On the Shape of things

From holography to elastica

Álvaro Véliz-Osorio (Jagiellonian University)

Iberian Strings 2017

Based on 1611.03462 with:

Piermarco Fonda (Leiden)

Vishnu Jejjala (Wits)

General question

Which shape a manifold is compelled to take when immersed in another one, provided it must extremize some functional?

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Did Virgilio forsee Ryu-Takayanagi?!?!

Geometric setup

Immersion $f : N \to (M, g)$

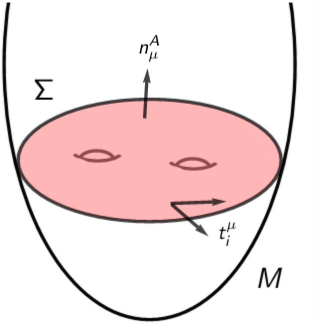
$$\Sigma = \{x^{\mu}(\sigma_i)|\ i = 1, \ldots, p\}$$

Indices:

Ambient $\mu,\ \nu = 1, \ldots, d$

Tangent $i,\ j = 1, \ldots, p$

Normal $A,\ B = 1, \ldots, (d - p)$



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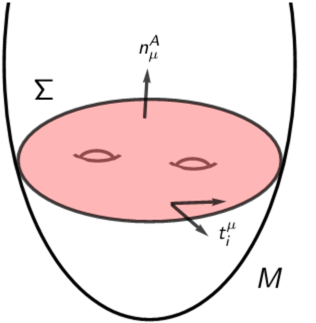

Induced metric

$$h_{ij} = g_{\mu\nu} t^\mu_i t^\nu_j$$

We can associate $\tilde{\nabla}_i$, $\mathcal{R}_{ijkl}$, $\mathcal{R}$, ...

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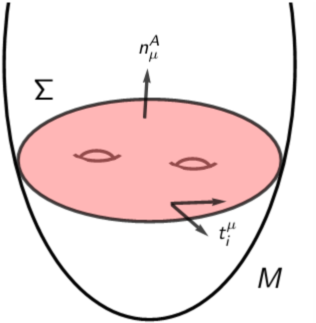

Projecting ambient tensors, example

$$R^A{}_{jik} = R_{\mu\nu\rho\sigma} n^{A\mu} t^\nu_j t^\rho_j t^\sigma_j$$

Extrinsic geometry

As one moves along Σ, how do normal vectors change?

$$t_i^\nu \nabla_\nu n^{\mu A} = K_{ij}^A t^{\mu j} - T_i^{AB} n_B^\mu \ ,$$

Extrinsic curvatures $K_{ij}^A = t_i^\mu t_j^\nu \nabla_\mu n_\nu^A$

Extrinsic torsions $T_i^{AB} = t_i^\mu n^{\nu A} \nabla_\mu n_\nu^B$

Under gauge transformations, $\mathcal{M}^A{}_B n_\mu^B$

$$K_{ij}^A \to \mathcal{M}^A{}_B K_{ij}^B \qquad T_i^{AB} \to \mathcal{M}^C{}_A \mathcal{M}^D{}_B \ T_i^{AB} + \eta^{AB} \mathcal{M}^C{}_A \partial_i \mathcal{M}^D{}_B$$

T_i^{AB} transform as connections, introduce $\tilde{D}_i{}^A{}_B$

Effective action

Generalizations of the area functional

$$S_0[\Sigma] = \lambda_0 \int_\Sigma d^p\sigma \, \sqrt{h} \; 1 = \lambda_0 \, \mathrm{Area}[\Sigma] \; .$$

Ex: Willmore functional, Canham-Helfrich, Dong functional, ...

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Contributions at second order

$$\lambda_1 \mathcal{R} + \lambda_2 R + \lambda_3 {R_A}^A + \lambda_4 {R_{AB}}^{AB} + \lambda_5 \mathrm{Tr} K_A \mathrm{Tr} K^A + \lambda_6 \mathrm{Tr} K^A K_A$$

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They aren't all independent, since

$$\mathcal{R} = R - 2R_A{}^A + R_{AB}{}^{AB} + \mathrm{Tr} K_A \mathrm{Tr} K^A - \mathrm{Tr}(K_A K^A)$$

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Mission: find the extrema of this functional

Shape equations

Complicated, yet they are completely expressed in terms of geometrical objects.

Simons, Yau, Yano, Chen, Carter, Guven, Capovilla, ...

Many interesting physical applications

Canham, Helfrich, Zhon-Chan, Boisseau-Letelier, Armas, ...

Some of interesting cases:

- Minimal submanifolds $\lambda_0 \neq 0$

$$\mathrm{Tr}\, K^A = 0$$

- Generalized Willmore $\lambda_5 \neq 0$

$$\mathrm{Tr} K_B \left[\mathrm{Tr} K^A \mathrm{Tr} K^B - 2\mathrm{Tr}\left(K^B K^A\right) - 2R^B{}_i{}^{Ai} \right] - 2\tilde{D}_i{}^B{}_C \tilde{D}^{iCA} \mathrm{Tr} K_B = 0$$

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...in their full glory

Shape equations

For arbitrary dimension and codimension, the extrema of the second order functional obey

$$\mathcal{E}^A = \lambda_0 \mathrm{Tr} K^A + \sum_{n=1}^{6} \lambda_n \mathcal{E}_n^A = 0$$

with

$$\begin{aligned}
\mathcal{E}_1^A &= \mathrm{Tr} K^A \mathcal{R} - 2\mathcal{R}^{ij} K_{ij}^A, \\
\mathcal{E}_2^A &= \mathrm{Tr} K^A R + n_\mu^A \nabla^\mu R, \\
\mathcal{E}_3^A &= \mathrm{Tr} K^A R_B{}^B + 2\tilde{D}_k^{BA} R_B^k + n_C^\mu n^{C\nu} n^{A\delta} \nabla_\delta R_{\mu\nu}, \\
\mathcal{E}_4^A &= \mathrm{Tr} K^A R_{CB}{}^{CB} + 4\tilde{D}_k^{BA} R^{kC}{}_{BC} + n_C^\mu n_B^\nu n^{C\rho} n^{B\sigma} n^{A\delta} \nabla_\delta R_{\mu\nu\rho\sigma}, \\
\mathcal{E}_5^A &= \mathrm{Tr} K_B \left[\mathrm{Tr} K^A \mathrm{Tr} K^B - 2\mathrm{Tr}\left(K^B K^A\right) - 2R^B{}_i{}^{Ai} \right] - 2\tilde{D}_i{}^B{}_C \tilde{D}^{iCA} \mathrm{Tr} K_B, \\
\mathcal{E}_6^A &= \mathrm{Tr} K^A \mathrm{Tr}\left(K_B K^B\right) - 2\left[\tilde{D}_i{}^C{}_B \tilde{D}_j{}^{BA} K_C^{ij} + \mathrm{Tr}\left(K^B K_B K^A\right) + K_B^{ij} R^B{}_j{}^A{}_i \right],
\end{aligned}$$

Shape equations in maximally symmetric spaces

Suppose that the ambient has

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \qquad R = \text{const}$$

with

$$R = \kappa\frac{d(d-1)}{L^2}\ , \qquad \kappa = 0, \pm 1\ .$$

Minimal submanifolds are extrema if

$$\lambda_1 = \lambda_6 \qquad \text{or} \qquad \mathcal{R}^{ij}K^A_{ij} = 0\ .$$

The second condition is true for curves $\mathcal{R}^{ij} = 0$ and surfaces

$$\mathcal{R}^{ij}K^A_{ij} = \frac{\mathcal{R}}{2}\mathrm{Tr}K^A = 0$$

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For $p > 2$, minimal submanifolds are NOT necessarily extrema

Extrema in maximally symmetric surfaces

Considering a curve γ in $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$

Glimpses of my shameful past

```
In[16]:= eomf = VariationalD[ℒ[ϕ[r]], ϕ[r], r] // FullSimplify
                                                  simplifica completa
```

Out[16]=

$$
\frac{1}{L^2\left(L^2+r^4\phi'[r]^2\right)^4\sqrt{\frac{L^2+r^4\phi'[r]^2}{r^2}}}
\Big(r\Big(-L^2\lambda_0\left(L^2+r^4\phi'[r]^2\right)^3\left(3L^2\phi'[r]+r^4\phi'[r]^3+L^2 r\,\phi''[r]\right)+
\lambda_5\Big(75L^4r^8\phi'[r]^5-3L^2r^{12}\phi'[r]^7-r^{16}\phi'[r]^9+L^2r^{13}\phi'[r]^6\phi''[r]-
5L^4r^4\phi'[r]^3\left(37L^2+r^6\phi''[r]\left(-15\phi''[r]+4r\,\phi^{(3)}[r]\right)\right)+
L^6\phi'[r]\left(18L^2-5r^6\phi''[r]\left(27\phi''[r]+4r\,\phi^{(3)}[r]\right)\right)+
L^4r^5\phi'[r]^2\left(-267L^2\phi''[r]+30r^6\phi''[r]^3+4L^2r^2\phi^{(4)}[r]\right)+
2L^4r^9\phi'[r]^4\left(54\phi''[r]+r\left(-10\phi^{(3)}[r]+r\,\phi^{(4)}[r]\right)\right)+
L^6r\left(46L^2\phi''[r]-5r^6\phi''[r]^3+2L^2r\left(10\phi^{(3)}[r]+r\,\phi^{(4)}[r]\right)\right)\Big)\Big)\Big)
$$

This is the shape equation for a curve in the Poincaré disk

Extrema in maximally symmetric surfaces

Using the shape equations instead

$$2\tilde{\Delta}\mathrm{Tr}k + \mathrm{Tr}k^3 - \left(\frac{\hat{\lambda}_0}{\lambda_5'} - \frac{2\kappa}{L^2}\right)\mathrm{Tr}k = 0$$

In arc-length parametrization

$$2\ddot{k} + k^3 - B\,k = 0\ , \qquad B = \left(\frac{\hat{\lambda}_0}{\lambda_5'} - \frac{2\kappa}{L^2}\right)$$

Two-step splitting

- First find out the extrinsic curvature
- Second invert this curvature to find γ

Extrema in maximally symmetric surfaces

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Perhaps this can be solved (?)

Extrema in maximally symmetric surfaces

Finding the extrinsic curvature

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Simplest solutions, $k = const$

$$\text{Geodesics} \quad k = 0 \quad \text{and} \quad \text{CMC} \quad k^2 = B$$

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Even these cannot be spotted straightaway in the old equation

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Simplest solutions, $k = const$

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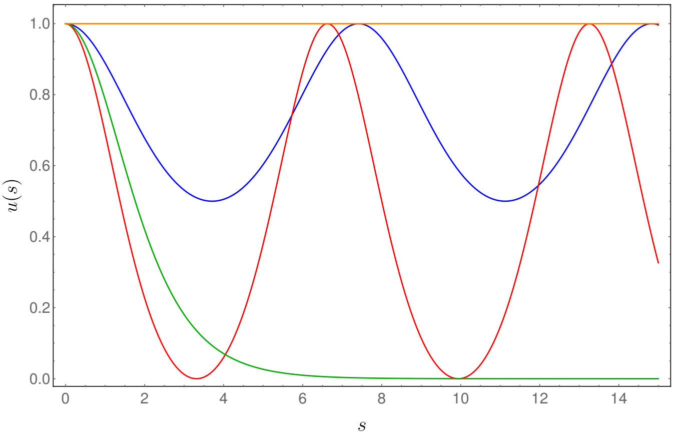
In fact, the full solution can be found !!

$$k^2(s) = \alpha\left[1 - \frac{\alpha-\gamma}{\alpha}\mathrm{sn}^2\!\left(\frac{1}{2}\sqrt{\alpha-\beta}\,s,\ \frac{\alpha-\gamma}{\alpha-\beta}\right)\right]$$

With $\mathrm{sn}(z,m)$ a Jacobi elliptic function and α, γ and β constants

Langer-Singer '84



Possible behaviour for $u(s) = k^2(s)$

- Orange: Constant mean curvature
- Red: Wavelike
- Blue: Orbitlike
- Green: Asymptotically geodesic

Extrema in $\mathbb{H}^2$

Now, we must invert $k(s)$

This is rather involved, yet attainable analytically

For hyperbolic geometry $\mathbb{H}^2$

$$ds^2 = \frac{1}{z^2}\left(dx^2 + dz^2\right)$$

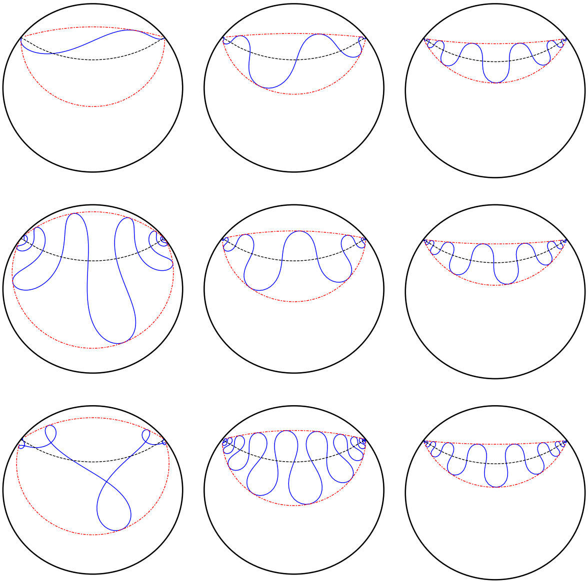
Wavy solutions in $\mathbb{H}^2$ read

$$z(s) = \frac{C}{2+\lambda}\,\frac{\exp\left[\sqrt{C^2 - 4(\lambda+1)}\left(\frac{s}{4} - \frac{2(C-2)}{4\sqrt{2C}(C+2)}\Pi\left[n, \varphi(s); m\right]\right)\right]}{\sqrt{(C+2)^2 - 4(C+2+\lambda)\mathrm{sn}^2\left(\sqrt{\frac{C}{2}}s, \frac{C+2+\lambda}{2C}\right)}}$$

Where C and λ are constants, while

$$\varphi(s) = \mathrm{amp}\left(\sqrt{\frac{C}{2}}s, \frac{C+2+\lambda}{2C}\right)$$

Extrema in $\mathbb{H}^2$



Application: holographic entanglement entropy

For field theories with an Einstein gravity dual

Area functional

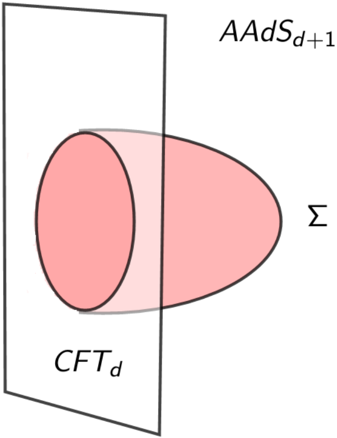
$$S_{\text{eff}}[\Sigma] = \frac{1}{4G_d} \int_\Sigma d^p\sigma \sqrt{h}$$

Minimize

$$\mathrm{Tr}\, K^A = 0$$

Evaluate on-shell

$$S_{\text{EE}}(A) = S_{\text{eff}}^{\text{on-shell}}[\Sigma] \,.$$



Ryu, Takayanagi '06

Application: holographic entanglement entropy

If the dual gravitational theory has h.c. corrections

$$\mathcal{L} = -2\Lambda + R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Then the EE functional

$$S_{\rm eff} = \frac{1}{4G_d}\int_\Sigma d^p\sigma\sqrt{h}\Big[1 + 2c_1 R + c_2\left(R_A{}^A - \frac{1}{2}\mathrm{Tr}K_A \mathrm{Tr}K^A\right) + 2c_3\left(R_{AB}{}^{AB} - \mathrm{Tr}(K^A K_A)\right)\Big]$$

Bhattacharrya, Sharma, Sinha; Camps; Dong '13

- To find the EE one must evaluate the functional on an extremum.
- Which of the possible extrema is not settled (minimal??)
- It would be nice to be able to scan the space of extrema.

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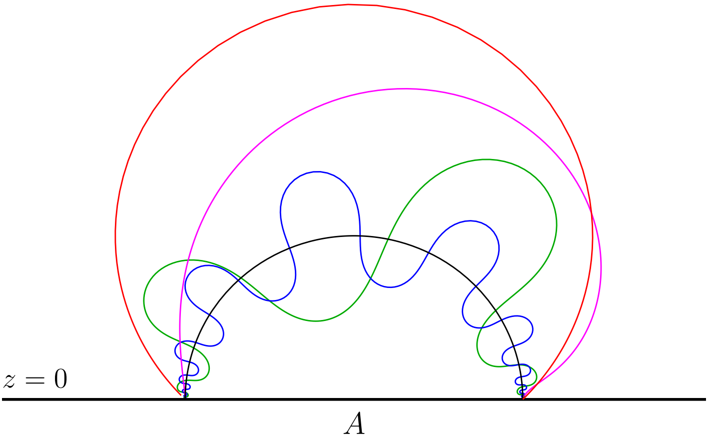
Clearly, the extrema are solutions to the shape equations

Entanglement entropy AdS_3

For a higher-derivative theory (such as NMG)



We know **all** the extrema

Entanglement entropy AdS_3/CFT_2

The EE for an interval is universal

$$S_{\rm EE}(A) = \frac{c}{3}\log\left(\frac{\ell}{\epsilon}\right) + \mathcal{O}(\epsilon)$$

Holzhey, Larsen, Wilczek '94

Using the fact that

$$c = \frac{L}{2G_3}\, g^{\mu\nu}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}\,,$$

Saida, Soda '00

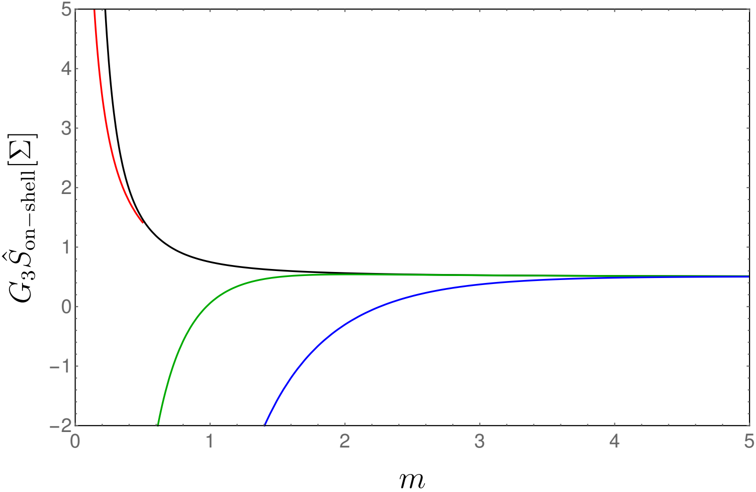
We show that

$$S_{\rm EE}(A) = S^{\rm Geo}_{\rm on-shell}[\Sigma]\,,$$

Are geodesics minimal? Let's compare

$$\hat{S}_{\rm on-shell}[\Sigma] = \ell\frac{d}{d\ell}\, S_{\rm on-shell}[\Sigma]$$

Comparing on-shell values in NMG



The geodesic's on-shell value is the black one (**not** minimal)

Interesting questions

- If not minimality then which criterium?
- What if geodesics are not extrema?
- What about perturbations? or curvature driven flows?
- Is there an information theoretic interpretaton of other extrema in terms of:
  - Length and differential entropy

    Czech, Hayden, Lashkari, Swingle '14
  - The surface/state correspondece

    Miyaji, Takayanagi '15
- How far can we get in higher-dimensional settings?

  Fonda, Véliz-Osorio ..in progress
- How far can we get in less symmetric ambients?

Epilogue

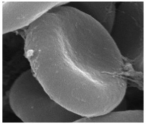
Elastica

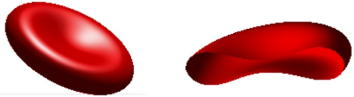
Two interesting set-ups

Canham-Helfrich

$p = 2, \; M = \mathbb{R}^3$

$$\mathcal{L} = \left[\sigma + \frac{k_c}{4}(\mathrm{Tr}K)^2 + \bar{k}_c \det K\right]$$

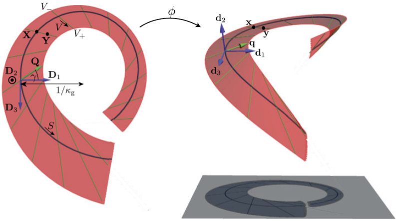




Sadowsky-Wünderlich

$p = 2, \; M = \mathbb{R}^3$

$$\mathcal{L} = \frac{(k_{\mathrm{FS}}^2 + \tau_{\mathrm{FS}}^2)^2}{k_{\mathrm{FS}}^2}\,.$$



(a) (b)

Elastica

Two interesting set-ups

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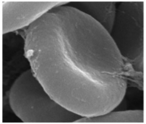

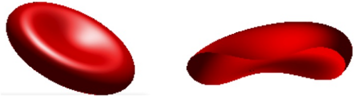

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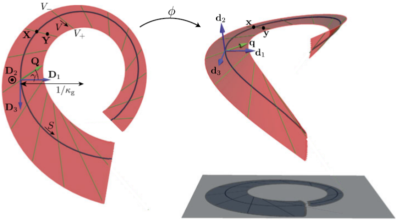
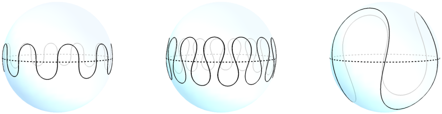
Judicious breaking of gauge invariance



(a) (b)

Interesting problems

- Elastica in (evolving) surfaces



- Embeddings of embeddings, interface theory on surfaces
- Minimal surfaces bounded by elastic lines

Giomi, Mahadevan '11

- Conformal maps: Minimal in $\mathrm{AdS}_d \longleftrightarrow$ Willmore in $\mathbb{R}^3$

Alexakis, Mazzeo '10; Fonda, Seminara, Tonni '15

- Generalized curvature flows

Fonda, Véliz-Osorio...in progress

Muito obrigado!!