Wall Crossing Invariants from Spectral Networks

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Goal of the talk:

A construction of the BPS monodromy for theories of class S,
directly from the Coulomb branch geometry

- Does not involve knowledge of the BPS spectrum
- Manifest wall-crossing invariance
- Topological nature and symmetries of the superconformal index

- The BPS monodromy $\mathbb{U}$ is of central importance in wall crossing. It is also a spectrum generating function, BPS state counting follows from knowledge of $\mathbb{U}$ [Kontsevich-Soibelman, Gaiotto-Moore-Neitzke, Dimofte-Gukov].

- Relation to various specializations of the superconformal index [Cecotti-Neitzke-Vafa, Iqbal-Vafa, Cordova-Shao, Cecotti-Song-Vafa-Yan].

- Graphs encoding $\mathbb{U}$ are an important link in the Network/Quiver correspondence

On Coulomb branches $\mathcal{B}$ of 4d $\mathcal{N} = 2$ gauge theories gauge symmetry is spontaneously broken to $U(1)^r$.

At **generic** $u \in \mathcal{B}$ the lightest charged particles are **BPS solitons** $|\psi\rangle = |\gamma, m\rangle \in \mathscr{H}_u^{\text{BPS}}$ characterized by charge $\gamma \in \mathbb{Z}^{2r+f}$ and spin $j_3 = m$

$$M\,|\psi\rangle = |Z_\gamma|\,|\psi\rangle\,, \quad \mathcal{Q}_\vartheta|\psi\rangle = 0 \qquad (\vartheta = \mathrm{Arg} Z_\gamma)\,.$$

$Z_\gamma(u)$ is topological, linear in γ, locally holomorphic in u.

Low energy dynamics on $\mathcal{B}$ admits a geometric description, involving a family of complex curves Σ_u fibered over $\mathcal{B}$ **[Seiberg-Witten]**

$$\gamma \in H_1(\Sigma_u, \mathbb{Z}) \qquad Z_\gamma = \frac{1}{\pi}\oint_\gamma \lambda$$

On $\mathbb{R}^3 \times S^1_R$ a 3d σ-model into $\mathcal{M} \to \mathcal{B}$, effective action receives quantum corrections $\sim e^{-2\pi R|Z_\gamma|}$ from BPS particles wrapping S^1_R.
The metric on $\mathcal{M}$ therefore encodes the BPS spectrum, which can be extracted with geometric tools like spectral networks **[Gaiotto-Moore-Neitzke]**.

BPS particles interact, forming boundstates

$$E_{bound} = |Z_{\gamma_1+\gamma_2}| - |Z_{\gamma_1}| - |Z_{\gamma_2}| \leq 0$$

Boundstates form/decay at $codim_{\mathbb{R}}$-1 marginal stability loci

$$MS(\gamma_1, \gamma_2) := \{u \in \mathcal{B} \mid \mathrm{Arg} Z_{\gamma_1}(u) = \mathrm{Arg} Z_{\gamma_2}(u)\}$$

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Jumps of the BPS spectrum are controlled by an $\mathrm{Arg}\, Z_\gamma$-ordered product of quantum dilogarithms **[Kontsevich-Soibelman]**

$$\prod_{\gamma,m}^{\mathrm{Arg}Z(u)\nearrow} \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)} = \prod_{\gamma,m}^{\mathrm{Arg}Z(u')\nearrow} \Phi((-y)^m X_\gamma)^{a_m(\gamma,u')}$$

- non-commutative: DSZ-twisted product $X_{\gamma_1} X_{\gamma_2} = y^{\langle\gamma_1,\gamma_2\rangle} X_{\gamma_1+\gamma_2}$
- BPS degeneracies $a_m(\gamma, u) = (-1)^m \dim \mathscr{H}^{BPS}_{u,\gamma,m}$ count $|\gamma, m\rangle$

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2d-4d system:

- 2d $\mathcal{N} = (2,2)$ theory on $\mathbb{R}^{1,1} \subset \mathbb{R}^{1,3}$
- chiral matter in a representation of a global symmetry G
- 4d vector multiplets couple to 2d chirals, gauging G

Vevs of 4d VM scalars on $\mathcal{B}$ correspond to twisted masses for 2d chirals. Therefore Coulomb moduli control the 2d effective superpotential $\widetilde{W}(u)$. For u generic, $\widetilde{W}(u)$ has a finite number of massive vacua $\widetilde{W}_i(u)$, $i = 1, \ldots, d$.

2d-4d BPS states: BPS field configurations interpolating between vacua (ij) on the defect, carrying both topological (2d) and flavor (4d) charges

$$Z_{ij,\gamma}(u) \sim \widetilde{W}_j(u) - \widetilde{W}_i(u) + Z_\gamma(u)\,, \qquad M_{ij,\gamma} = |Z_{ij,\gamma}|\,.$$

[Hanany-Hori, Dorey, Gaiotto, Gaiotto-Moore-Neitzke, PL, Gaiotto-Gukov-Seiberg]

2d-4d vacua are fibered nontrivially over the space of 4d vacua $\mathcal{B}$.
Both the chiral ring and central charges $Z_{ij,\gamma}$ depend on u, through $\widetilde{W}(u)$.

2d-4d wall-crossing: 2d-4d BPS states can form boundstates

$$(ij,\gamma') + (jk,\gamma'') \to (ik,\gamma)$$

$$E_{bind} = |Z_{ij,\gamma'} + Z_{jk,\gamma''}| - |Z_{ij,\gamma'}| - |Z_{jk,\gamma''}| \leq 0$$

Marginal stability occurs when $\mathrm{Arg}\, Z_{ij,\gamma'}(u) = \mathrm{Arg}\, Z_{jk,\gamma''}(u)$, the 2d-4d BPS spectrum depends on u.

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2d-4d mixing: Boundstates of solitons of opposite type mix with 4d BPS states

$$(ij,\gamma') + (ji,\gamma'') \to (ii,\gamma) \sim \gamma$$

in this way the surface defect **probes the 4d BPS spectrum**.

To compute 2d-4d mixing, introduce a formal generating series of 2d-4d BPS states preserving $\mathcal{Q}_\vartheta$:

$$F(\vartheta, u) = \sum_{ij,\gamma} \Omega(\vartheta, u, ij, \gamma; y)\, X_{ij,\gamma}$$

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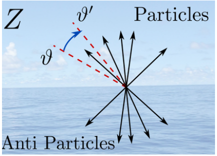
Dependence on ϑ: $F(\vartheta, u)$ is piecewise-constant in ϑ,
jumps across 4d BPS rays $\vartheta = \mathrm{Arg}\, Z_\gamma$
[Gaiotto-Moore-Neitzke]



$$F(\vartheta', u) = \left[\prod \Phi((-y)^m X_\gamma)^{a_m(\gamma)}\right] F(\vartheta, u) \left[\prod \Phi((-y)^m X_\gamma)^{a_m(\gamma)}\right]^{-1}$$

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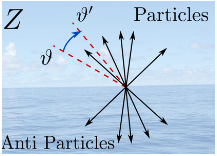

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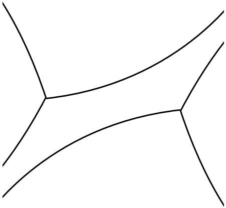
4d BPS degeneracies $a_m(\gamma)$ control jumps in ϑ (at fixed u). Conversely, comparing $F(\vartheta, u)$ to $F(\vartheta + \pi, u)$ gives the **whole 4d BPS spectrum at u**:

$$F(\vartheta + \pi, u) = \mathbb{U}\, F(\vartheta, u) \mathbb{U}^{-1}\,.$$

1. For **canonical defects** of Class $\mathcal{S}$ theories, the generating function $F(\vartheta, u)$ is computed by the combinatorics of networks on the (Class $\mathcal{S}$) UV curve

- The shape of a network is controlled by the **geometry** of Σ_u, and by an angle ϑ
- Edges carry **soliton data** counting 2d-4d BPS states. $\Omega(\vartheta, u, ij, \gamma; y)$ determined by global topology
- **Finite edges** appear at $\vartheta = \mathrm{Arg} Z_\gamma$, corresponding to 4d BPS states



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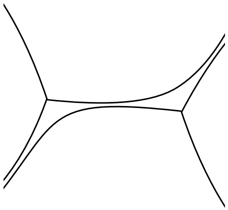

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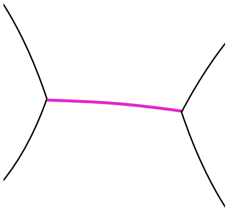

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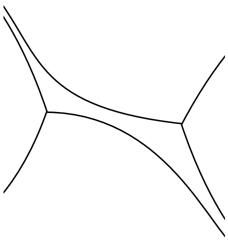

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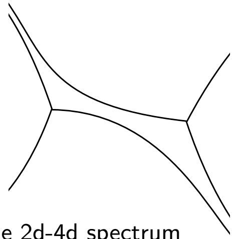

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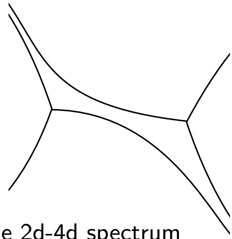

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Then use spectral networks to compute $F(\vartheta, u)$, $F(\vartheta + \pi, u)$ and obtain $\mathbb{U}$.

- still choosing a chamber of $\mathcal{B}$, with some 4d BPS spectrum
- still difficult, due to complexity of 2d-4d wall crossing

Marginal Stability

Let $\mathcal{B}_c \subset \mathcal{B}$ be a locus where central charges of **all 4d BPS particles** have **the same phase**

$$\mathcal{B}_c := \{u \in \mathcal{B}\,,\ \mathrm{Arg}\, Z_\gamma(u) = \mathrm{Arg}\, Z_{\gamma'}(u) \equiv \vartheta_c(u)\}$$

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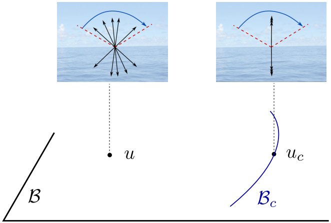
central charges of 2d-4d states are **phase-resolved**.

At $u_c \in \mathcal{B}_c$ the generating function of 2d-4d $\mathcal{Q}_\vartheta$-BPS states is well defined

$$F(\vartheta, u_c) = \sum_{ij,\gamma} \Omega(\vartheta, u_c, ij, \gamma; y)\, X_{ij,\gamma}$$



at u : $\quad F' = \left[\prod \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)}\right] \cdot F \cdot \left[\prod \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)}\right]^{-1}$

at u_c : $\quad F' = \mathbb{U} \cdot F \cdot \mathbb{U}^{-1}$

- $F(\vartheta, u_c)$ exhibits a **single jump** at ϑ_c which captures the **full BPS monodromy**
- From the viewpoint of 2d-4d states nothing special happens at the critical locus: can "parallel transport" both F and F' to $\mathcal{B}_c$
- Redefining $\mathbb{U}$ as the jump $F \to F'$, **extends its definition to $\mathcal{B}_c$**

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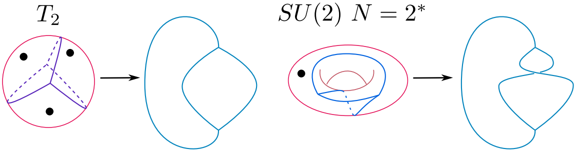
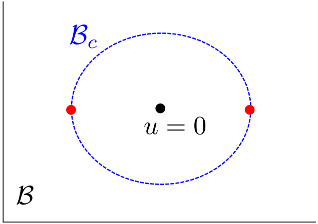
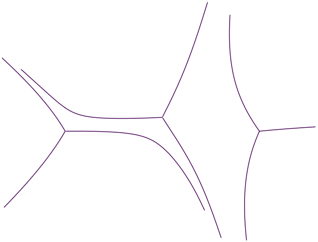
The **graph** topology, together with a notion of framing, **determine** $\mathbb{U}$.



First Example: Argyres-Douglas





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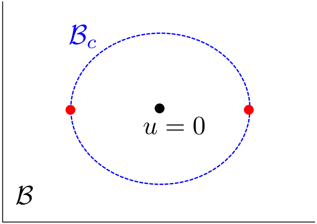

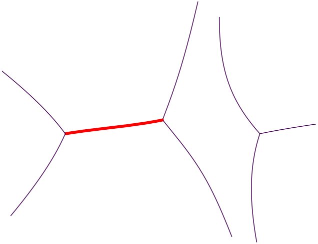

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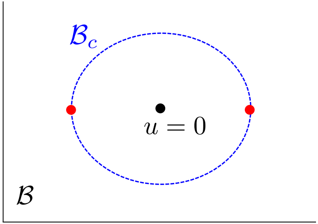

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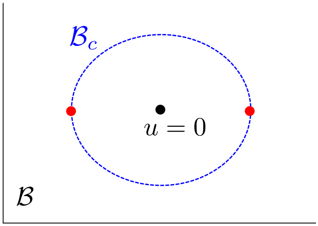

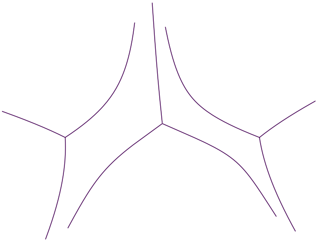

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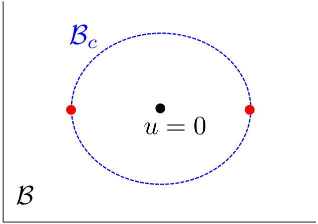

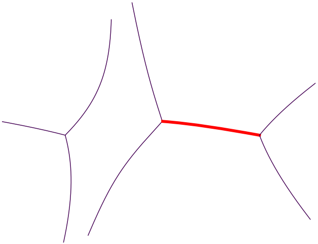

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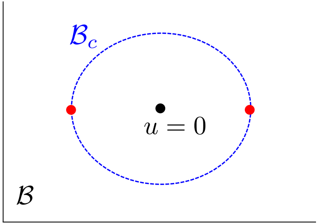

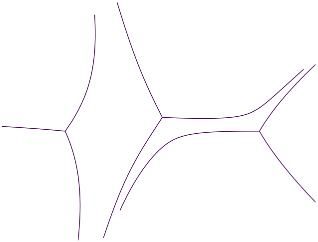

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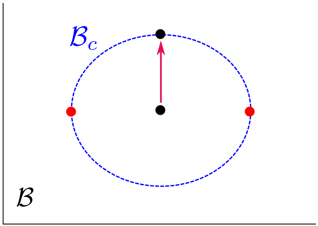

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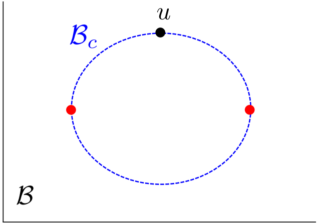

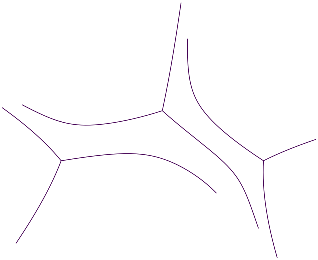

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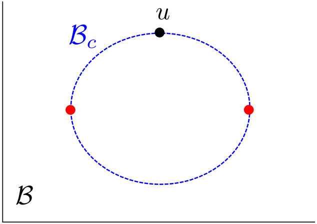

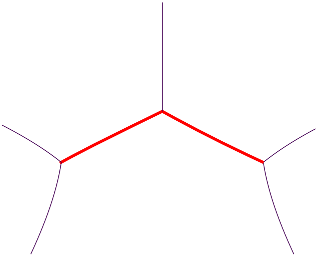

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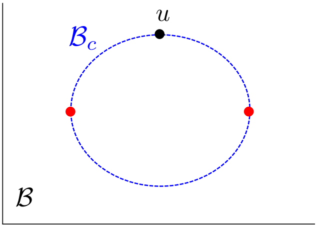
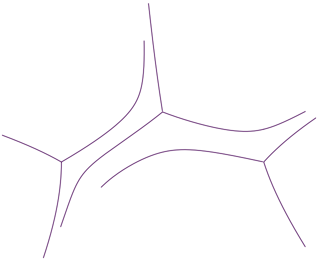




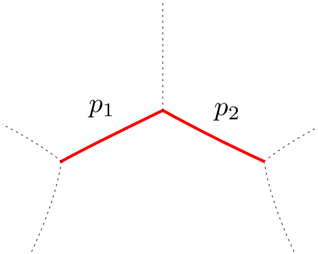
First Example: Argyres-Douglas

The graph has 2 edges, each contributes an equation

$$F'_p = \mathbb{U}\, F_p\, \mathbb{U}^{-1}$$

with

$$\begin{aligned} F_{p_1} &= 1 + y^{-1} X_{\gamma_1} + y^{-1} X_{\gamma_1+\gamma_2} \\ F_{p_2} &= 1 + y^{-1} X_{\gamma_2} \\ F'_{p_1} &= 1 + y^{-1} X_{\gamma_1} \\ F'_{p_2} &= 1 + y^{-1} X_{\gamma_2} + y^{-1} X_{\gamma_1+\gamma_2} \end{aligned}$$



Together, they determine the monodromy

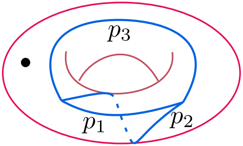
$$\begin{aligned} \mathbb{U} &= 1 - \frac{y}{(y)_1}\left(X_{\gamma_1} + X_{\gamma_2}\right) + \frac{y^2}{(y)_1^2} X_{\gamma_1+\gamma_2} + \frac{y^2}{(y)_2}\left(X_{2\gamma_1} + X_{2\gamma_2}\right) + \ldots \\ &= \Phi(X_{\gamma_1})\Phi(X_{\gamma_2}) \end{aligned}$$

Second Example: $SU(2)$ $N = 2^*$

The graph has three edges p_1, p_2, p_3;
each contributes one equation

$$F'_p = \mathbb{U}\, F_p\, \mathbb{U}^{-1}$$

with



$$F_{p_1} = \frac{1+X_{\gamma_1}+\left(y+y^{-1}\right)X_{\gamma_1+\gamma_3}+X_{\gamma_1+2\gamma_3}+\left(y+y^{-1}\right)X_{\gamma_1+\gamma_2+2\gamma_3}+X_{\gamma_1+2\gamma_2+2\gamma_3}+X_{2\gamma_1+2\gamma_2+2\gamma_3}}{\left(1-X_{2\gamma_1+2\gamma_2+2\gamma_3}\right)^2}$$

$$F'_{p_1} = \frac{1+X_{\gamma_1}+\left(y+y^{-1}\right)X_{\gamma_1+\gamma_2}+X_{\gamma_1+2\gamma_2}+\left(y+y^{-1}\right)X_{\gamma_1+2\gamma_2+\gamma_3}+X_{\gamma_1+2\gamma_2+2\gamma_3}+X_{2\gamma_1+2\gamma_2+2\gamma_3}}{\left(1-X_{2\gamma_1+2\gamma_2+2\gamma_3}\right)^2}$$

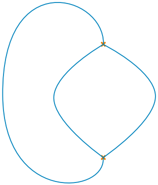
$F_{p_{2,3}}$ & $F'_{p_{2,3}}$ are obtained by cyclic $\mathbb{Z}_3$ shifts of $\gamma_1, \gamma_2, \gamma_3$.

The solution:

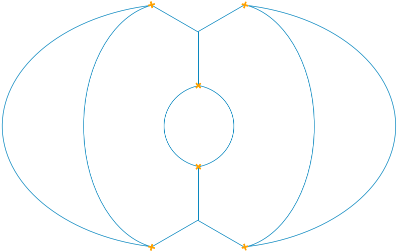
$$\mathbb{U} = \left(\prod^{\nearrow}_{n\geq 0} \Phi\left(X_{\gamma_1+n(\gamma_1+\gamma_2)}\right)\right)$$

$$\times \Phi\left(X_{\gamma_3}\right) \Phi\left((-y)X_{\gamma_1+\gamma_2}\right)^{-1} \Phi\left((-y)^{-1}X_{\gamma_1+\gamma_2}\right)^{-1} \Phi\left(X_{2\gamma_1+2\gamma_2+\gamma_3}\right)$$

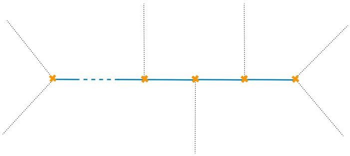
$$\times \left(\prod^{\searrow}_{n\geq 0} \Phi(X_{\gamma_2+n(\gamma_1+\gamma_2)})\right)$$



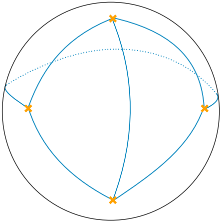
T_2



T_3



$A_k \ AD$



$SU(2) \ N_f = 4$

Framing

F_p, F'_p are computed from the graph by simple rules, based on

- the **topology** of a graph
- the **framing**: a cyclic ordering of edges at each node

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- the **topology** of a graph
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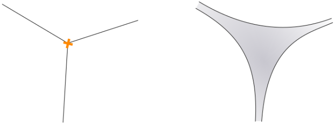
Graphs of A_1 theories are trivalent: topology and framing define a **ribbon graph**.



To each ($\overline{\mathbb{Q}}$-algebraic) Riemann surface C is associated a holomorphic map $\mathfrak{B} : C \to \mathbb{P}^1$, with ramification at $0, 1, \infty$ [Belyi].
The preimage $\mathfrak{B}^{-1}([0,1])$ is a ribbon graph on C, a dessin d'enfants [Grothendieck].
The ribbon graph is the union of critical leaves of a foliation on C by a Strebel differential [Harer, Mumford, Penner, Thurston, Mulase-Penkava].

Symmetries of a graph: automorphisms preserving both its topology and framing, they are **inherited by $\mathbb{U}$**.

These symmetries are often **hidden** by the Kontsevich-Soibelman factorization $\mathbb{U} = \prod \Phi(X)$. Instead they become **manifest on the graph** (Ex. $\mathbb{Z}_3$ symmetry in $\mathcal{N} = 2^*$).

Graph symmetries

Symmetries of a graph: automorphisms preserving both its topology and framing, they are **inherited by** $\mathbb{U}$.

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Graph symmetries show that $\mathbb{U}$ shares important properties of the superconformal index.

- Punctures on C encode **global symmetries** of a Class $\mathcal{S}$ theory [Gaiotto, Chacaltana-Distler-Tachikawa].
- The index is computed by correlators of a TQFT on C [Gadde-Pomoni-Rastelli-Razamat], it is a symmetric function of the flavor fugacities.
- Symmetries of the graph **permute punctures**, implying that $\mathbb{U}$ is a **symmetric** function of the corresponding **flavor fugacities**, like the index.

1. To a class S theory associate a **canonical "critical graph"** on the UV curve, emerging from a degenerate spectral network at $\mathcal{B}_c$.

2. A new definition of the BPS monodromy, encoded by the **topology and framing** of the graph.

3. Does not use BPS spectrum. **Manifest invariance** under wall-crossing. At the critical locus $\mathcal{B}_c$ the BPS spectrum is ill-defined.

4. Simpler than computing $\mathbb{U}$ by using BPS spectra. **Symmetries** of $\mathbb{U}$ are manifest from the graph.

Open questions

- Existence conditions for the critical locus $\mathcal{B}_c$ where the critical graph emerges
- Equivalence relations among graphs: different topology, same $\mathbb{U}$ on different components of $\mathcal{B}_c$
- Constructive approach by gluing graphs **[Gabella-PL in progress]**
- Relation to BPS quivers **[Gabella-PL-Park-Yamazaki in progress]**