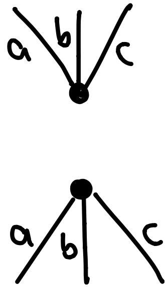

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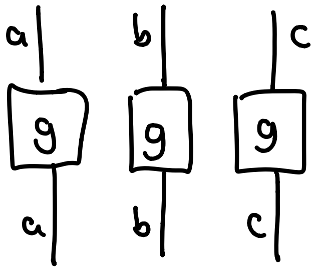

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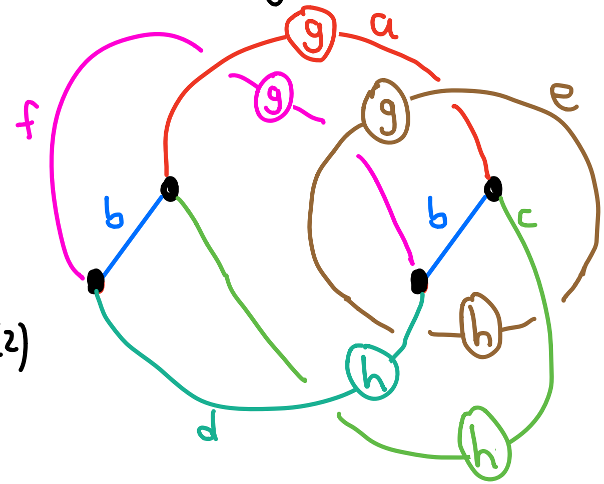

$$dg\,dh$$

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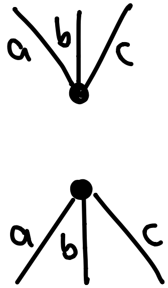

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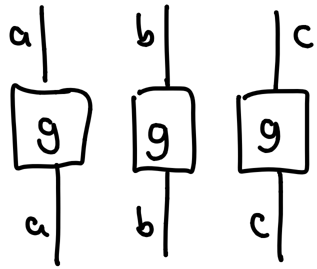

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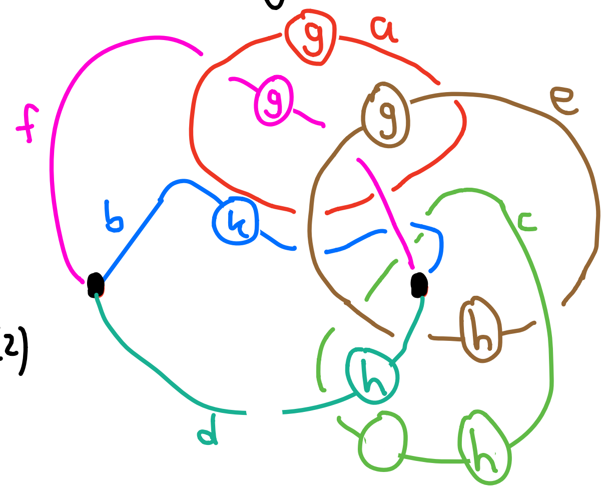

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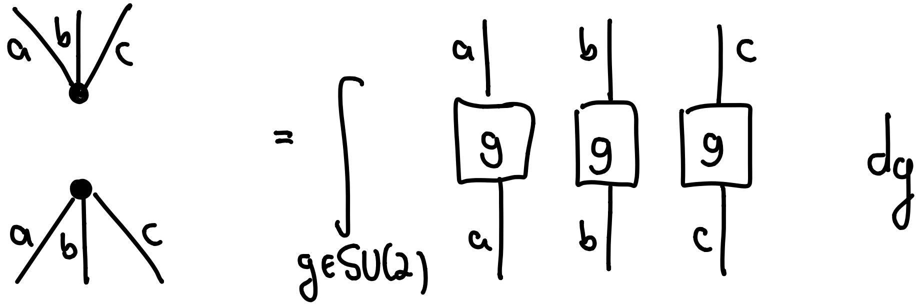

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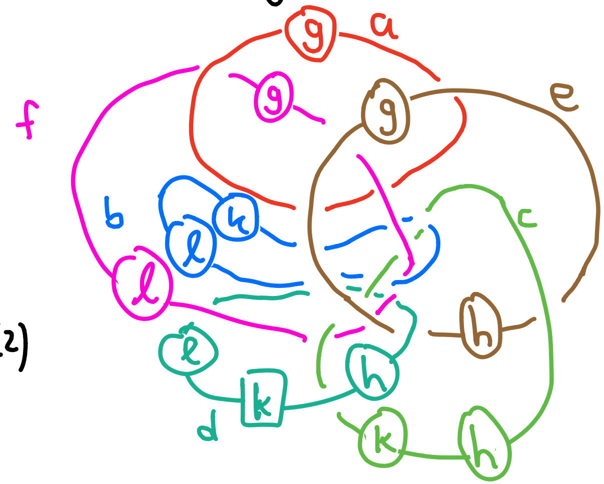

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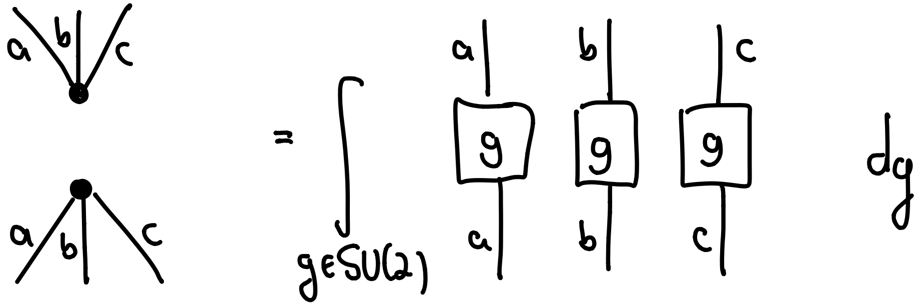
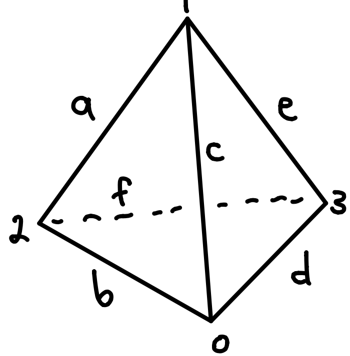


allows us to compute the square of the 6j symbol as a group integral:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2 = \int_{SU(2)^4} \prod_{i<j} \chi_{m_{\overline{ij}}}\left(g_i g_j^{-1}\right) dg_0\, dg_1\, dg_2\, dg_3$$



where $m_{12} = a$, $\overline{23} = 12$ etc.

2. The quest for a lattice gauge theory description of the Turaev-Viro model

In 1989, Witten made the celebrated assertion that the Jones polynomial of a knot could be calculated as a <u>path integral</u>:

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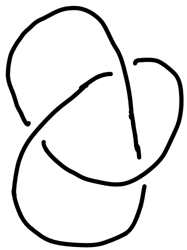

$K \subseteq S^3$

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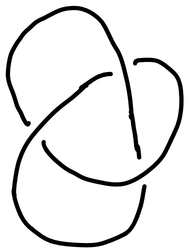

$K \subseteq S^3$

$$\underbrace{V_K}_{\text{Jones polynomial of } K,\ \in \mathbb{Z}[q,q^{-1}]}$$

$= q + q^3 - q^4$ for $K =$ trefoil above

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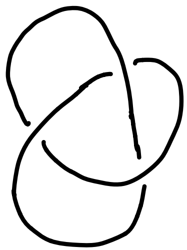

$K \subseteq S^3$

$$V_K\Big|_{q = e^{\frac{2\pi i}{k+2}}}$$

k an arbitrary positive integer

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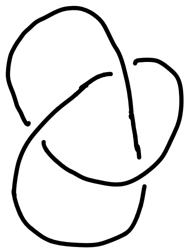

$K \subseteq S^3$

$$V_K \Big|_{q = e^{\frac{2\pi i}{k+2}}} = \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } S^3}} \mathrm{Tr}\left(\mathrm{Hol}_A(K)\right) e^{ikCS(A)} \, \mathcal{D}A$$

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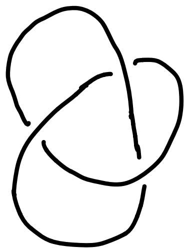

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Chern-Simons invariant of A

$$CS(A) = \frac{1}{4\pi}\int_{S^3} \mathrm{Tr}\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

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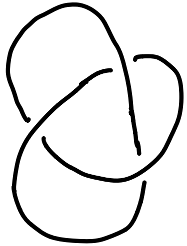

$K \subseteq S^3$

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holonomy of the connection around the knot $\in SU(2)$

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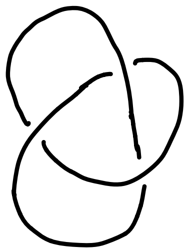

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$K \subseteq S^3$

$$\text{Nth coloured Jones polynomial}\Big|_{q = e^{\frac{2\pi i}{k+2}}} = \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } S^3}} \mathrm{Tr}\left(\rho\left(\mathrm{Hol}_A(K)\right)\right) e^{ikCS(A)} \, \mathcal{D}A$$

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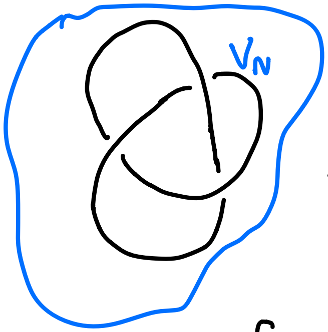
M



$K \subseteq M$

Knot invariant of K in M

$$= \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } M}} \mathrm{Tr}\Big(\rho\big(\mathrm{Hol}_A(K)\big)\Big)\, e^{ik\,CS(A)}\, \mathcal{D}A$$

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$$\int_{\substack{\text{all } SU(2) \text{ connections} \\ A \text{ on } M}} e^{ikCS(A)} \, \mathcal{D}A$$

giving a direct definition of this functional integral is the central question in mathematical physics

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this talk

$$\int_{\substack{\text{all } SU(2) \text{ connections} \\ A \text{ on } M}} e^{ikCS(A)}\, \mathcal{D}A \;:=\; \underbrace{RT_k(M)}_{\text{Reshetikhin-Turaev invariant of } M}$$

Reshetikhin-Turaev invariant of M

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Pick a triangulation Δ of M.

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$$TV_k(M) = \sum_{\{j\}} \prod_{\text{edges}} (-1)^{2j} \operatorname{qdim}(V_j) \prod_{\text{triangles}} (-1)^{j_1+j_2+j_3} \prod_{\text{tetrahedra}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_q$$

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a sum over labellings of edges of Δ by irreps of $U_q sl_2$

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the 6j symbols for $\mathrm{Rep}(U_q\mathfrak{sl}_2)$

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geometric: should use G itself, and irreps of G (not $U_q sl_2$) for knot invariants

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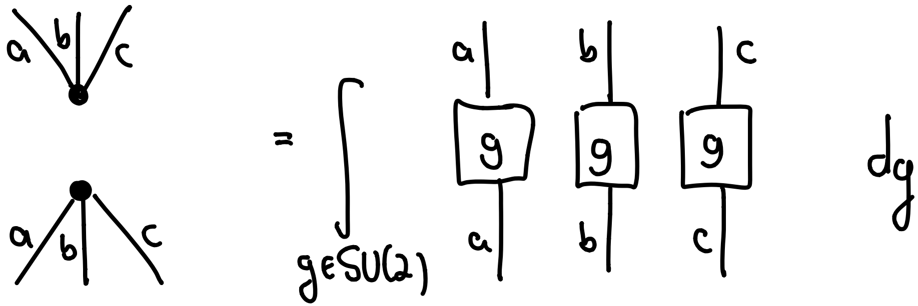
What we'd <u>really</u> like is a version of the "fusing identity"



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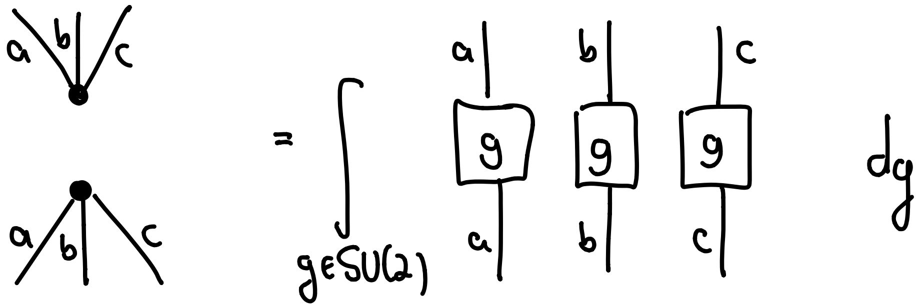

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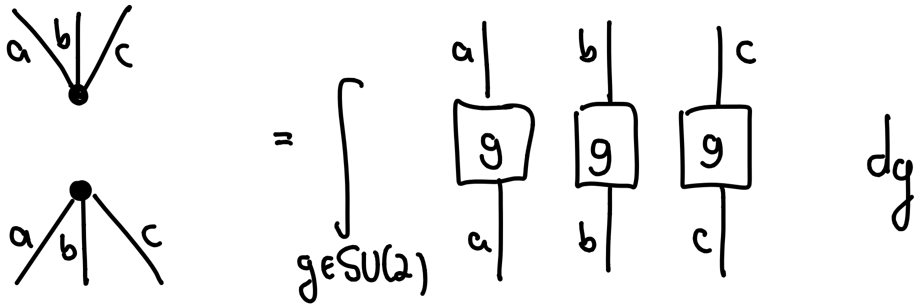
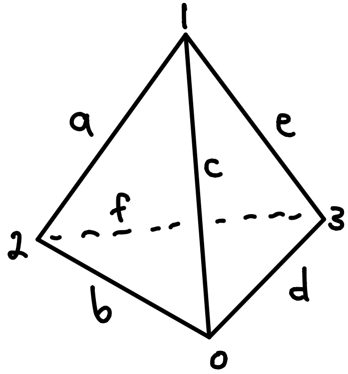


valid when V_a, V_b and V_c are irreps of $U_q sl_2$. Sadly, I don't know how to even formulate this.

Let's try something weaker.

Can we generalize the integral formula for the <u>classical</u> 6j symbols,

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2 = \int_{SU(2)^4} \prod_{i<j} \chi_{m_{\overline{ij}}}\left(g_i g_j^{-1}\right) dg_0\, dg_1\, dg_2\, dg_3$$



where $m_{12} = a$, $\overline{23} = 12$ etc.

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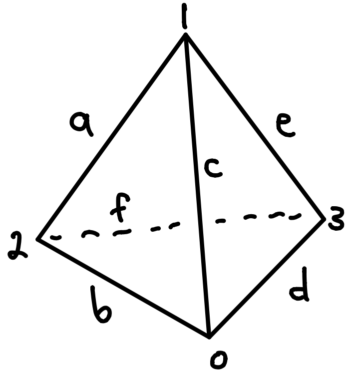

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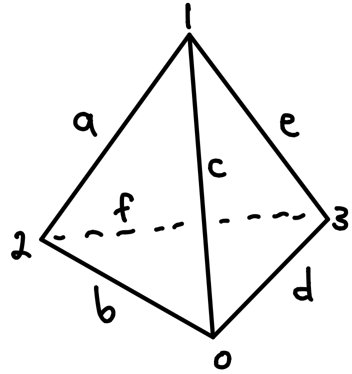

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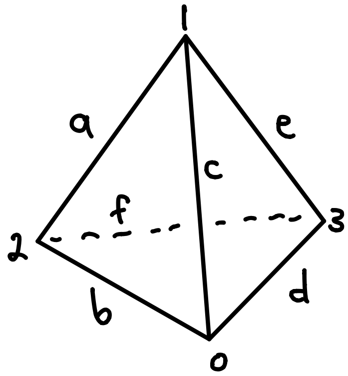

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?

3. The quest for an integral formula for the quantum 6j symbols

We need to figure out what "$CS(g_0, g_1, g_2, g_3)$" means.

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a class in here is a natural assignment for all manifolds M

$$\begin{array}{c} P \\ \downarrow \\ M \end{array} \longmapsto \text{class in } H^4(M;\mathbb{Z})$$

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2nd Chern class, generator

$$\begin{array}{c} P \\ \downarrow \\ M \end{array} \longmapsto \frac{1}{8\pi^2} \mathrm{Tr}(F \wedge F) \in H^{4 \; \text{integral}}_{De Rham}(M; \mathbb{R})$$

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$\downarrow$ M   Flat connection

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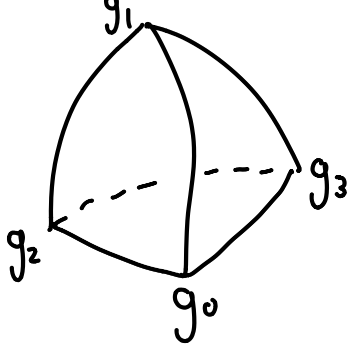
$$c_2 \longmapsto CS_2$$

$$CS_2(A) = \frac{1}{8\pi^2} \mathrm{Tr}\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right) \in H^3(M;\mathbb{R}) / H^3(M;\mathbb{Z})$$

Theorem (Cheeger-Simons 1985) In the bar resolution model for $H^3_{grp}(SU(2); \mathbb{R}/\mathbb{Z})$, CS_2 is given by the group 3-cocycle

$$\mathrm{vol}(g_0, g_1, g_2, g_3) := \text{volume of spherical tetrahedron in } S^3 \text{ with vertices at } g_0, g_1, g_2, g_3$$



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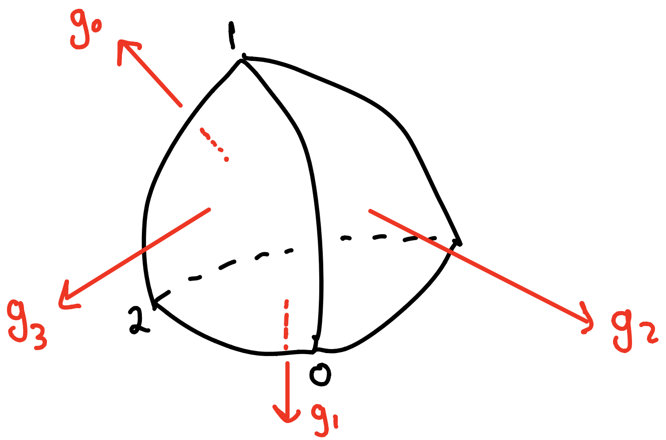
$T(g_0, g_1, g_2, g_3) =$ 

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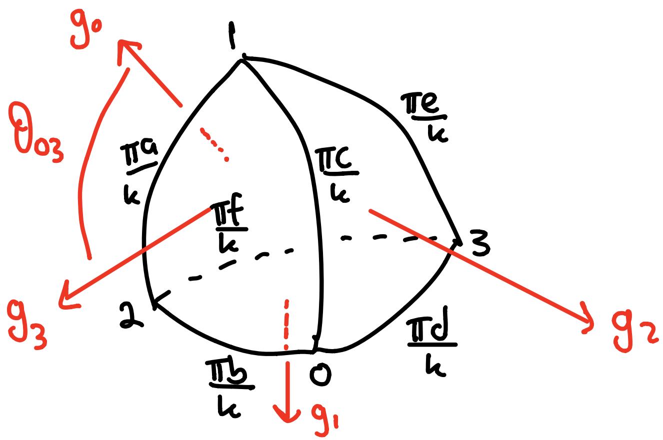
$T(g_0, g_1, g_2, g_3) =$ 

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I asked my PhD student Hosana Ranaivomanana to investigate this.

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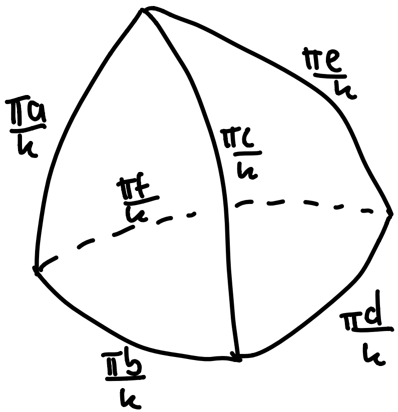

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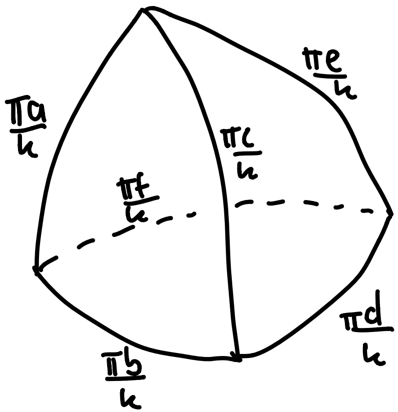
$\alpha = \frac{a}{k}$ (fixed ratio)

etc.

Note: using integer spins convention now



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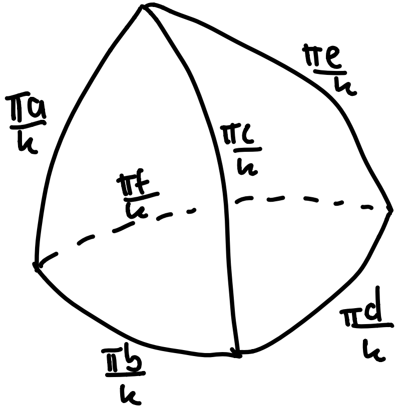
length of edge e



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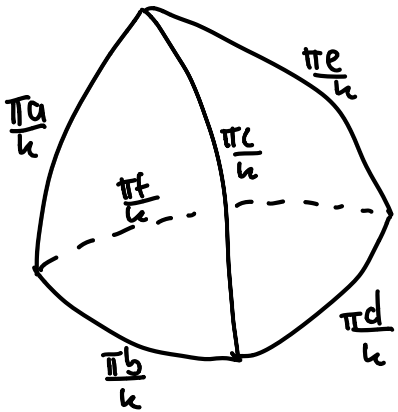
exterior dihedral angle at edge e



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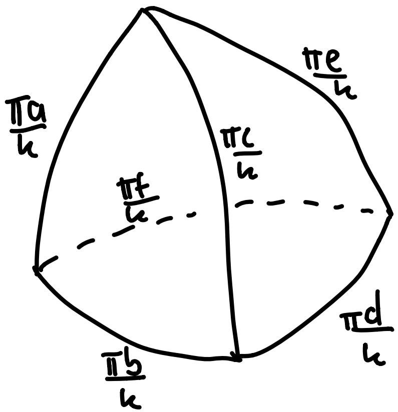
volume of the tetrahedron



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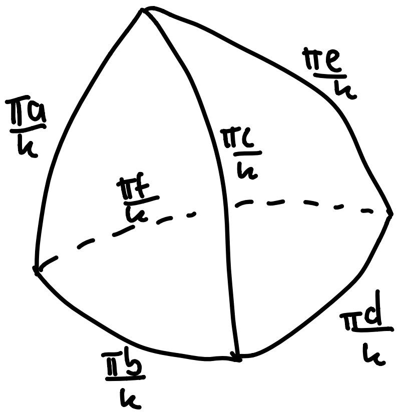
4×4 Gram matrix

$$G_{ij} = \cos(\ell_{ij})$$



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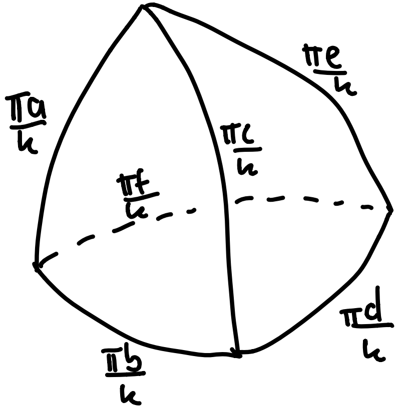
an error! should be a +



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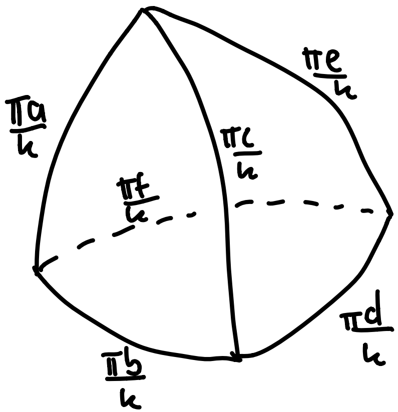
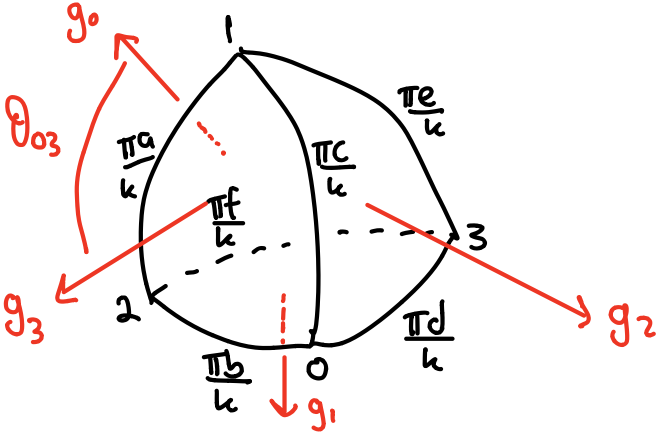


Does our integral match this asymptotic behaviour?

$$\left\{\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right\}^2_{q=e^{\frac{\pi i}{k+2}}} \cong \int_{SU(2)^4} \prod_{i<j} \frac{\sin\left((m_{ij}+1)\theta_{ij}\right)}{\sin\theta_{ij}}\, e^{\frac{2ik\,\mathrm{Vol}(T)}{\pi}}\, dg_0\, dg_1\, dg_2\, dg_3$$

$T(g_0, g_1, g_2, g_3) =$



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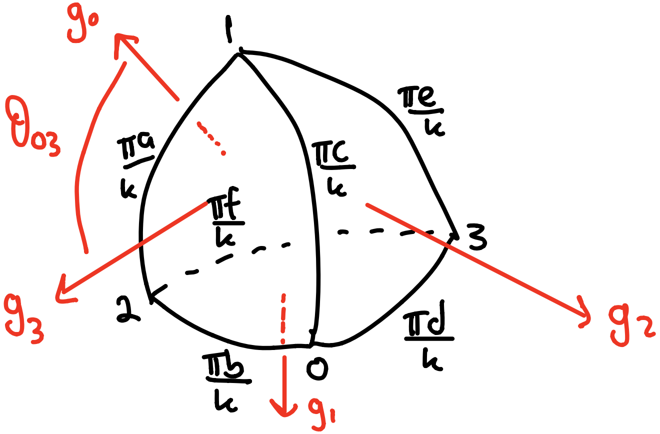
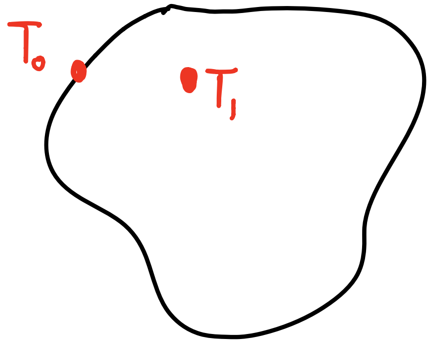
$T(g_0, g_1, g_2, g_3) =$ 



Space of tetrahedra in S^3

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}^2_{q=e^{\frac{\pi i}{k+2}}} \cong \int_{SU(2)^4} \prod_{i<j} \frac{\sin\left((m_{ij}+1)\theta_{ij}\right)}{\sin\theta_{ij}} \, e^{\frac{2ik\,\mathrm{Vol}(T)}{\pi}} \, dg_0\, dg_1\, dg_2\, dg_3$$

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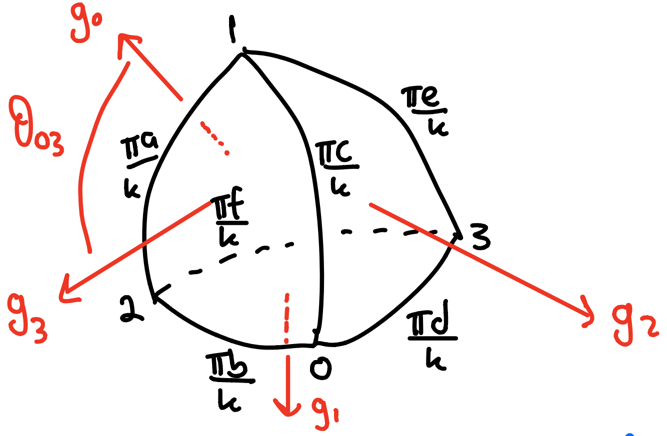

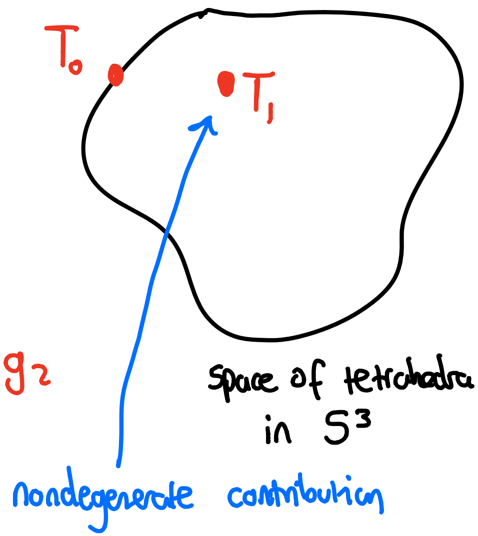

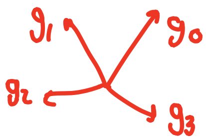

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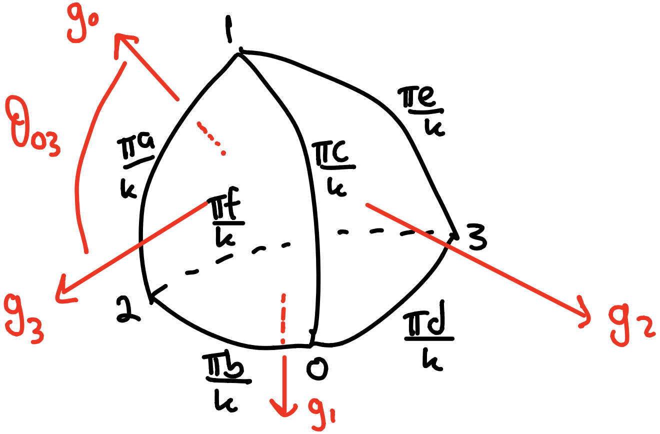

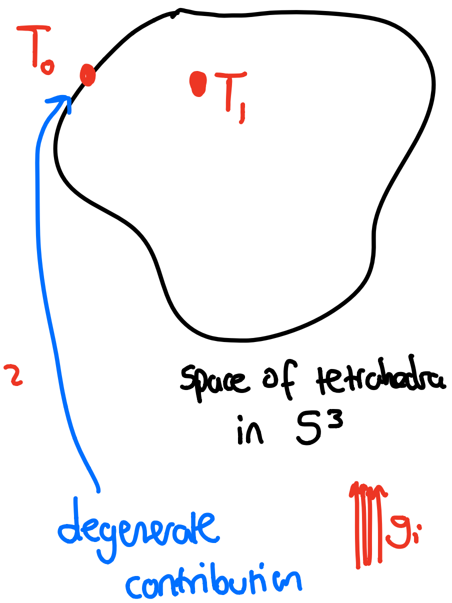

The known asymptotics of the square of the quantum 6j symbols (Taylor and Woodward's formula) can also be split up into two contributions:

$$\begin{Bmatrix} k\alpha & k\beta & k\gamma \\ k\delta & k\epsilon & k\zeta \end{Bmatrix}_{q=e^{\frac{\pi i}{k+2}}} \sim \sqrt{\frac{4\pi^2}{k^3\sqrt{\det G}}}\cos\left(\sum_e (k l_e + 1)\frac{\theta_e}{2} + \frac{k}{\pi}V + \frac{\pi}{4}\right)$$

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Hosana's results so far:

$$\int_{SU(2)^4} \prod_{i<j} \frac{\sin((m_{ij}+1)\theta_{ij})}{\sin\theta_{ij}} e^{\frac{2ik\,\mathrm{Vol}(T)}{\pi}} \sim \begin{pmatrix}\text{deg.} \\ \text{contribution}\end{pmatrix} - \frac{\pi^2}{4k^3\sqrt{\det G}} \cos\left(\sum_e (k l_e + 1)\theta_e + 2kV\right)\Bigg]$$

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See also:

Bartlett and Ranaivomanana, Reciprocity of the Wigner derivative for spherical tetrahedra (arXiv:2012.10609)