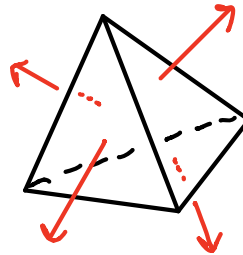


# Asymptotics of the classical and quantum symbols

Bruce Bartlett  
(j/w Hosana Ranaivomana)

$$\left\{ \begin{array}{ccc} \dot{j}_1 & \dot{j}_2 & \dot{j}_{12} \\ \dot{j}_3 & \dot{j}_{123} & \dot{j}_{23} \end{array} \right\}$$

$\sim$



TQFT club seminar, Instituto Superior Técnico, 21 May 2021

1. The  $b_j$  symbols



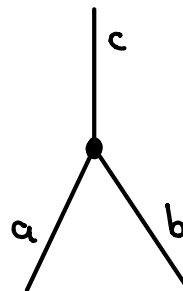
Recall that up to isomorphism, there is a unique irreducible rep  $V_j$  of  $SU(2)$  of dimension  $2j+1$  for each  $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ .

Recall that up to isomorphism, there is a unique irreducible rep  $V_j$  of  $SU(2)$  of dimension  $2j+1$  for each  $j=0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ . Moreover:

$$\text{Hom}_{\text{Rep } SU(2)}(V_c, V_a \otimes V_b) = \begin{cases} \mathbb{C} \cdot f_{ab}^c & \text{if } |a-b| \leq c \leq a+b \\ 0 & \text{otherwise} \end{cases}$$



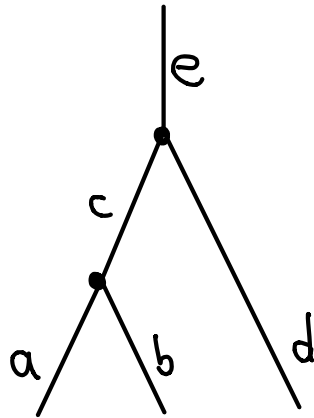
graphical calculus  
~~~~~→



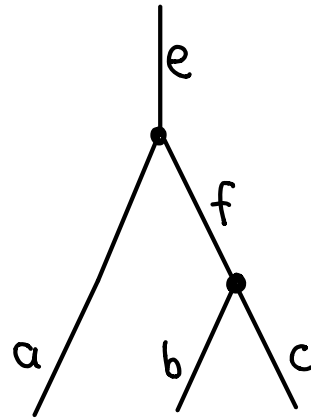
So there are two natural bases for

$$\text{Hom}_{\text{Rep SU}(2)}(V_e, V_a \otimes V_b \otimes V_d)$$

and the classical  $6j$  symbols are defined as the change-of-basis coefficients:



$$= \sum_f \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{\in \mathbb{R}}$$



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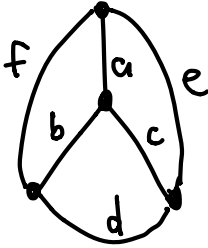
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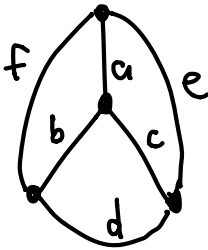
Mathematically, the  $6j$  symbols encode the associator on  $\text{Rep} \text{SU}(2)$ .

From this definition, it is not hard to see that the  $b_j^o$  symbol can be computed via string diagrams as the "Mercedes graph":

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \left( \begin{matrix} \text{normalization} \\ \text{factors} \end{matrix} \right) \times$$


The diagram is a graph with three vertices arranged in a triangle. Each vertex is connected to the other two by straight lines. Additionally, each vertex is connected to a central point by a straight line. The edges are labeled as follows: the top-left edge is labeled 'f', the top-right edge is labeled 'e', the bottom-left edge is labeled 'b', the bottom-right edge is labeled 'c', and the bottom edge is labeled 'd'. The top edge is labeled 'a'.

From this definition, it is not hard to see that the  $b_j^o$  symbol can be computed via string diagrams as the "Mercedes graph":

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$$\stackrel{\substack{\text{(Racah,} \\ \text{1942)}}}{=} \Delta(a,b,c) \Delta(c,d,e) \Delta(a,e,f) \Delta(b,d,f) \sum_n \frac{(-1)^n (n+1)!}{(n-a-b-c)! (n-c-d-e)! \dots (b+c+e+f-n)!}$$

$$\Delta(a,b,c) = \sqrt{\frac{(a+b-c)! (a-b+c)! (-a+b+c)!}{(a+b+c+1)!}}$$

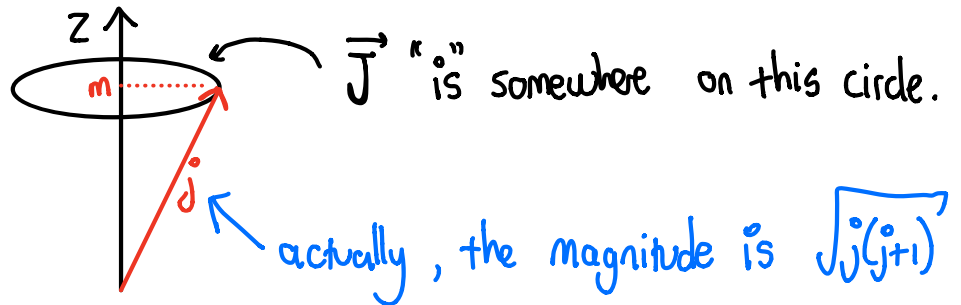
In physics, one thinks of  $V_j$  as spanned by

$$|j; m\rangle, \quad m = j^0, j^0-1, \dots, -j^0$$

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$$|j; m\rangle, \quad m = j, j-1, \dots, -j$$

Each basis vector  $|j; m\rangle$  is to be thought of as the quantum-mechanical avatar of an unknown vector  $\vec{J}$  in  $\mathbb{R}^3$  with magnitude  $j$  and z-component  $m$ .





This gives a physical interpretation of the  $6j$  symbol

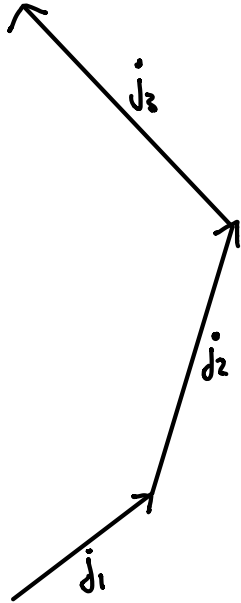
$$\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{pmatrix}$$

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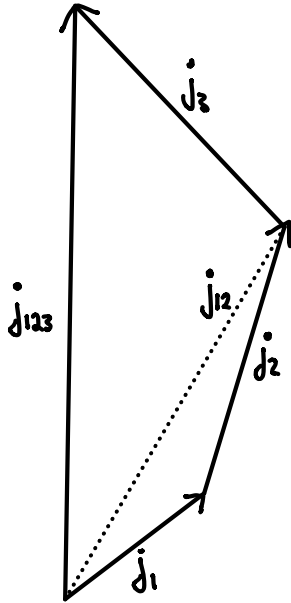
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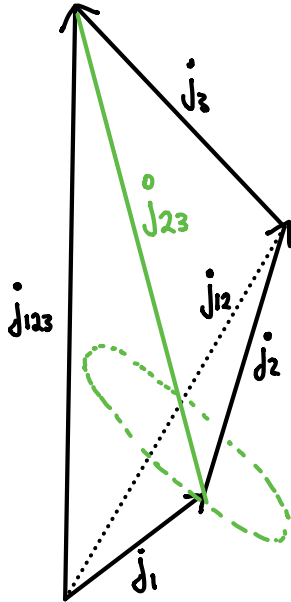
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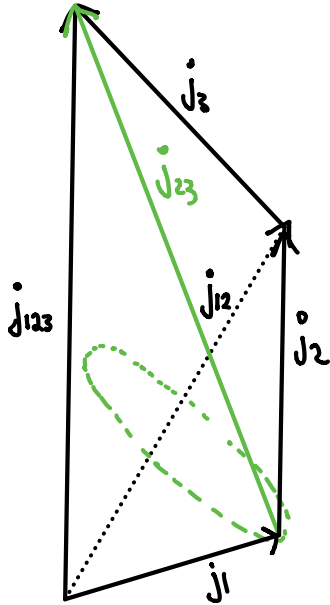
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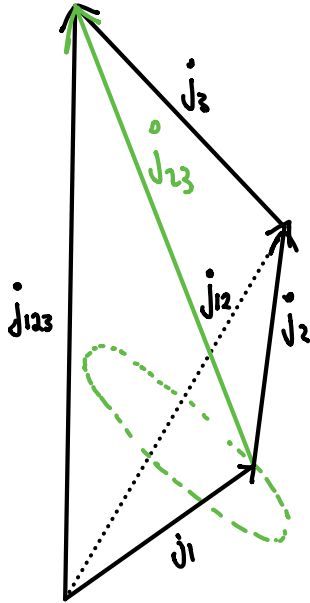
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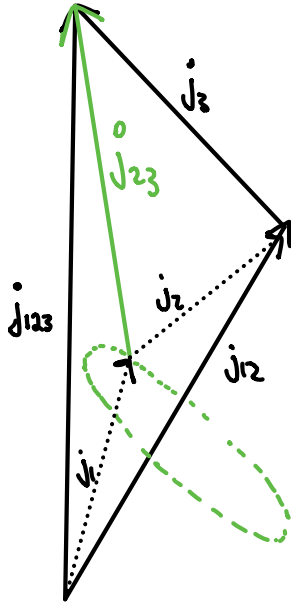
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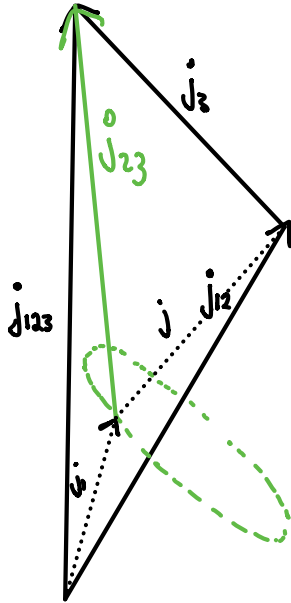
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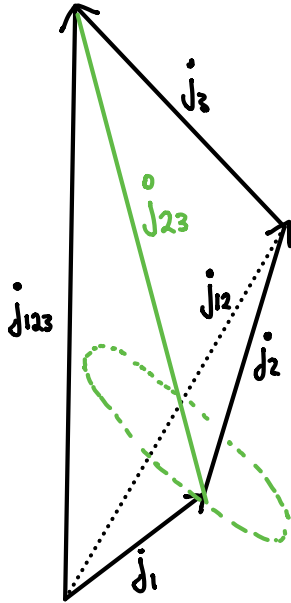




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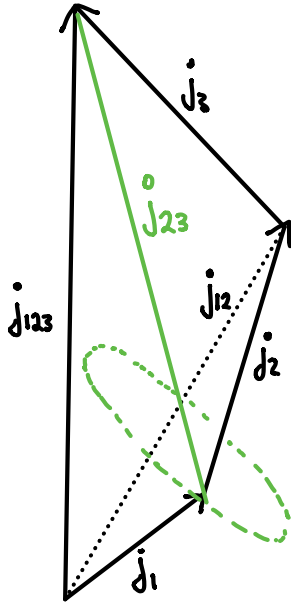
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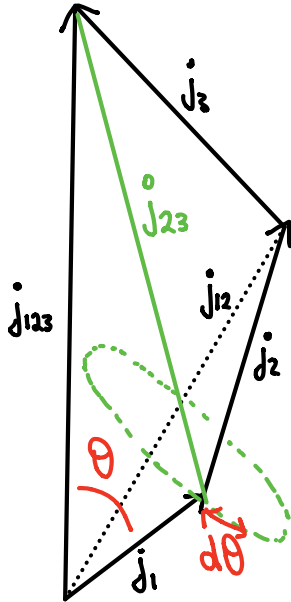


Wigner (1959) All vectors  $\vec{J}_{23}$  on the green circle should be uniformly probable.

This gives a physical interpretation of the  $b_j^\circ$  symbol

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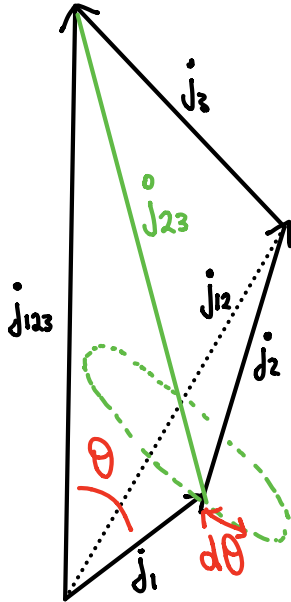
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i.e.  $\frac{1}{2\pi} d\theta = P(j_{23}^\circ) dj_{23}^\circ$

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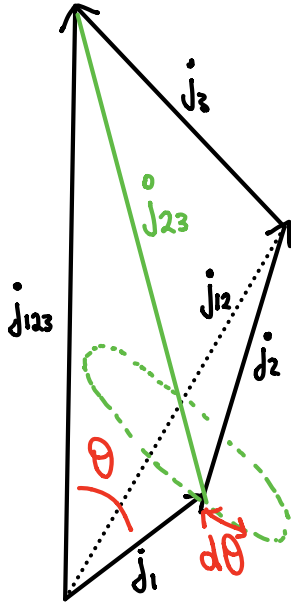
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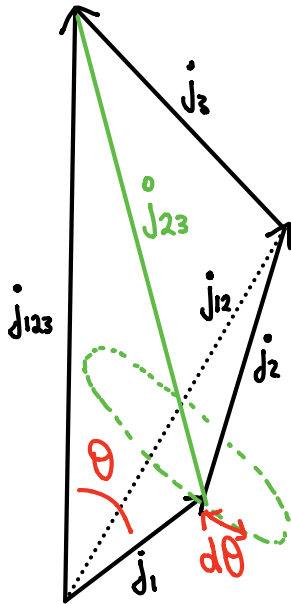
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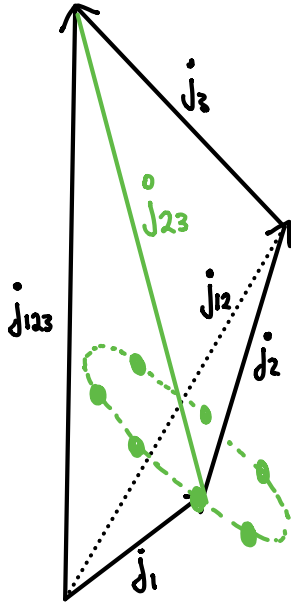
Wigner  
derivative

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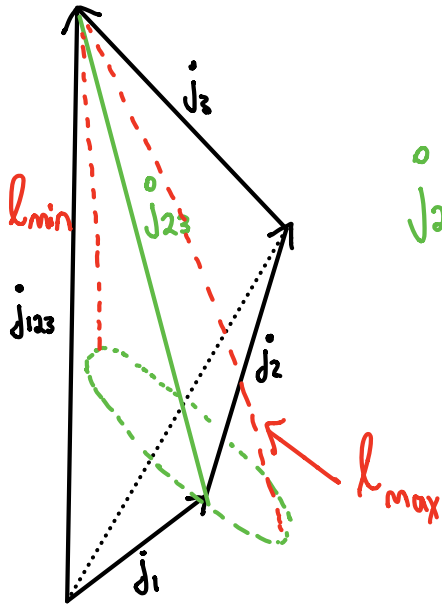
Wigner (1959) For large  $j$ 's,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{Bmatrix}^2 \sim \frac{1}{24\pi V}$$

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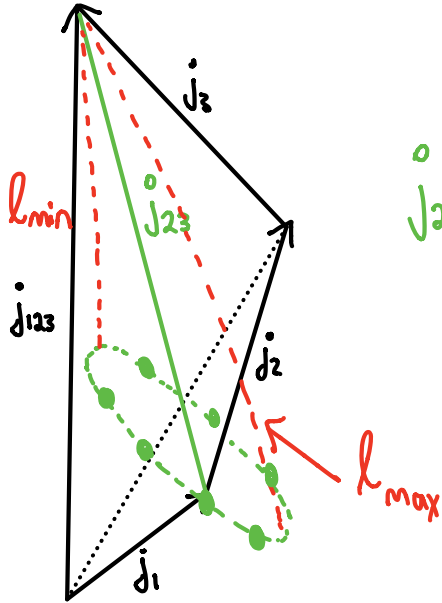
$$j_{23} \in \left( \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2} \right) \cap [l_{\min}, l_{\max}]$$



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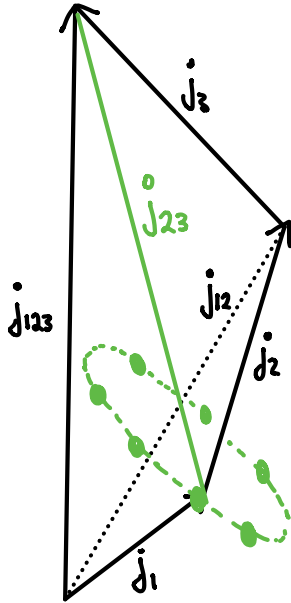


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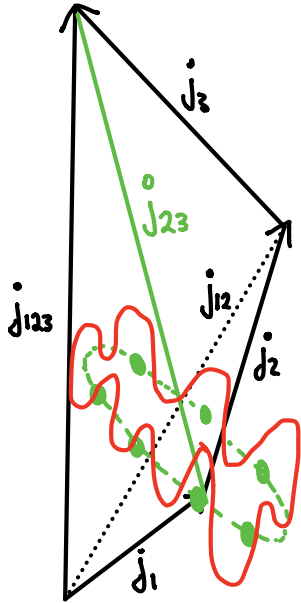
Ponzano-Regge (1968) For large  $j$ 's,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_{123} & j_{23} \end{Bmatrix}^2 \sim \frac{1}{12\pi V} \cos^2 \left( \sum_e j_e \theta_e + \frac{\pi}{4} \right)$$

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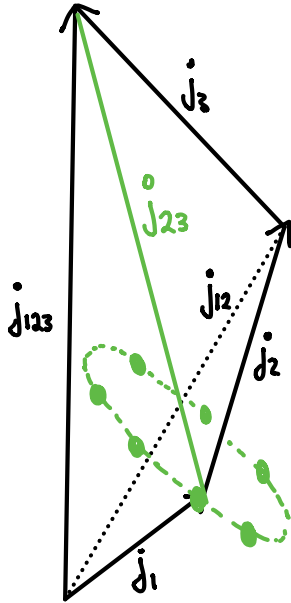
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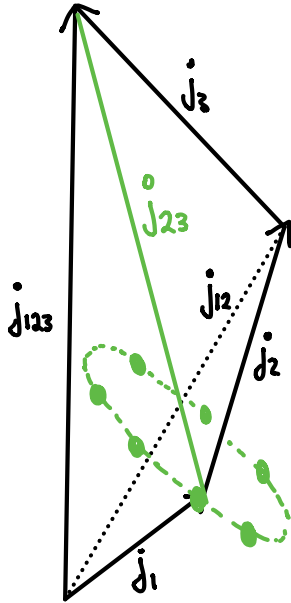
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edges of the tetrahedron

This gives a physical interpretation of the  $6j$  symbol

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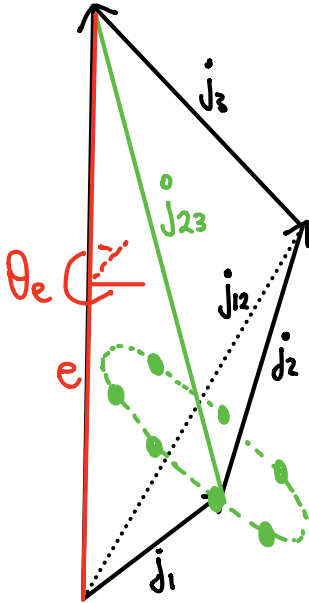
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length of edge

This gives a physical interpretation of the  $6j$  symbol

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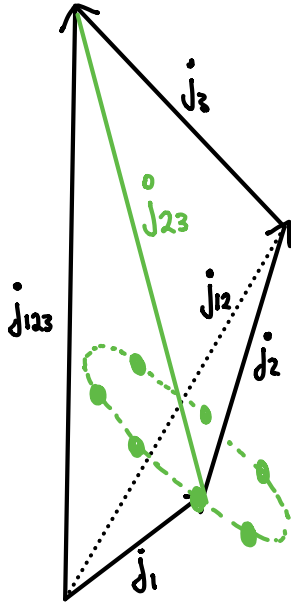
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exterior dihedral angle  
at edge

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Proved by Roberts (1999) via geometric quantization.

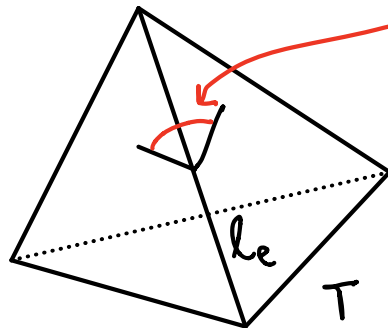
## 2. The Ponzano-Regge model



Wait!

Regge (1961) In a Riemannian 3-manifold  $(M, g)$  constructed by gluing tetrahedra together along their faces,

$$\int_M R \text{vol} = 2 \sum_{\text{edges } e} l_e \sum_{\substack{\text{tetrahedra } T \\ \text{incident to } e}} (\pi - \theta_{T,e})$$



= interior dihedral  
angle at edge  $e$   
in  $T$

This suggests that the quantum gravity idea of a "sum over all geometries" can be implemented in a discrete way in 3d, using the  $6j$  symbols for  $SU(2)$ !

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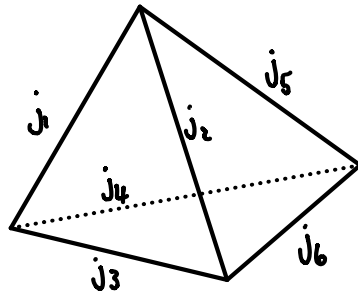
### Ponzano-Regge model

$$Z = \sum_{\{j\}} \prod_{\text{edges}} (-1)^{2j} (2j+1) \prod_{\text{triangles}} (-1)^{j_1 + j_2 + j_3} \prod_{\text{tetrahedra}} \begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix}$$

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### Ponzano-Regge model

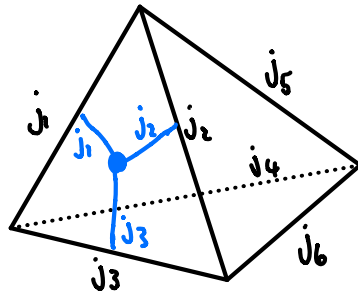
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This suggests that the quantum gravity idea of a "sum over all geometries" can be implemented in a discrete way in 3d, using the  $b_j$  symbols for  $SU(2)$ !

### Ponzano-Regge model

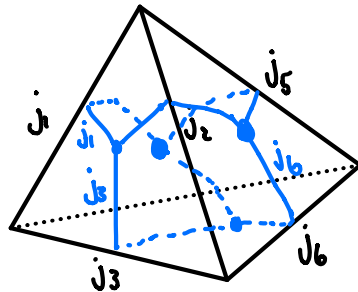
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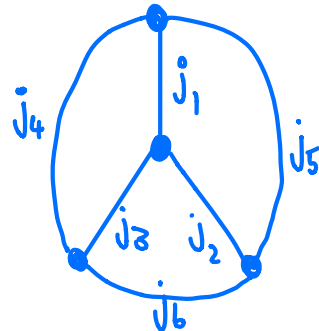
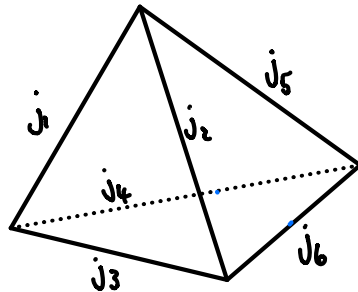
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a sum over all  
assignments of spins (i.e. irreps of  $SU(2)$ )  
to the edges of the triangulation

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$$= \int \mathcal{D}g \, e^{i \int_M R \, \text{vol}_g} \quad "$$

metrics  $g$   
on  $M$

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Ponzano-Regge model

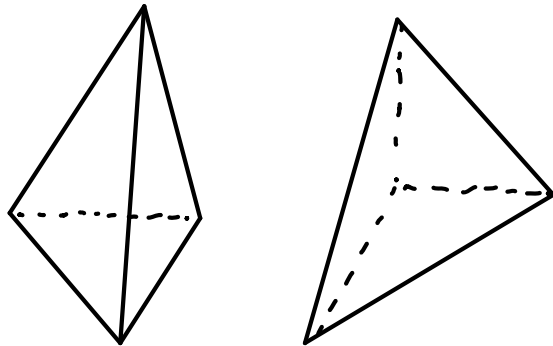
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Problem: for many triangulated 3-manifolds, this sum diverges.

This discrete sum over irreps of  $SU(2)$  assigned to the edges of the polyhedral decomposition  $\Delta$  can be rewritten as an integral over all connections (group elements assigned to the edges) on the dual polyhedral decomposition  $\Delta^*$  !

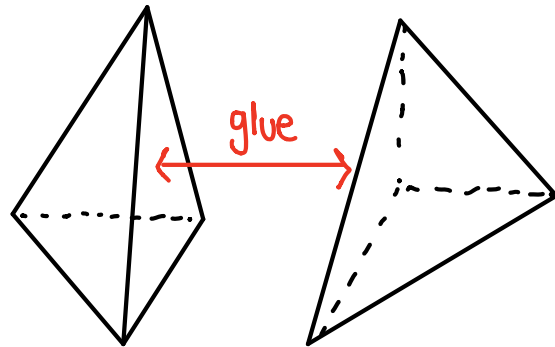
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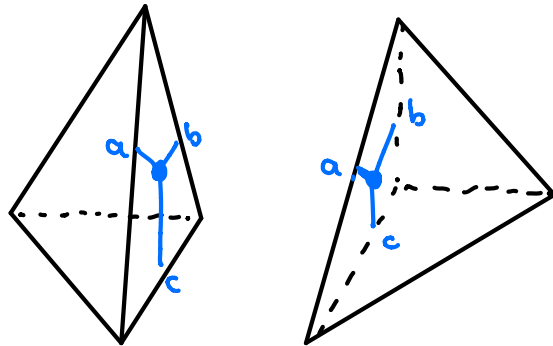
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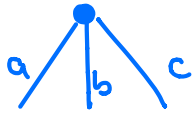
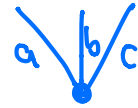


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Now use:



$= \int_{g \in \text{SU}(2)}$



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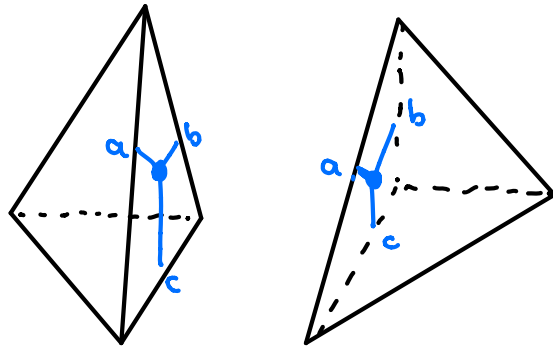
$$= \int_{g \in \text{SU}(2)} dg$$

This is simply the well-known representation theory identity that the projection  $P: V \rightarrow V$  of a representation onto its trivial subspace is given by:

$$= \frac{1}{|G|} \sum_{g \in G} P_v(g)$$

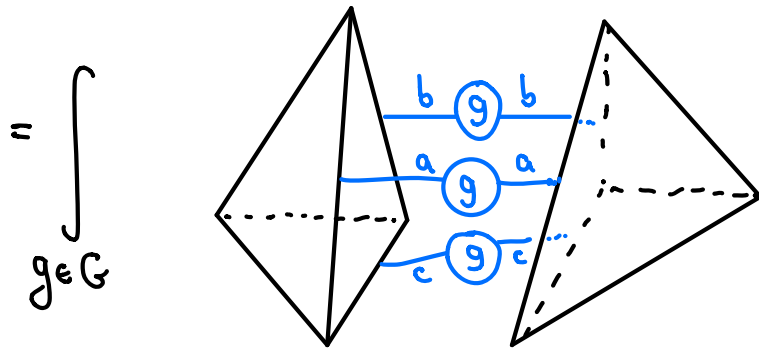
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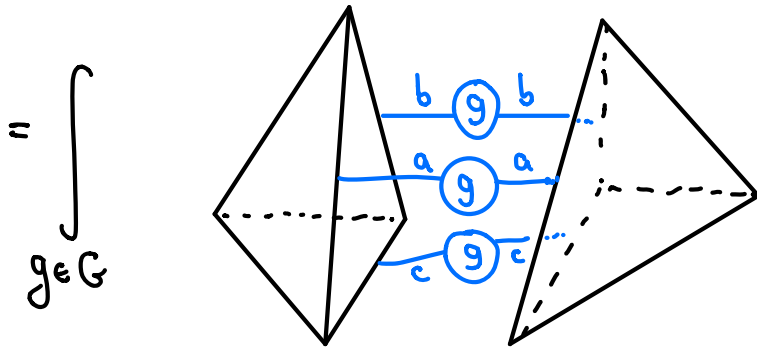
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$$= \int_{g \in G}$$

Use:

$$\sum_a \dim(V_a) \chi_a(g) = \delta(g)$$

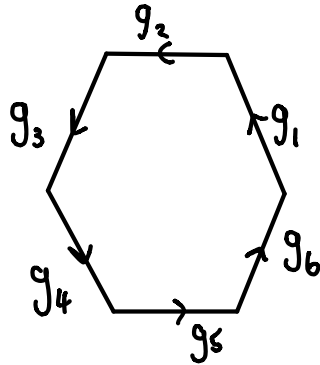
We obtain:

$$Z_{\text{Ponzano-Regge}} = \int \prod_{f \in \text{faces}(\Delta^*)} \delta(\text{holonomy around } f) \, d\{g_e\}_{\{g_e, e \in \text{edges}(\Delta^*)\}}$$

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holonomy =  $g_6 g_5 g_4 g_3 g_2 g_1$

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(Barrett and Naish-Guzman, 2008)

$$= \int_{[\rho] \in \text{Hom}(\pi_1 M, G)/G} R_{[\rho]}$$

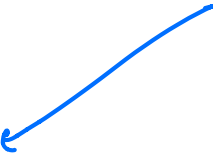
Reidemeister torsion

We obtain:

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$$\begin{aligned} & \left( \text{Barrett and Naish-Guzman, 2008} \right) \\ &= \int_{[\rho] \in \text{Hom}(\pi_1 M, G)/G} R_{[\rho]} \end{aligned}$$

Reidemeister torsion



providing  $H^2(\Delta^*, [\rho])$  vanishes for all  $[\rho]$ .



Let's observe something else.

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$$\begin{array}{c} \diagup a \\ \bullet \\ \diagdown c \end{array} \quad \begin{array}{c} \diagdown b \\ \bullet \\ \diagup a \end{array} \quad \begin{array}{c} \diagup c \\ \bullet \\ \diagdown b \end{array}$$

$$= \int_{g \in \text{SU}(2)} \begin{array}{c} a \\ | \\ \boxed{g} \\ | \\ a \end{array} \begin{array}{c} b \\ | \\ \boxed{g} \\ | \\ b \end{array} \begin{array}{c} c \\ | \\ \boxed{g} \\ | \\ c \end{array} dg$$

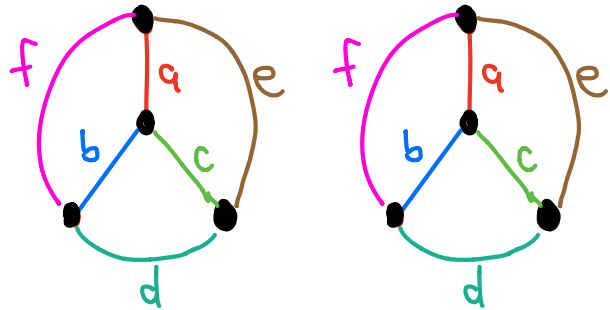
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$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2 =$$



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$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2 = \int_{g \in \text{SU}(2)} \text{Diagram} dg$$

The diagram consists of two trivalent vertices. The left vertex has edges labeled  $b$  (blue),  $c$  (green), and  $d$  (cyan). The right vertex has edges labeled  $b$  (blue),  $c$  (green), and  $d$  (cyan). External edges connect the vertices: a magenta edge labeled  $f$  connects the bottom of the left vertex to the bottom of the right vertex; a red edge labeled  $a$  connects the top of the left vertex to the top of the right vertex; a brown edge labeled  $e$  connects the top of the left vertex to the bottom of the right vertex; and a magenta edge labeled  $g$  connects the top of the right vertex to the bottom of the left vertex. Each of these four external edges has a small circle containing the letter  $g$  next to it. The entire expression is integrated over  $g \in \text{SU}(2)$  with a measure  $dg$ .

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$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2 = \int_{g, h \in \text{SU}(2)} \text{Diagram} dg dh$$

The diagram shows two trivalent vertices connected by three lines (blue, green, magenta). Each vertex is also connected to three external lines (red, brown, cyan). The external lines are labeled with group elements  $g$  and  $h$  in circles. The magenta line is labeled  $f$ , the blue line is labeled  $b$ , and the green line is labeled  $d$ . The red line is labeled  $a$ , the brown line is labeled  $e$ , and the cyan line is labeled  $c$ . The diagram is integrated over  $g, h \in \text{SU}(2)$  with measure  $dg dh$ .

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allows us to compute the square of the  $b_j$  symbol as a group integral:

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2 = \int_{g,h,k \in \text{SU}(2)} \text{Diagram} dg dh dk$$

The diagram shows a complex graph with two vertices (black dots) and several loops. The edges are colored and labeled: a red loop at the top with vertices  $g$  and  $g$ ; a blue loop on the left with vertex  $h$ ; a green loop at the bottom with vertices  $h$  and  $h$ ; and a brown loop on the right with vertices  $g$  and  $h$ . The edges are labeled with  $a, b, c, d, e, f$  at various points. The integral is over  $g, h, k \in \text{SU}(2)$  with measure  $dg dh dk$ .

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$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2 = \int_{g, h, k, l \in \text{SU}(2)} \text{diagram} dg dh dk dl$$

The diagram is a complex graph with nodes and edges colored to match the labels in the symbol. It features several loops and paths, with nodes labeled  $g, h, k, l$  and edges labeled with the letters  $a, b, c, d, e, f$ . The graph is connected and represents the square of the symbol as a group integral over  $\text{SU}(2)$  elements.

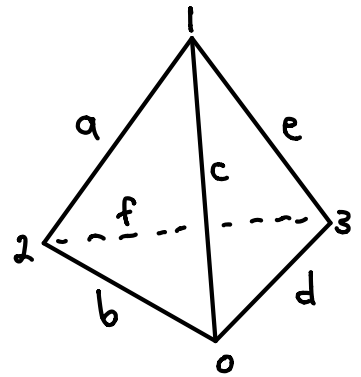
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allows us to compute the square of the  $b_j$  symbol as a group integral:

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2 = \int_{\text{SU}(2)^4} \prod_{i < j} \chi_{m_{\overline{ij}}} (g_i g_j^{-1}) dg_0 dg_1 dg_2 dg_3$$

where  $m_{12} = a$ ,  $\overline{23} = 12$  etc.

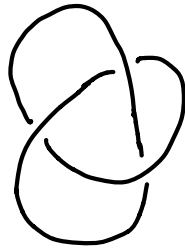




2. The quest for a lattice gauge theory  
description of the Turaev-Viro model

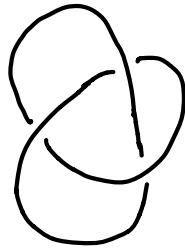
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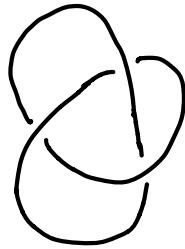
$$K \subseteq S^3$$

$$V_K$$

Jones polynomial of  $K$ ,  $\in \mathbb{Z}[q, q^{-1}]$

$$= q + q^3 - q^4 \quad \text{for } K = \text{trefoil above}$$

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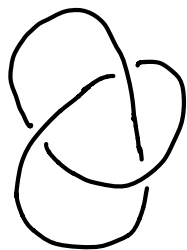


$$K \subseteq S^3$$

$$V_K \Big|_{q = e^{\frac{2\pi i}{K+2}}}$$

$K$  an arbitrary positive integer

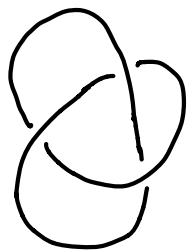
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$$V_K \Big|_{q = e^{\frac{2\pi i}{K+2}}} = \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } S^3}} \text{Tr} \left( H_A(K) \right) e^{ik \text{CS}(A)} \mathcal{D}A$$

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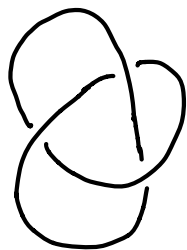
$$V_K \Big|_{q = e^{\frac{2\pi i}{K+2}}} = \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } S^3}} \text{Tr} \left( \text{Hol}_A(K) \right) e^{ik \text{CS}(A)} \mathcal{D}A$$

Chern-Simons  
invariant  
of  $A$



$$\text{CS}(A) = \frac{1}{4\pi} \int_{S^3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

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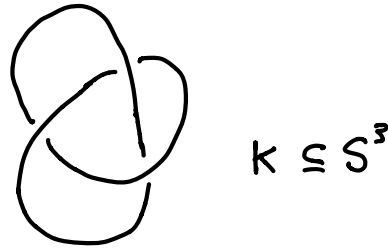
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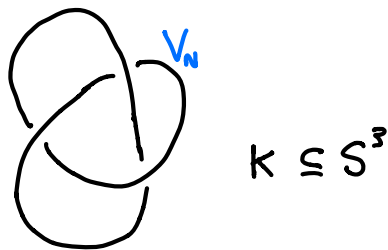
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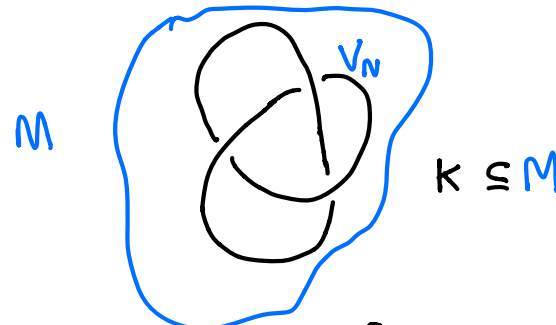
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$$\left. \begin{array}{l} \text{Nth coloured} \\ \text{Jones polynomial} \end{array} \right|_{q = e^{\frac{2\pi i}{k+2}}} = \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } S^3}} \text{Tr}(\rho(\text{Hol}_A(K))) e^{ikCS(A)} \mathcal{D}A$$

$$CS(A) = \frac{1}{4\pi} \int_{S^3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

In 1989, Witten made the celebrated assertion that the Jones polynomial of a knot could be calculated as a path integral:



$$= \int_{\substack{\text{all } SU(2)\text{-connections} \\ A \text{ on } M}} \text{Tr} \left( \rho \left( \text{Hol}_A(K) \right) \right) e^{iK \text{CS}(A)} \mathcal{D}A$$

knot invariant of  $K$  in  $M$

$$\text{CS}(A) = \frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

If we don't bother with a knot, then the path integral gives a topological invariant of  $M$ :

$$\int_{\text{all } SU(2) \text{ connections } A \text{ on } M} e^{ikCS(A)} \mathcal{D}A$$

giving a direct definition of this functional integral is the central question in mathematical physics

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this talk

$$:= \underbrace{RT_k(M)}$$

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A certain discrete sum over irreps of  $U_q \mathfrak{sl}_2$  at  $q = e^{\frac{2\pi i}{k+2}}$ .

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a sum over labellings  
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the  $6j$  symbols for  $\text{Rep}(U_q \mathfrak{sl}_2)$

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| algebraic:                   | State sum model                | dual lattice gauge theory description                             |
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| State sum model | dual lattice gauge theory description                                                                                                                                                              |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Ponzano-Regge   | Barrett and Naish-Guzman                                                                                                                                                                           |
| Turaev-Viro     | <p>geometric: should use <math>G</math> itself, and irreps of <math>G</math> (not <math>U_q\mathfrak{sl}_2</math>) for knot invariants</p> <p>? " <math>\int e^{ik CSA } \mathcal{D}A</math> "</p> |

What we'd really like is a version of the "fusing identity"

$$\begin{array}{c} a \backslash b / \\ \bullet \\ c \end{array} = \int_{g \in \text{SU}(2)} \begin{array}{c} a \\ | \\ \boxed{g} \\ | \\ a \end{array} \begin{array}{c} b \\ | \\ \boxed{g} \\ | \\ b \end{array} \begin{array}{c} c \\ | \\ \boxed{g} \\ | \\ c \end{array} dg$$

Below the first diagram is another diagram with a central dot and three lines extending downwards, labeled  $a$ ,  $b$ , and  $c$  from left to right.

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The diagram on the left shows two trivalent vertices. The top vertex has lines labeled  $a$ ,  $b$ , and  $c$  meeting at a central dot. The bottom vertex is its mirror image. The right side of the equation shows an integral over  $g \in \mathrm{SU}(2)$  of three separate boxes, each labeled  $g$ . Each box has two vertical lines passing through it, with the top and bottom lines labeled  $a$ ,  $b$ , and  $c$  respectively. The integral is denoted by  $\int_{g \in \mathrm{SU}(2)}$  and the measure is  $dg$ .

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The diagram on the left shows a vertex with three lines: line 'a' from the top-left, line 'b' from the top-right, and line 'c' from the bottom. The diagram on the right shows an integral over  $g \in \mathrm{SU}(2)$  of three separate boxes, each labeled 'g'. The first box has line 'a' entering from the top and line 'a' exiting from the bottom. The second box has line 'b' entering from the top and line 'b' exiting from the bottom. The third box has line 'c' entering from the top and line 'c' exiting from the bottom. The integral is denoted by  $\int_{g \in \mathrm{SU}(2)}$  and the measure is  $dg$ .

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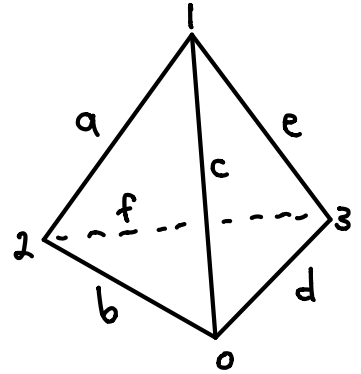
Let's try something weaker.

Can we generalize the integral formula for the classical 6j symbols,

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}^2 = \int \prod_{i < j} \chi_{m_{\bar{ij}}} (g_i g_j^{-1}) dg_0 dg_1 dg_2 dg_3$$

$SU(2)^4$

where  $m_{12} = a$ ,  $\bar{23} = 12$  etc.



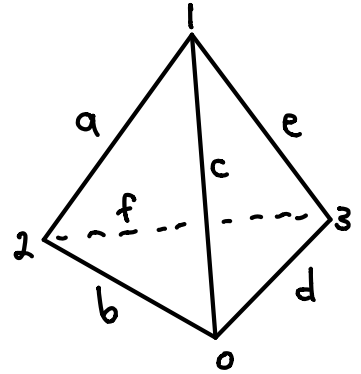
to the quantum 6j symbols?

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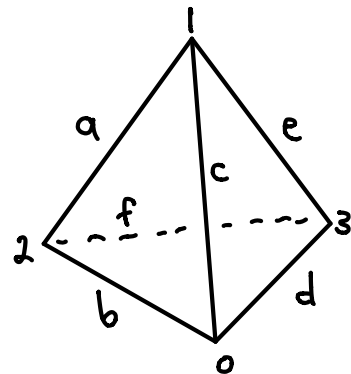


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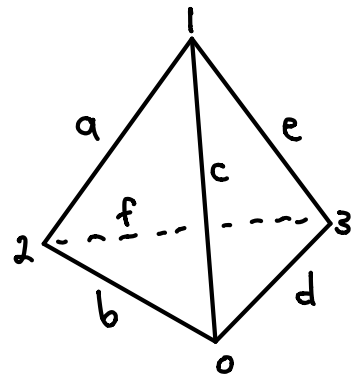
to the quantum 6j symbols? (Recall this formula followed from the "fusing identity.")

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn}}}^2 = \int_{SU(2)^4} \dots ?$$

Can we generalize the integral formula for the classical  $6j$  symbols,

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to the quantum  $6j$  symbols? (Recall this formula followed from the "fusing identity.")

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn}}} \stackrel{?}{=} \int_{SU(2)^4} \prod_{i < j} \chi_{m_{\bar{i}j}}(g_i g_j^{-1}) e^{i k \text{CS}(g_0, g_1, g_2, g_3)} dg_0 dg_1 dg_2 dg_3$$

↑ ?

3. The quest for an integral formula  
for the quantum b<sub>j</sub> symbols

We need to figure out what " $CS(g_0, g_1, g_2, g_3)$ " means.

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a class in  $C_2 \mapsto$   
here is a natural assignment for all manifolds  $M$

$$\begin{array}{c} P \\ \downarrow \\ M \end{array} \mapsto \text{class in } H^4(M; \mathbb{Z})$$



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2nd Chern  
class, generator  $\longrightarrow C_2 \longmapsto$

$$\begin{array}{c} P \\ \downarrow \\ M \end{array} \longmapsto \frac{1}{8\pi^2} \text{Tr}(F \wedge F) \overset{\text{integral}}{\in} H^4_{\text{De Rham}}(M; \mathbb{R})$$

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 a class in here is a natural assignment

$$\begin{array}{ccc} (P, A) & & \text{class in} \\ \downarrow & \uparrow \text{Flat connection} & H^3(M; \mathbb{R}/\mathbb{Z}) \\ M & & \end{array} \quad \longmapsto$$

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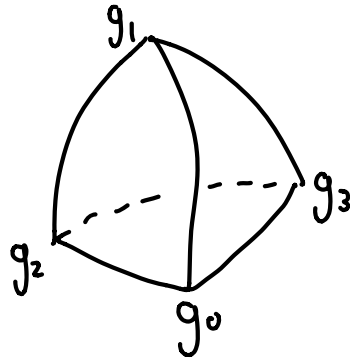
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$$CS_2(A) = \frac{1}{8\pi^2} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \in H^3(M; \mathbb{R}) / H^3(M; \mathbb{Z})$$

Theorem (Cheeger-Simons 1985) In the bar resolution model for  $H_{\text{grp}}^3(\text{SU}(2); \mathbb{R}/\mathbb{Z})$ ,  $\text{CS}_2$  is given by the group 3-cocycle

$\text{vol}(g_0, g_1, g_2, g_3) :=$  volume of spherical tetrahedron  
in  $S^3$  with vertices at  $g_0, g_1, g_2, g_3$

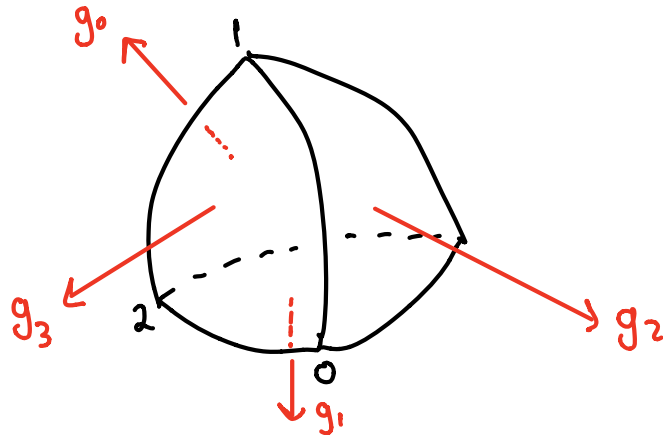


This caused me to speculate that as  $k \rightarrow \infty$  and the spins  $a, b, c, d, e, f \rightarrow \infty$  with the ratios  $\frac{a}{k}, \dots, \frac{f}{k}$  held fixed,

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$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}_{q=e^{\frac{\pi i}{k+2}}}^2 \cong \int_{SU(2)^4} \prod_{i < j} \chi_{m_{ij}}(g_i g_j^{-1}) e^{\frac{2ik \text{Vol}(T)}{\pi}} dg_0 dg_1 dg_2 dg_3$$

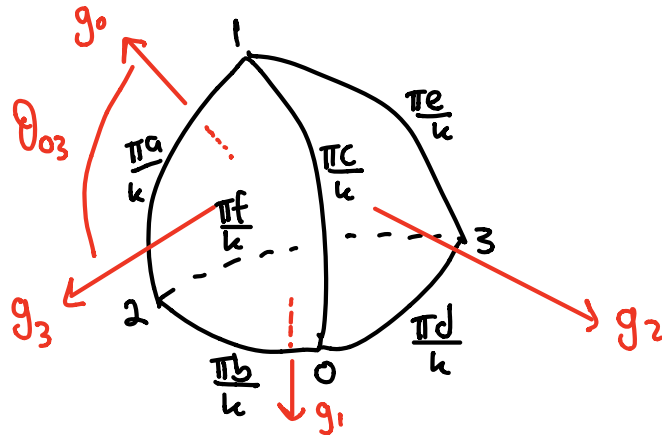
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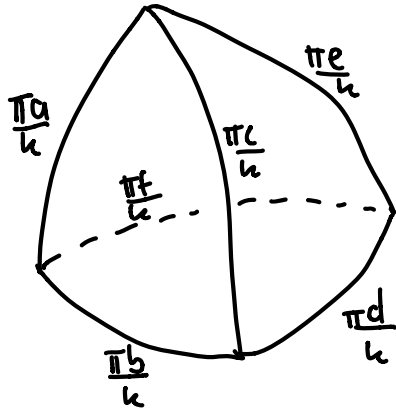
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I asked my PhD student Hosana Ranaivomanana to investigate this.

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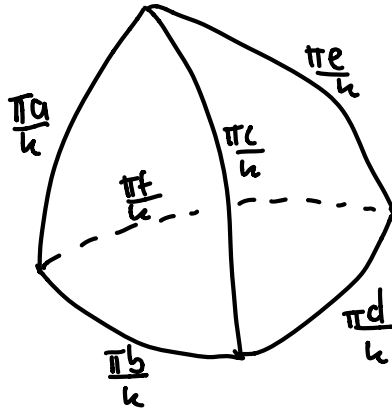
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$$\alpha = \frac{a}{k} \quad (\text{fixed ratio})$$

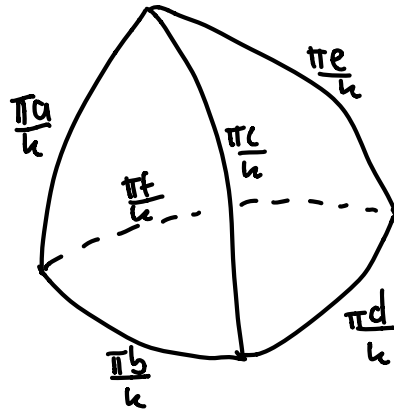
etc.

Note: using integer  
spins convention now



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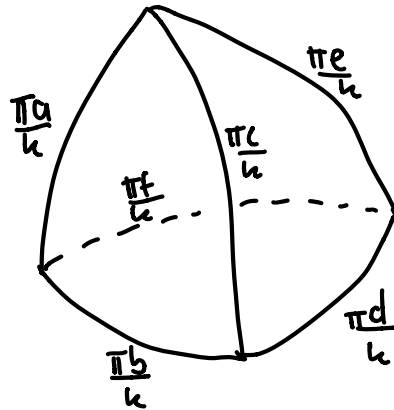


length of  
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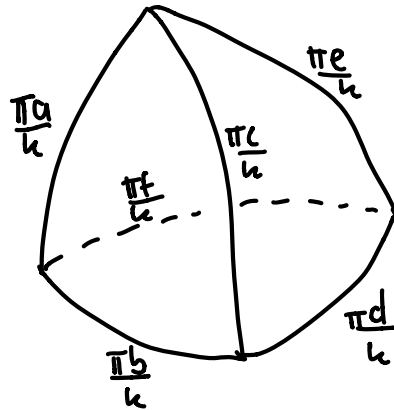
$$\begin{pmatrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_5 \end{pmatrix}_{q=e^{\frac{\pi i}{k\alpha_2}}} \sim \sqrt{\frac{4\pi^2}{k^3 \sqrt{\det G}}} \cos \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} - \frac{k}{\pi} V + \frac{\pi}{4} \right)$$

exterior dihedral  
 angle at edge  $e$



The point is that there is a known asymptotic formula for the quantum  
 b<sub>j</sub> symbols in terms of the geometry of a spherical tetrahedron, due to Taylor-Woodward:

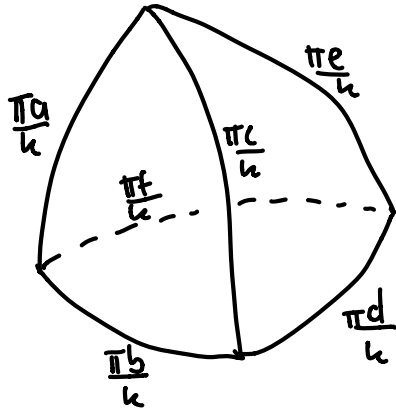
$$\begin{pmatrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{pmatrix}_{q=e^{\frac{\pi i}{k\alpha_2}}} \sim \sqrt{\frac{4\pi^2}{k^3 \sqrt{\det G}}} \cos \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} - \frac{k}{\pi} V + \frac{\pi}{4} \right)$$



Volume of the  
tetrahedron

The point is that there is a known asymptotic formula for the quantum  
 b<sub>j</sub> symbols in terms of the geometry of a spherical tetrahedron, due to Taylor-Woodward:

$$\begin{pmatrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{pmatrix}_{q=e^{\frac{\pi i}{k\alpha_2}}} \sim \sqrt{\frac{4\pi^2}{k^3 \sqrt{\det G}}} \cos \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} - \frac{k}{\pi} V + \frac{\pi}{4} \right)$$

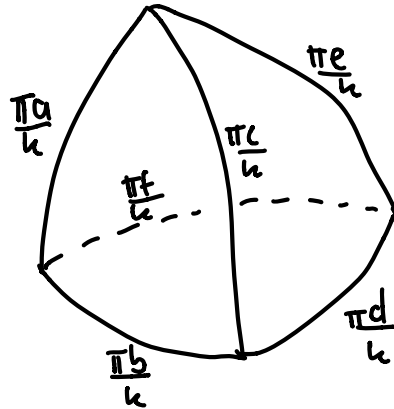


4x4 Gram matrix  
 $G_{ij} = \cos(l_{ij})$



The point is that there is a known asymptotic formula for the quantum  
 b<sub>j</sub> symbols in terms of the geometry of a spherical tetrahedron, due to Taylor-Woodward:

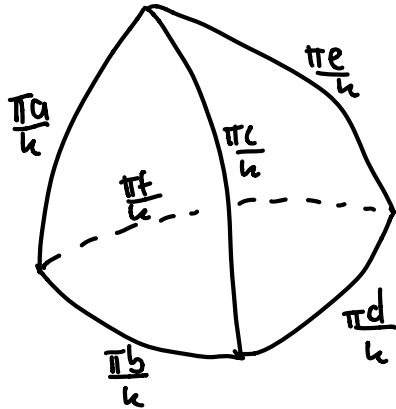
$$\begin{pmatrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{pmatrix}_{q=e^{\frac{\pi i}{k\alpha_2}}} \sim \sqrt{\frac{4\pi^2}{k^3 \sqrt{\det G}}} \cos \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} - \frac{kV}{\pi} + \frac{\pi}{4} \right)$$



an error! should  
 be a +

The point is that there is a known asymptotic formula for the quantum  
 b<sub>j</sub> symbols in terms of the geometry of a spherical tetrahedron, due to Taylor-Woodward:

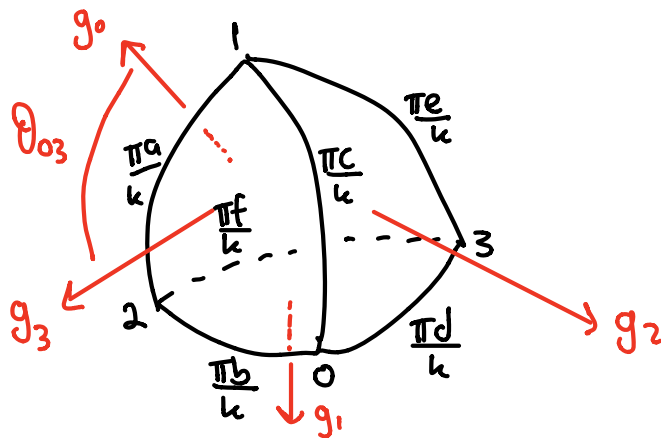
$$\left\{ \begin{matrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{matrix} \right\}_{q=e^{\frac{\pi i}{k\alpha_2}}} \sim \sqrt{\frac{4\pi^2}{k^3 \sqrt{\det G}}} \cos \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} + \frac{k}{\pi} V + \frac{\pi}{4} \right)$$



Does our integral match this asymptotic behaviour?

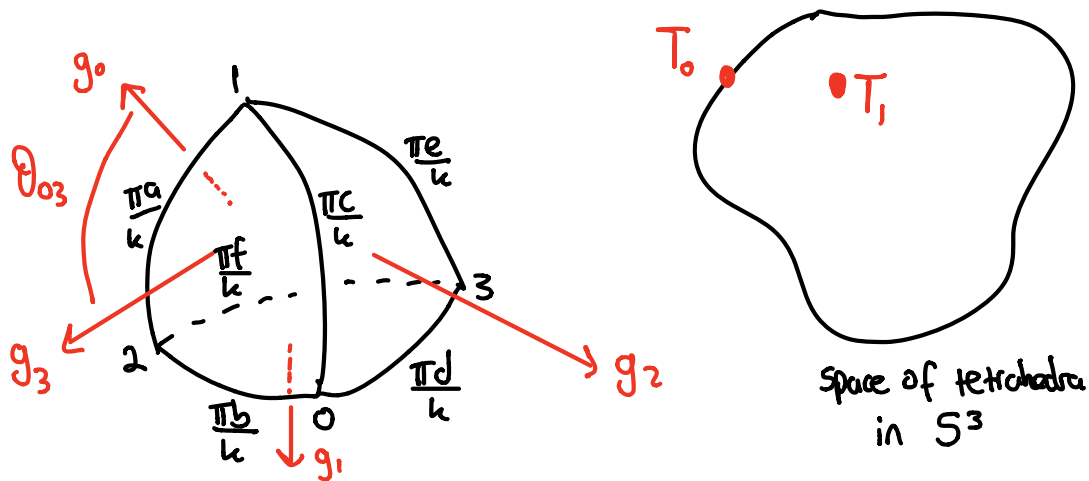
$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\}^2_{q=e^{\frac{\pi i}{kn}}} \cong \int_{SU(2)^4} \prod_{i < j} \frac{\sin((m_{ij}+1)\theta_{ij})}{\sin \theta_{ij}} e^{\frac{2ik \text{Vol}(T)}{\pi}} dg_0 dg_1 dg_2 dg_3$$

$$T(g_0, g_1, g_2, g_3) =$$



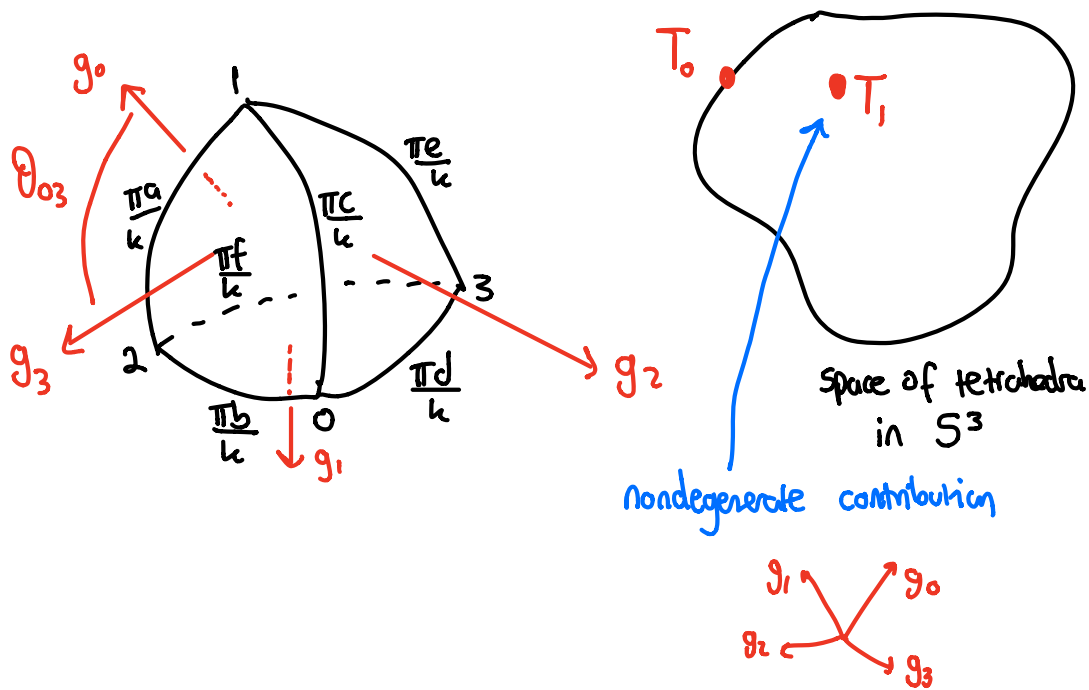
$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn}}}^2 \cong \int_{SU(2)^4} \prod_{i < j} \frac{\sin((m_{ij}+1)\theta_{ij})}{\sin \theta_{ij}} e^{\frac{2ik \text{Vol}(T)}{\pi}} dg_0 dg_1 dg_2 dg_3$$

$$T(g_0, g_1, g_2, g_3) =$$



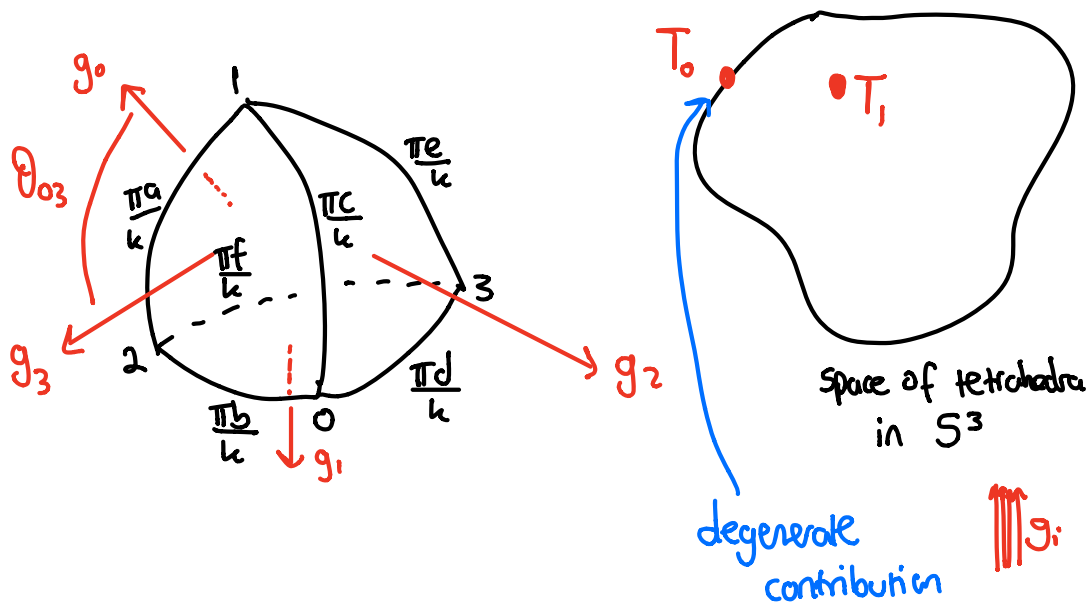
$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn}}}^2 \cong \int_{SU(2)^4} \prod_{i < j} \frac{\sin((m_{ij}+1)\theta_{ij})}{\sin \theta_{ij}} e^{\frac{2ik \text{Vol}(T)}{\pi}} dg_0 dg_1 dg_2 dg_3$$

$$T(g_0, g_1, g_2, g_3) =$$



$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn}}}^2 \cong \int_{SU(2)^4} \prod_{i < j} \frac{\sin((m_{ij}+1)\theta_{ij})}{\sin \theta_{ij}} e^{\frac{2ik \text{Vol}(T)}{\pi}} dg_0 dg_1 dg_2 dg_3$$

$$T(g_0, g_1, g_2, g_3) =$$



The known asymptotics of the square of the quantum b<sub>j</sub> symbols (Taylor and Woodward's formula) can also be split up into two contributions:

$$\begin{pmatrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{pmatrix}_{q=e^{\frac{\pi i}{kn^2}}} \sim \sqrt{\frac{4\pi^2}{k^3 \sqrt{\det G}}} \cos \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} + \frac{k}{\pi} V + \frac{\pi}{4} \right)$$

The known asymptotics of the square of the quantum  $b_j$  symbols (Taylor and Woodward's formula) can also be split up into two contributions:

$$\left\{ \begin{matrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn^2}}}^2 \sim \frac{4\pi^2}{k^3 \sqrt{\det G}} \cos^2 \left( \sum_e (k l_e + 1) \frac{\theta_e}{2} + \frac{k}{\pi} V + \frac{\pi}{4} \right)$$



The known asymptotics of the square of the quantum b<sub>j</sub> symbols (Taylor and Woodward's formula) can also be split up into two contributions:

$$\left\{ \begin{matrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{matrix} \right\}_{q=e^{\frac{i\hbar}{k r^2}}}^2 \sim \frac{2\pi^2}{k^3 \sqrt{\det G}} \left[ 1 - \sin \left( \sum_e (k l_e + 1) \theta_e + 2kV \right) \right]$$

The known asymptotics of the square of the quantum  $b_j$  symbols (Taylor and Woodward's formula) can also be split up into two contributions:

$$\left\{ \begin{matrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_\zeta \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn^2}}}^2 \sim \frac{2\pi^2}{k^3 \sqrt{\det G}} - \frac{2\pi^2}{k^3 \sqrt{\det G}} \sin \left( \sum_e (k l_e + 1) \theta_e + 2kV \right)$$

Hosana's results so far:

$$\int_{\text{SU}(2)^4} \prod_{i < j} \frac{\sin((m_{ij} + 1) \theta_{ij})}{\sin \theta_{ij}} e^{\frac{2ik \text{Vol}(T)}{\pi}} \sim \left( \begin{matrix} \text{deg.} \\ \text{contribution} \end{matrix} \right) - \frac{\pi^2}{4k^3 \sqrt{\det G}} \cos \left( \sum_e (k l_e + 1) \theta_e + 2kV \right)$$

The known asymptotics of the square of the quantum 6j symbols (Taylor and Woodward's formula) can also be split up into two contributions:

$$\left\{ \begin{matrix} k_\alpha & k_\beta & k_\gamma \\ k_\delta & k_\epsilon & k_5 \end{matrix} \right\}_{q=e^{\frac{\pi i}{kn^2}}}^2 \sim \frac{2\pi^2}{k^3 \sqrt{\det G}} - \frac{2\pi^2}{k^3 \sqrt{\det G}} \sin \left( \sum_e (k l_e + 1) \theta_e + 2kV \right)$$

Hosana's results so far:

$$\int_{\text{SU}(2)^4} \prod_{i<j} \frac{\sin((m_{ij}+1)\theta_{ij})}{\sin \theta_{ij}} e^{\frac{2ik \text{Vol}(T)}{\pi}} \sim \left( \begin{matrix} \text{deg.} \\ \text{contribution} \end{matrix} \right) - \frac{\pi^2}{4k^3 \sqrt{\det G}} \cos \left( \sum_e (k l_e + 1) \theta_e + 2kV \right)$$

See also:

Bartlett and Ranaivomanana, Reciprocity of the Wigner derivative for spherical tetrahedra (arXiv:2012.10609)







































































































































































