A holographic view of symmetry – as shadow of topological order

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2021/04/26 QM^3

Ji Wen arXiv:1912.13492

Kong Lan Wen Zhang Zheng arXiv:2003.08898; arXiv:2005.14178

Simons Collaboration on Ultra-Quantum Matter

Three kinds of many-body quantum systems

- Three kinds of quantum matter:

  (1) No low energy excitations (insulator, superconductor) $\rightarrow$ **trivial**
  (2) A few low energy modes (superfluid, crystal) $\rightarrow$ **simple**
  (3) Infinity low energy modes (Fermi liquids) $\rightarrow$ **complicated**

- **Topological orders** are "trivial" state of matter.
  Since they are trivial, now we start to have a systematic understanding and classification of all topological orders.

- **Next big question**: How to gain a systematic understanding and classification the "simple" gapless quantum states of matter *(which can be strongly correlated)*?

\- **Simple**: a **finite** number of **linear** modes $\omega \propto k$ (*ie* conformal field theory).

\- **Complicated**: low energy effective field theory has **infinite** number of fields.

Wen arXiv:2101.08772

1+1D (quantum) transverse Ising model

- Strongly correlated gapless state in **transverse Ising model**

$$H_{\text{Is}} = -\sum_i (J Z_i Z_{i+1} - h X_i), \quad J, h > 0.$$

where X, Y, Z are Pauli matrices.

- The Z_2 symmetry

$$U_{\mathbb{Z}_2} = \prod_i X_i, \quad |\uparrow\rangle \leftrightarrow |\downarrow\rangle$$

- We will always restrict ourselves in the **symmetric sub-Hilbert space**

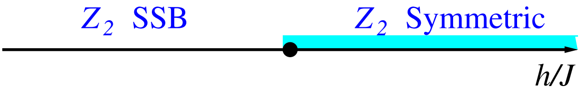
$$\mathcal{V}_{\text{symm}} = \{|\psi\rangle \mid U_{Z_2}|\psi\rangle = |\psi\rangle\}$$

which will be important later.

1+1D Ising critical point
– a strongly correlated “simple” gapless state

Two ways to view the critical point at $h = J$
within such a symmetric sub-Hilbert space $\mathcal{V}_{\text{symm}}$:

- From the symmetric state $|\cdots \to\to\to\to\to \cdots\rangle$ at $h \gg J$. The excitations are even spin flips $|\cdots \to \leftarrow \to\to \leftarrow \cdots\rangle$.
- The spin flips, denoted as e, have the fusion rule
  $$e \otimes e = \mathbf{1}, \qquad (\mathbf{1} = \text{the null excitation})$$
  The fusion rule implies the mod-2 conservation and the $\mathbb{Z}_2$ symm.
- As we decrees h/J, we have more and more spin flips $\to$ **condensation** of spin flips $\to$ spontaneous $\mathbb{Z}_2$ symmetry breaking.
- The transition point is the **Ising critical point**.



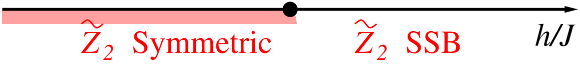
1+1D Ising critical point

- From the symmetry-breaking state in **symmetric subspace** $\mathcal{V}_{\text{symm}}$: $|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle + |\cdots\downarrow\downarrow\downarrow\downarrow\downarrow\cdots\rangle$ at $h \ll J$. The excitations are even domain walls $|\cdots\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\cdots\rangle + |\cdots\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\cdots\rangle$
- The domain walls, denoted as m, have the fusion rule

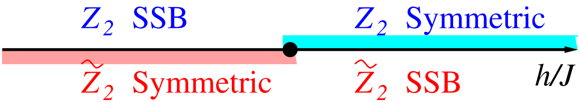
$$m \otimes m = \mathbf{1}, \quad (\mathbf{1} = \text{the null excitation})$$

The fusion rule implies a mod-2 conservation of domain walls, and a new $\widetilde{\mathbb{Z}}_2$ **dual symmetry**.

- As we increase h/J, we have more and more domain walls $\to$ **condensation** of domain walls $\to$ spontaneous $\widetilde{\mathbb{Z}}_2$ dual-symmetry breaking *(and restoration of the $\mathbb{Z}_2$ symmetry)*.
- The transition point is the same Ising critical point.



Critical point has both symmetry & dual symmetry



- The critical point touches the $\mathbb{Z}_2$ symmetric phase and touches the $\widetilde{\mathbb{Z}}_2$ dual-symmetric phase. It has no spin flip condensation, nor domain wall condensation. Ji-Wen, arXiv:1912.13492

  We say **the critical point has both $\mathbb{Z}_2$ symmetry and $\widetilde{\mathbb{Z}}_2$ dual-symmetry**, almost by definition.

- Putting the two symmetries together, we say the critical point has a **categorical symmetry** $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$, which means
  (1) The system (*ie* Hamiltonian) has the categorical symmetry, and
  (2) the categorical symmetry is not spontaneously broken.
- The $\mathbb{Z}_2$-symmetry and the dual $\widetilde{\mathbb{Z}}_2$-symmetry are not independent. There is a "mutual statistics" between them.
  So **the categorical symmetry is not** $\mathbb{Z}_2 \times \widetilde{\mathbb{Z}}_2$.

Patch symmetry transformation

To write down the transformations of symmetry and dual symmetry, **within** the symmetric subspace $\mathcal{V}_{\text{min}}$ and to see the "mutual statistics" between them, we introduce the **patch symmetry transformation**.

- Patch symmetry transformation for $\mathbb{Z}_2$ symmetry is given by

$$U_{\mathbb{Z}_2}(i_1, i_2) = \prod_{i_1 \le i \le i_2} X_i$$

- The **symmetric local operators** O_i acting on several spins near site-i:

$$O_i U_{\mathbb{Z}_2}(i_1, i_2) = U_{\mathbb{Z}_2}(i_1, i_2) O_i, \quad i \text{ far away from } i_1, i_2.$$

- The **symmetric Hamiltonian** (for example, the transverse Ising model) is a sum of symmetric local operators.

Dual Ising model and dual symm. transformation

To write down dual symmetry transformation, we go to the dual representation of the transverse Ising model

$$X_i \to \widetilde{X}_{i-\frac{1}{2}} \widetilde{X}_{i+\frac{1}{2}}, \qquad Z_i Z_{i+1} \to \widetilde{Z}_{i+\frac{1}{2}},$$

$$\widetilde{H}_{DW} = -\sum_i (J \widetilde{Z}_{i+\frac{1}{2}} + h \widetilde{X}_{i-\frac{1}{2}} \widetilde{X}_{i+\frac{1}{2}})$$

The model and the dual model are equivalent within the symmetric sub-Hilbert spaces of each model.

- The $\widetilde{\mathbb{Z}}_2$ dual-symmetry transformation (the **patch operator**)

$$\widetilde{U}_{\widetilde{\mathbb{Z}}_2}(i_1, i_2) = \prod_{i_1 \le i \le i_2} \widetilde{Z}_{i+\frac{1}{2}}$$

- The original $\mathbb{Z}_2$ symmetry transformation (the patch operator)

$$U_{\mathbb{Z}_2}(i_1, i_2) = \widetilde{X}_{i_1+\frac{1}{2}} \widetilde{X}_{i_2+\frac{1}{2}}$$

The $\mathbb{Z}_2$ patch symmetry transformation has a trivial bulk in the dual model $\to$ **any local operators in the dual formulation are**

Symmetry and dual symmetry in Ising model

- Transverse Ising model

$$H_{\text{Is}} = -\sum_i (J Z_i Z_{i+1} - h X_i), \quad J, h > 0.$$

$$U_{\mathbb{Z}_2}(i_1, i_2) = \prod_{i_1 \le i \le i_2} X_i, \qquad \widetilde{U}_{\widetilde{\mathbb{Z}}_2}(i_1, i_2) = Z_{i_1} Z_{i_2}.$$

- Dual Ising model

$$\widetilde{H}_{DW} = -\sum_i (J \widetilde{Z}_{i+\frac{1}{2}} + h \widetilde{X}_{i-\frac{1}{2}} \widetilde{X}_{i+\frac{1}{2}})$$

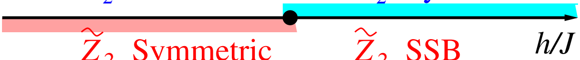
$$U_{\mathbb{Z}_2}(i_1, i_2) = \widetilde{X}_{i_1+\frac{1}{2}} \widetilde{X}_{i_2+\frac{1}{2}} \qquad \widetilde{U}_{\widetilde{\mathbb{Z}}_2}(i_1, i_2) = \prod_{i_1 \le i \le i_2} \widetilde{Z}_{i+\frac{1}{2}}$$

- The two are equivalent within the symmetric sub Hilbert space.
- The patch operator for the dual $\widetilde{\mathbb{Z}}_2$-symmetry creates a pair of $\mathbb{Z}_2$-symmetry charge (a pair of $\mathbb{Z}_2$-symmetry order parameter). The patch operator for the $\mathbb{Z}_2$-symmetry creates a pair of dual $\widetilde{\mathbb{Z}}_2$-symmetry charge (a pair of dual $\widetilde{\mathbb{Z}}_2$-symmetry order parameter)

Relations between symmetry and dual symmetry

$\langle U_{\mathbb{Z}_2}(i_1,i_2)\rangle \sim \mathrm{e}^{-|i_1-i_2|/\xi}$ $\quad$ $\langle U_{\mathbb{Z}_2}(i_1,i_2)\rangle \sim \text{cont.} \neq 0$

Z_2 SSB $\quad$ Z_2 Symmetric



$\widetilde{Z}_2$ Symmetric $\quad$ $\widetilde{Z}_2$ SSB

$\langle \widetilde{U}_{\widetilde{\mathbb{Z}}_2}(i_1,i_2)\rangle \sim \text{cont.} \neq 0$ $\quad$ $\langle \widetilde{U}_{\widetilde{\mathbb{Z}}_2}(i_1,i_2)\rangle \sim \mathrm{e}^{-|i_1-i_2|/\xi}$

- At the critical point both $\langle U_{\mathbb{Z}_2}(i_1,i_2)\rangle$ and $\langle \widetilde{U}_{\widetilde{\mathbb{Z}}_2}(i_1,i_2)\rangle$ are not non-zero constants $\to$ both e (spin flips) and m (domain walls) are not condensed.
- A gapped state must spontaneously break part of the categorical symmetry $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ (for example, $\mathbb{Z}_2$ or $\widetilde{\mathbb{Z}}_2$). A state with unbroken categorical symmetry must be gapless. Levin, arXiv:1903.09028

\- The symmetry and the dual symmetry are not independent

$$U_{\mathbb{Z}_2}(i_1,i_2)\widetilde{U}_{\widetilde{\mathbb{Z}}_2}(j_1,j_2) = -\widetilde{U}_{\widetilde{\mathbb{Z}}_2}(j_1,j_2)U_{\mathbb{Z}_2}(i_1,i_2), \quad i_1 \ll j_1 \ll i_2 \ll j_2.$$

We say they have a **mutual π statistics** $\to$ the above results.

What is so special about gapless states

- The gapless states (the critical point) are very special.

- either the low energy excitations become **non-interacting** *(via the renormalization group flow)*
- or the interaction flow to a very **special form**, so that the low energy interaction is perfectly balanced not to open energy gap.

- Supporting this point of view, we often see a larger low-energy **emergent** symmetry in gapless state (which are not spontaneously broken).

  The new insight: **the emergent symmetry may have non-trivial "mutual statistics".**

- *What is so special about a gapless state?* The full emergent symmetry is a **categorical symmetry** with **non-trivial "mutual statistics"**. Such categorical symmetry may be the key to systematically understand (and might even classify) the strongly correlated simple gapless states (CFTs).

Algebraic symmetry

The spontaneous-$\mathbb{Z}_2$-symmetry-breaking critical point of 1+1D Ising model have a $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ categorical symmetry.

What is the categorical symmetry for 1+1D spontaneous-G-symmetry-breaking critical point?

- The G-symmetry charges are G-representations R_q. The G-charge conservation is encoded in the fusion rule $R_{q_1} \otimes R_{q_2} = \oplus_{q_3} N^{q_3}_{q_1 q_2} R_{q_3}$. (*eg* spin-$\frac{1}{2}$ $\otimes$ spin-$\frac{1}{2}$ $=$ spin-0 $\oplus$ spin-1) (Mathematically, the G-charge conservation is described by a **fusion category** $\mathcal{R}\mathrm{ep}_G$).
- The dual $\widetilde{G}$-symmetry charges are G-SSB domain walls, label by $g \in G$ ($|\cdots g_- g_- g_- \overbrace{g_- g_+}^{g_+ = g g_-} g_+ g_+ g_+ \cdots\rangle$) The $\widetilde{G}$-charge conservation is encoded in the fusion rule $g_1 \times g_2 = g_3$, where $g_3 = g_1 g_2$. (Mathematically, the $\widetilde{G}$-charge conservation is described by a fusion category $\mathcal{V}\mathrm{ec}_G$).

Algebraic symmetry

- The dual $\widetilde{G}$-symmetry is not described by a group.

- In a dual model with degrees freedom $g_{ij} \in G$ living on the links, the ground state with no $\widetilde{G}$-charge: $|\{g_{ij}=1\}\rangle = |\cdots 11111\cdots\rangle$.

- A state with a $\widetilde{G}$-charge: $|\cdots 11g11\cdots\rangle$.

- A $\widetilde{G}$-symmetry transformations: $\widetilde{U}_q = \mathrm{Tr}(\prod_i R_q(g_{i,i+1})$ are given by irreducible representations of G, labeled by q

- $\widetilde{U}_q$'s do not form a group: $\widetilde{U}_{q_1} \otimes \widetilde{U}_{q_2} = \sum_{q_3} N^{q_3}_{q_1 q_2} \widetilde{U}_{q_3}$.

The symmetry $\widetilde{G}$ is called **Algebraic symmetry**.

Kong, Lan, Wen, Zhang, Zheng; arXiv:2005.14178

It is also called **non-invertible defect line**

Chang Lin Shao Wang Yin, arXiv:1802.04445

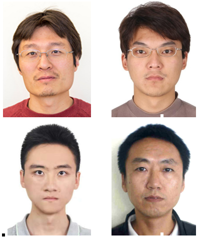
or **fusion category symmetry** Thorngren Wang, arXiv:1912.02817

- The G-symmetry is described fusion category $\mathcal{R}\mathrm{ep}_G$. Algebraic $\widetilde{G}$-symmetry is described fusion category $\mathcal{V}\mathrm{ec}_G$.

Algebraic higher symmetry

- In $n+1$D spacetime, the G-symmetry charges are point-like excitations carrying G-representations R_q.

Kong Lan Wen Zhang Zheng, arXiv:2005.14178



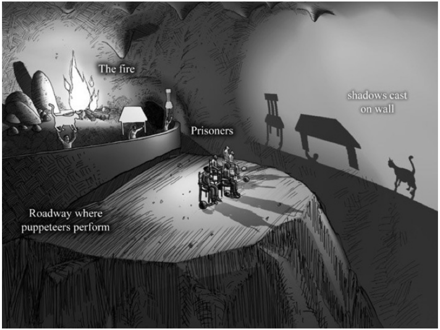
- The dual $\widetilde{G}$-symmetry charges are $n-1$-dimensional domain walls labeled by $g \in G$. The fusion of the $n-1$-dimensional excitations gives rise to an **algebraic higher symmetry**, denoted as $\widetilde{G}^{(n-1)}$. The transformation of the $\widetilde{G}^{(n-1)}$-symmetry is given by Wilson line operators: $\widetilde{U}_q(S^1) = \mathrm{Tr}(\prod_{\langle ij\rangle \in S^1} R_q(g_{ij}))$.

Ji Wen arXiv:1912.13492

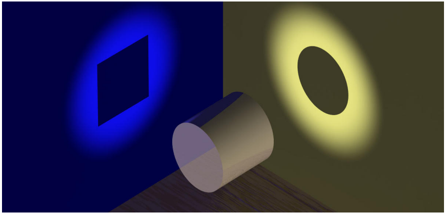
- The SSB critical point of G-symmetry has the categorical symmetry $G \vee \widetilde{G}^{(n-1)}$ with non trivial "mutual non-Abeian statistics". The critical point has gapless 0-dimensional operators carrying G-symmetry charges and gapless $n-1$-dimensional operators carrying $\widetilde{G}^{(n-1)}$-symmetry charges.

Categorical symmetry lives in one higher dimension

To gain a deeper understanding of categorical symmetry $G \vee \widetilde{G}^{(n)}$, such as their non-trivial "mutual (non-Abelian) statistics", we realized that categorical symmetry actually lives in one higher dimension.



Categorical symmetry
$\rightarrow$ symmetry and dual symmetry



Holographic theory of symmetry & dual symmetry

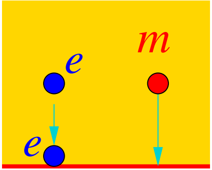
Consider a 2+1D Z_2 gauge theory, which have 4 kind of excitations

$\mathbf{1}$ (null excitations), $\quad e$ ($\mathbb{Z}_2$-charge, boson),

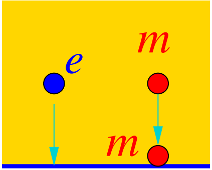
m ($\mathbb{Z}_2$-flux, boson), $\quad f = e \otimes m$ (charge-flux, fermion).

e and m have mutual π statistics between them.

- The Z_2 gauge theory has a gapped boundary induced by m condensation on the boundary. The boundary excitations are described by even number of e particles with fusion $e \otimes e = \mathbf{1}$. $\to$ the $\mathbb{Z}_2$-symmetric phase ($\widetilde{\mathbb{Z}}_2$ symm. broken).



- The Z_2 gauge theory has another gapped boundary induced by e condensation. The boundary excitations are described by even number of m particles with fusion $m \otimes m = \mathbf{1}$. $\to$ the $\mathbb{Z}_2$ symm. broken phase ($\widetilde{\mathbb{Z}}_2$ symmetric).



Holographic theory of symmetry & dual symmetry

The transition from one gapped boundary (m condensed) to another gapped boundary (e condensed) $\to$ a gapless critical critical point – the $\mathbb{Z}_2$ spontaneous symm. breaking critical point.

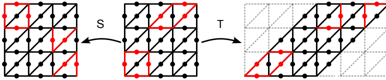
- The $\mathbb{Z}_2$ symmetry-breaking critical point is a gapless boundary of a 2+1D $\mathbb{Z}_2$ gauge theory, that has no m condensation, nor e condensation (*ie* has the full $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ categorical symmetry).
- **Categorical symm. = topol. order in one higher dimension**
  1+1D $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$ categorical symmetry = 2+1D Z_2 gauge theory.
  In the bulk, $e \times e = \mathbf{1} \to \mathbb{Z}_2$ symm. $m \times m = \mathbf{1} \to \widetilde{\mathbb{Z}}_2$ dual symm.
- A low energy effective field have emergent symmetries, and anomalies. Within the symmetric sub-Hilbert space, we may view all those emergent symmetries as **non-invertible gravitational anomalies**, since $\mathcal{V}_{\text{symm}} \neq \otimes_i \mathcal{V}_i$. Ji Wen, arXiv:1905.13279

Gravitational anomaly = topological order in one higher dimension = Categorical symmetry . Wen, arXiv:1303.1803

How categorical symmetry determine gapless state

In other words, how topological order in one higher dimension determine its gapless boundary.

- **What is the data to describe 2+1D topological order?**
- **What is the data to describe 1+1D gapless state?**
- **How the two data are related?**



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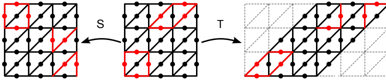


Universal wavefunction overlap
$\to S, T$ matrix $\to$ the data for 2+1D topological order.

- Consider the degenerate ground states $|\Psi_\alpha\rangle$ on torus T^2 for a topo. order. Consider two maps, $\hat{S} = 90^\circ$ rotation and $\hat{T} =$ Dehn twist.

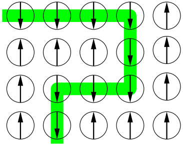
The S, T-matrix from the overlap

$$\hat{S}: \int \prod_i \mathrm{d}x_i \mathrm{d}y_i \ \Psi^*_\alpha(\{x_i, y_i\})\Psi_\beta(\{y_i, -x_i\}) = \mathrm{e}^{-\text{Area}/\xi_S^2} S_{\alpha,\beta}$$

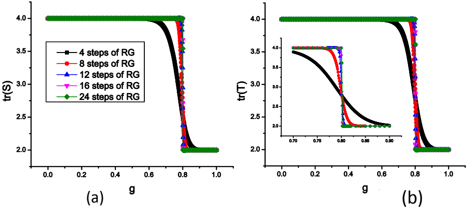
$$\hat{T}: \int \prod_i \mathrm{d}x_i \mathrm{d}y_i \ \Psi^*_\alpha(\{x_i, y_i\})\Psi_\beta(\{x_i + y_i, y_i\}) = \mathrm{e}^{-\text{Area}/\xi_T^2} T_{\alpha,\beta}$$

Heidar-Wen arXiv:1401.0518, He-Heidar-Wen arXiv:1401.5557

Computing universal wave function overlap



- A topological order of Z_2 gauge theory $\Psi(\ ) = 1$ (equal weight superposition of all spin-down loops).
- Another topological order $\Psi(\ ) = (-)^{\#\text{ of loops}}$ → The double semion topo. order described by mutual $U^2(1)$ CS theory
- For the $\mathbb{Z}_2$ topological order:
  $\Psi_1(\ ) = g^{\text{string-length}}$
  $\Psi_2(\ ) = (-)^{W_x} g^{\text{str-len}}$
  $\Psi_3(\ ) = (-)^{W_y} g^{\text{str-len}}$
  $\Psi_4(\ ) = (-)^{W_x+W_y} g^{\text{str-len}}$
- $g < 0.8$ small-loop phase
  $|\Psi_\alpha\rangle$ are the same state
- $g > 0.8$ large-loop phase
  $|\Psi_\alpha\rangle$ are four diff. states
- For the double semion topo. order:



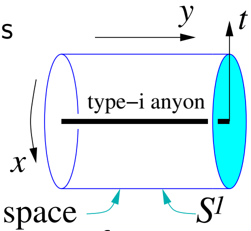
g=0.802

$$S = \begin{pmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{pmatrix} \quad T = \begin{pmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{pmatrix} \qquad S = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \quad T = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}$$

(c)

$$S = \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&-1 \end{pmatrix}, \; T = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix}.$$

How to understand different degenerate ground states on torus?



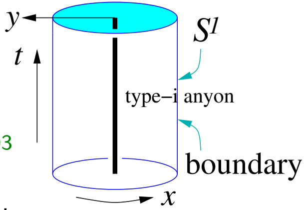
- A spacetime evolution M^3, that produces a torus $T^2 = \partial M^3$, produces a ground state on T^2.
- Different spacetime evolutions M^3's, that produce the same torus $T^2 = \partial M^3$, may produce different degenerate ground states.
- In particular, the embeded worldline of different types of anyon gives rise to different spacetime evolutions and different degenerate ground states $|\Psi_i\rangle$ on torus.
- This gives rise to an orthorgonal basis of the the ground state subspace, the anyon basis $|\Psi_i\rangle$, labeled by anyon types i.
- Under the modular transformations S, T they transform as

$$|\Psi_i\rangle \to S_{ij}|\Psi_j\rangle, \qquad |\Psi_i\rangle \to T_{ij}|\Psi_j\rangle$$

What is the data to describe 1+1D gapless states (gapless edge of topo. order) Ji & Wen arXiv:1905.13279

- A powerful data to describe 1+1D gapless state of a lattice model is the partition function $Z(\mathrm{i}\beta+\delta x)=\mathrm{Tr} e^{-\beta H+\mathrm{i}\delta x P}$, which is **modular invariant**
  $T: Z(\tau)=Z(\tau+1),\quad S: Z(\tau)=Z(-1/\tau).$

- *1+1D lattice system → anomaly free theory*



- Boundary of 2+1D topological order → 1+1D anomalous theory, Wen arXiv:1303.1803
  which has several partition functions $Z_i(\tau)$ labeled by the anyon types i of the 2+1D bulk topo. order. $Z_i(\tau)$ **is not modular invariant but modular covariant**:
  $T^{\text{bdry}}: \ Z_i(\tau+1)=T^{\text{bdry}}_{ij}Z_j(\tau),\qquad S^{\text{bdry}}: \ Z_i(-1/\tau)=S^{\text{bdry}}_{ij}Z_j(\tau).$
  The S^{bdry}, T^{bdry} is the data that characterize the gapless boundary.

The relation between the data for bulk topo. order and the data for its gapless boundary

The relation is very simple: $S = S^{\mathbf{bdry}},\ \ T = T^{\mathbf{bdry}}$

- 2+1D Z_2 topological order (*ie* $\mathbb{Z}_2$ gauge thepry) has 4 types of anyons $1, e, m, f$. It is characterized by data

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

- Its gapless boundary has a 4-component partition function $Z_{\mathbf{1}}(\tau)$, $Z_e(\tau)$, $Z_m(\tau)$, and $Z_f(\tau)$ that satisfy (where $i, j = 1, e, m, f$):
  $T^{\text{bdry}}:\ Z_i(\tau+1) = T_{ij} Z_j(\tau), \quad S^{\text{bdry}}:\ Z_i(-1/\tau) = S_{ij} Z_j(\tau),$
- **Partition function for 1+1D gapless (CFT) state has a form**
  $Z_i(\tau, \bar\tau) = \sum_{\alpha,\beta} \bar\chi_\alpha(\bar\tau) M_i^{\alpha\beta} \chi_\beta(\tau), \quad \chi_\alpha(\tau) =$ conformal characters

Solving the modular covariant condition
$\to$ Several possible gapless boundaries:

- Ising CFT boundary (minimal model $(3,4)$): $c=\bar c=\frac{1}{2}$

$$\begin{pmatrix} Z_{\mathbf{1}}(\tau,\bar\tau) \\ Z_e(\tau,\bar\tau) \\ Z_m(\tau,\bar\tau) \\ Z_f(\tau,\bar\tau) \end{pmatrix} = \begin{pmatrix} |\chi_0^{\text{Is}}(\tau)|^2+|\chi_{\frac{1}{2}}^{\text{Is}}(\tau)|^2 \\ |\chi_{\frac{1}{16}}^{\text{Is}}(\tau)|^2 \\ |\chi_{\frac{1}{16}}^{\text{Is}}(\tau)|^2 \\ \chi_0^{\text{Is}}(\tau)\bar\chi_{\frac{1}{2}}^{\text{Is}}(\bar\tau)+\chi_{\frac{1}{2}}^{\text{Is}}(\tau)\bar\chi_0^{\text{Is}}(\bar\tau) \end{pmatrix},$$

$Z_{\mathbf{1}}(\tau,\bar\tau)=|\chi_0^{\text{Is}}(\tau)|^2+|\chi_{\frac{1}{2}}^{\text{Is}}(\tau)|^2 \to$ all the critical exponents.
Most stable $\to$ critical point of $\mathbb{Z}_2$ symmetry breaking.

- Minimal model $(4,5)$ CFT boundary: $c=\bar c=\frac{7}{10}$

$$\begin{pmatrix} Z_{\mathbf{1}} \\ Z_e \\ Z_m \\ Z_f \end{pmatrix} = \begin{pmatrix} |\chi_0^{m4}|^2+|\chi_{\frac{1}{10}}^{m4}|^2+|\chi_{\frac{3}{5}}^{m4}|^2+|\chi_{\frac{3}{2}}^{m4}|^2 \\ |\chi_{\frac{7}{16}}^{m4}|^2+|\chi_{\frac{3}{80}}^{m4}|^2 \\ |\chi_{\frac{7}{16}}^{m4}|^2+|\chi_{\frac{3}{80}}^{m4}|^2 \\ \chi_0^{m4}\bar\chi_{\frac{3}{2}}^{m4}+\chi_{\frac{1}{10}}^{m4}\bar\chi_{\frac{3}{5}}^{m4}+\chi_{\frac{3}{5}}^{m4}\bar\chi_{\frac{1}{10}}^{m4}+\chi_{\frac{3}{2}}^{m4}\bar\chi_0^{m4} \end{pmatrix}$$

Less stable $\to$ multi-critical point in $\mathbb{Z}_2$ symmetric model

Emergent maximal categorical symmetry

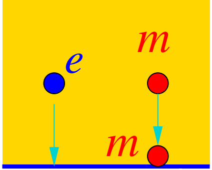
We have seen that the categorical symmetry (*ie* 2+1 $\mathbb{Z}-2$ gauge theory) does not unique determine the critical points, via the modular convariant condition:

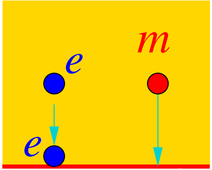
$$T: \; Z_i(\tau+1) = T_{ij} Z_j(\tau), \quad S: \; Z_i(-1/\tau) = S_{ij} Z_j(\tau),$$

But we still hope categorical symmetry fully characterize simple gapless states, and uniquely determine the simple gapless states.

- This can be acheiched by introducing a notion of **maximal categorical symmetry** – the categorical symmetry of the maximal emergent symmetry.

- The $\mathbb{Z}_2$ symmetry (and the associated categorical symmetry $\mathbb{Z}_2 \vee \widetilde{\mathbb{Z}}_2$) discussed above is a lattice (high energy) symmetry. The Ising critical point has a larger emergent symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^{em}$, where $\mathbb{Z}_2^{em}$ is the e-m exchange symmetry (*ie* self-dual symmetry).





- A larger emergent symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^{em} \to$ a larger **maximal categorical symmetry** = 2+1D double-Ising topological order.
- Ising critical point = gapless boundary of the 2+1D double-Ising topological order.
- The boundary of the 2+1D double-Ising topological order is described by a 9-component partition function, since the double-Ising topological order has 9-types of topological excitaions.
- For the *gapless* boundary (*ie* the Ising critical point), the 9-component partition function is given by

Chen, Jian, Kong, You, Zheng, arXiv:1903.12334

$$Z_{hh'}(\tau, \bar\tau) = \chi_h(\tau)\bar\chi_{h'}(\bar\tau), \qquad h, h' = 0, \frac{1}{2}, \frac{1}{16}.$$

- We propose that **the (emergent) maximal categorical symmetry fully characterize simple gapless states, and uniquely determine the simple gapless states**

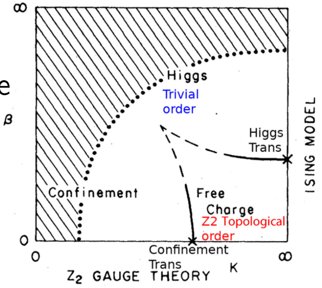
Ji-Wen, arXiv:1912.13492, Kong, Lan, Wen, Zhang, Zheng; arXiv:2005.14178

Apply categorical symm. to 3+1D Z_2 gauge theory

- Transitions from Z_2 topological order (Z_2 guage theory) to trivial product state:

- Higgs transition induced by Z_2 point-charge condensation. *(same as 3+1D Ising trans)* The Ising CFT has a Z_2 symmetry.

- Confinement transition induced by Z_2 flux-string condensation. The transition is first order. If we fine tune to make it a critical point, we get a confinement CFT with a $Z^{(1)}$ symmetry.



Fradkin-Shenker, PRD **19** 3682 (79)

- *The Ising CFT =?= The confinement CFT*
  They might have the same emergent symmetries.

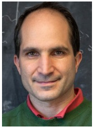
- The Ising CFT has categorical symmetry $Z_2 \vee Z_2^{(2)}$
  The confinement CFT has categorical symmetry $Z_2^{(1)} \vee Z_2^{(1)}$

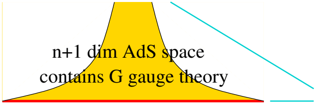
Ji-Wen arXiv:1912.13492

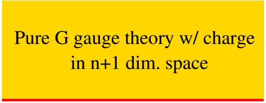
Apply categorical symmetry to AdS/CFT duality

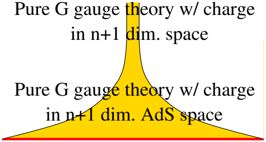
Maldacena arXiv:hep-th/9711200; Witten arXiv:hep-th/9802150

- **Witten**: "for gauge theory, suppose the AdS theory has a gauge group G, [...] Then in the scenario of [13], the group G is a global symmetry group of the conformal field theory on the boundary."





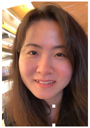




- G-symm.-breaking-transition CFT has a categorical symm. described by the G-gauge theroy in one higher dimension, *which uniquely determines the bulk theory* → A proposal: Pure G-gauge theory (w/ charge fluc. & gravity) in $(n+1)$-dim. AdS space = CFT at the G-symm.-breaking-transition in n-dim. space, not other CFT's with G-symmetry.

Ji-Wen arXiv:1912.13492

Summary



- A **gapless state** is very special, and has a lot of emergent symmetries. The full (?) emergent symmetry is the **maximal categorical symmetry** Ji Wen arXiv:1912.13492
  - Categorical symmetry = Topological order in one higher dimension = (non-invertible) gravitational anomaly
  - Maximal categorical symmetry (*ie* topogical order in one higher dimension) may largely determines a gapless state.
- We can classify all **gapped liquid phases** in systems with a **categorical symmetry**. Such a classification includes
  - Spontaneous symmetry breaking states.
  - SETs with algebraic higher symmetry
  - SPTs with algebraic higher symmetry
  - Gauge the algebraic higher symmetry
  - Anomalous algebraic higher symmetry

Kong Lan Wen Zhang Zheng arXiv:2005.14178