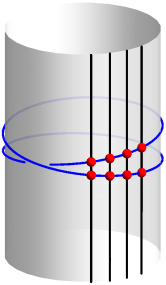
A Puzzle Solved by Massless Modes in AdS_3 Integrability

Michael Abbott

Uniwersytet Jagielloński, Iberia Norte



Mostly about 1512.08761 (PRD 2016)
with Inês Aniceto

Técnico Lisboa, January 2017

Programme

1. One of the nicest results in AdS/CFT:

   Quantum integrability, the dressing phase,
   and tests using macroscopic spinning strings.

2. A problem we were stuck on for 3 years:

   Generalising from $AdS_5 \times S^5$ to $AdS_3 \times S^3 \times T^4$:
   find a different phase, but this gives a mismatch.

3. How this led us to some new physics:

   The limitations of the Bethe Ansatz, and
   first unavoidable appearance of massless modes.

Planar AdS/CFT

4D $\mathcal{N} = 4$ Super Yang–Mills

$g_{\mathrm{YM}}^2 N = \lambda \ll 1$
(and $N = \infty$)

Spectrum Δ_i:

$$\langle O_i(x) O_j(y) \rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

$$\Delta - J_1 - J_2 \equiv E$$

IIB strings in $AdS_5 \times S^5$

$R^2/\alpha' = \sqrt{\lambda} \gg 1$
(and $g_{\mathrm{string}=0}$)

Metsaev–Tseytlin $\mathbb{Z}_4$ coset action:
$PSU(2,2|4)/SO(1,4) \times SO(5)$

$$\int^L d^2\sigma \, \mathrm{STr}\left[j_\mu^{(2)} j^{(2)\mu} + \epsilon^{ab} j_a^{(1)} j_b^{(3)}$$

Planar AdS/CFT ... Triality

4D $\mathcal{N}=4$ Super Yang–Mills

$g^2_{\rm YM} N = \lambda \ll 1$
(and $N=\infty$)

Spectrum Δ_i:

$$\langle O_i(x) O_j(y)\rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

$$\Delta - J_1 - J_2 \equiv E$$

Integrable system

$h = \sqrt{\lambda}/4\pi \quad \forall \lambda$

Asymptotic (i.e. $L \gg 1$) Bethe equations:

$$e^{iLp_i} = \prod_{j\neq i}^{K} S(p_i, p_j)$$

Input two-particle S-matrix, and dispersion relation.
Output:

$$E_{\{p_1\cdots p_K\}} = \sum_j E(p_j)$$

IIB strings in $AdS_5 \times S^5$

$R^2/\alpha' = \sqrt{\lambda} \gg 1$
(and $g_{\rm string=0}$)

Metsaev–Tseytlin $\mathbb{Z}_4$ coset action:
$PSU(2,2|4)/SO(1,4)\times SO(5)$

$$\int^L d^2\sigma\, \mathrm{STr}\left[j^{(2)}_\mu j^{(2)\mu} + \epsilon^{ab} j^{(1)}_a j^{(3)}_b \right.$$

... plus corrections $\mathcal{O}(e^{-L/h})$ at strong coupling

Planar AdS/CFT … Triality

4D $\mathcal{N}=4$ Super Yang–Mills

$g_{\mathrm{YM}}^2 N = \lambda \ll 1$
(and $N = \infty$)

Spectrum Δ_i:

$$\langle O_i(x) O_j(y) \rangle = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

$$\Delta - J_1 - J_2 \equiv E$$

Integrable system

$h = \sqrt{\lambda}/4\pi \quad \forall \lambda$

Asymptotic (i.e. $L \gg 1$) Bethe equations:

$$e^{iLp_i} = \prod_{j \neq i}^{K} S(p_i, p_j)$$

Input two-particle S-matrix, and dispersion relation.
Output:

$$E_{\{p_1 \cdots p_K\}} = \sum_j E(p_j)$$

IIB strings in $AdS_5 \times S^5$

$R^2/\alpha' = \sqrt{\lambda} \gg 1$
(and $g_{\mathrm{string}=0}$)

Metsaev–Tseytlin $\mathbb{Z}_4$ coset action:
$PSU(2,2|4)/SO(1,4) \times SO(5)$

$$\int^L d^2\sigma \, \mathrm{STr}\left[j_\mu^{(2)} j^{(2)\mu} + \epsilon^{ab} j_a^{(1)} j_b^{(3)}$$

Compare these two.

… plus corrections $\mathcal{O}(e^{-L/h})$ at strong coupling

$su(2)$ Sector

One-loop dilatation operator
= Heisenberg spin chain H_2

$$\mathrm{Tr}\,(ZZXZ\ldots) = \downarrow\downarrow\uparrow\downarrow\ldots$$

With K impurities = magnons,
$p_j = -i\log(x_j^+/x_j^-)$ constrained by

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}$$

and $\sum_j p_j = 0$. Energy is then

$$E = \sum_k \frac{h}{2i}\left[x_k^+ - \frac{1}{x_k^+} - \text{c.c.}\right] = \sum_k \sqrt{1 + 4h^2 \sin^2(p_k/2)}$$

[Bethe, 1931] [Minahan & Zarembo, 2002]
[Beisert, Kristjansen, Staudacher, 2003]

$su(2)$ Sector: Weak and Strong Coupling

One-loop dilatation operator
= Heisenberg spin chain H_2

$$\mathrm{Tr}\,(ZZXZ\ldots) = \downarrow\downarrow\uparrow\downarrow\ldots$$

With K impurities = magnons,
$p_j = -i\log(x_j^+/x_j^-)$ constrained by

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}$$

and $\sum_j p_j = 0$. Energy is then

$$E = \sum_k \frac{h}{2i}\left[x_k^+ - \frac{1}{x_k^+} - \text{c.c.}\right] = \sum_k \sqrt{1 + 4h^2 \sin^2(p_k/2)}$$

[Bethe, 1931] [Minahan & Zarembo, 2002]
[Beisert, Kristjansen, Staudacher, 2003]

Classical strings in $\mathbb{R}\times S^3$

also integrable:
≈ complex sine-gordon

$su(2)$ Sector: Weak and Strong Coupling

One-loop dilatation operator
= Heisenberg spin chain H_2

$$\text{Tr}\,(ZZXZ\ldots) = \downarrow\downarrow\uparrow\downarrow\ldots$$

With K impurities = magnons,
$p_j = -i\log(x_j^+/x_j^-)$ constrained by

Classical strings in $\mathbb{R}\times S^3$

also integrable:
≈ complex sine-gordon

Described by almost the same Bethe equations, but with AFS phase

[Arutyunov, Frolov, Staudacher, 2004]

$$\sigma = 1+\mathcal{O}(\lambda^4) \qquad \left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\left(\frac{1-\frac{1}{x_k^+x_j^-}}{1-\frac{1}{x_k^-x_j^+}}\right)\sigma^2(x_k,x_j) \qquad \sigma = e^{i\theta}$$

$$\theta(x_k,x_j) \;=\; h\sum_{s\geq 2} Q_{[s}(x_k)Q_{s+1]}(x_j)$$

and $\sum_j p_j = 0$. Energy is then

$$E = \sum_k \frac{h}{2i}\left[x_k^+ - \frac{1}{x_k^+} - \text{c.c.}\right] = \sum_k \sqrt{1+4h^2\sin^2(p_k/2)}$$

[Bethe, 1931] [Minahan & Zarembo, 2002]
[Beisert, Kristjansen, Staudacher, 2003]

$su(2)$ Sector: Weak and Strong Coupling

One-loop dilatation operator
= Heisenberg spin chain H_2

$$\mathrm{Tr}\,(ZZXZ\ldots) = \downarrow\downarrow\uparrow\downarrow\ldots$$

With K impurities = magnons,
$p_j = -i\log(x_j^+/x_j^-)$ constrained by

Classical strings in $\mathbb{R}\times S^3$

also integrable:
≈ complex sine-gordon

Described by almost the same Bethe equations, but with AFS phase

[Arutyunov, Frolov, Staudacher, 2004]

$\sigma = 1+\mathcal{O}(\lambda^4)$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\left(\frac{1-\frac{1}{x_k^+x_j^-}}{1-\frac{1}{x_k^-x_j^+}}\right)\sigma^2(x_k,x_j)$$

$\sigma = e^{i\theta}$

$$\theta(x_k,x_j) = h\sum_{s\geq 2} Q_{[s}(x_k)Q_{s+1]}(x_j) + \sum_{r,s\geq 2} c_{r,s}Q_rQ_s + \mathcal{O}(1/h)$$

and $\sum_j p_j = 0$. Energy is then

$$E = \sum_k \frac{h}{2i}\left[x_k^+ - \frac{1}{x_k^+} - \text{c.c.}\right] = \sum_k \sqrt{1+4h^2\sin^2(p_k/2)}$$

[Bethe, 1931] [Minahan & Zarembo, 2002]
[Beisert, Kristjansen, Staudacher, 2003]

Quantum effects: HL phase $c_{r,s}$

[Hernandez, Lopez, 2006]

+ higher terms $\mathcal{O}(1/h)$

Semiclassical Spinning Strings

Classical solution in $S^3 \subset \mathbb{C}^2$ [Arutyunov, Russo, Tseytlin]

$$Z_1 = \frac{1}{\sqrt{2}} e^{i\mathscr{J}\tau + im\sigma}, \qquad Z_2 = \frac{1}{\sqrt{2}} e^{i\mathscr{J}\tau - im\sigma}, \qquad t = \kappa\tau$$

with charges $J_1 = J_2 = \frac{1}{2}\sqrt{\lambda}\,\mathscr{J} \gg 1$.

Semiclassical Spinning Strings

Classical solution in $S^3 \subset \mathbb{C}^2$ [Arutyunov, Russo, Tseytlin]

$$Z_1 = \frac{1}{\sqrt{2}} e^{i\mathcal{J}\tau + im\sigma}, \qquad Z_2 = \frac{1}{\sqrt{2}} e^{i\mathcal{J}\tau - im\sigma}, \qquad t = \kappa\tau$$

with charges $J_1 = J_2 = \frac{1}{2}\sqrt{\lambda}\,\mathcal{J} \gg 1$.

Quantum correction to its energy:

$$\delta E = \frac{1}{2\kappa} \sum_r^{8+8} (-1)^{F_r} \sum_n^{\infty} w_n^r$$

frequencies $w_n^A = \sqrt{n^2 + \kappa^2}$ and $w_n^{S\pm} = \sqrt{n^2 + 2\mathcal{J}^2 \pm 2\sqrt{n^2(\mathcal{J}^2 + m^2) + \mathcal{J}^4}}$ etc.

Semiclassical Spinning Strings

Classical solution in $S^3 \subset \mathbb{C}^2$ [Arutyunov, Russo, Tseytlin]

$$Z_1 = \frac{1}{\sqrt{2}} e^{i\mathcal{J}\tau + im\sigma}, \qquad Z_2 = \frac{1}{\sqrt{2}} e^{i\mathcal{J}\tau - im\sigma}, \qquad t = \kappa\tau$$

with charges $J_1 = J_2 = \frac{1}{2}\sqrt{\lambda}\,\mathcal{J} \gg 1$.

Quantum correction to its energy:

$$\delta E = \frac{1}{2\kappa} \sum_r^{8+8} (-1)^{F_r} \sum_n^{\infty} w_n^r$$

frequencies $w_n^A = \sqrt{n^2 + \kappa^2}$ and $w_n^{S\pm} = \sqrt{n^2 + 2\mathcal{J}^2 \pm 2\sqrt{n^2(\mathcal{J}^2 + m^2) + \mathcal{J}^4}}$ etc.
lead to via [Beisert & Tseytlin]'s resummation:

$$\delta E^{\text{int}} = -\frac{m^6}{3\mathcal{J}^5} + \frac{m^8}{3\mathcal{J}^7} - \frac{49m^{10}}{120\mathcal{J}^9} + \frac{2m^{12}}{5\mathcal{J}^{11}} + \dots$$

Now recover this from Bethe equations...

Semiclassical Bethe Equations

Charges $J_2 = J_2 \gg 1$ implies half the spins are flipped, on a long chain.

Many Bethe roots ($\sim \sqrt{\lambda}$) but all in a line — one-cut resolvent:

$$G(\tilde{x}) = \sum_k \frac{1}{\tilde{x} - \tilde{x}_k} \frac{\tilde{x}_k^2}{\tilde{x}_k^2 - \tilde{g}^2} = 2\pi m - \frac{\sqrt{1 + (4\pi m \tilde{x})^2} - \sqrt{1 + (4\pi m \tilde{g})^2}}{2(\tilde{x} - \tilde{g}^2 / \tilde{x})}$$

Semiclassical Bethe Equations

Charges $J_2 = J_2 \gg 1$ implies half the spins are flipped, on a long chain.

Many Bethe roots ($\sim \sqrt{\lambda}$) but all in a line — one-cut resolvent:

$$G(\tilde{x}) = \sum_k \frac{1}{\tilde{x} - \tilde{x}_k} \frac{\tilde{x}_k^2}{\tilde{x}_k^2 - \tilde{g}^2} = 2\pi m - \frac{\sqrt{1 + (4\pi m \tilde{x})^2} - \sqrt{1 + (4\pi m \tilde{g})^2}}{2(\tilde{x} - \tilde{g}^2 / \tilde{x})}$$

Energy correction due to arbitrary one-loop dressing phase is

$$\delta E = \frac{m^4 c_{1,2}}{4 \mathscr{J}^3} + \frac{m^6 \left(-4 c_{1,2} - c_{1,4} + c_{2,3}\right)}{16 \mathscr{J}^5} + \frac{m^8 \left(15 c_{1,2} + 5 c_{1,4} + 2 c_{1,6} - 5 c_{2,3} - 2 c_{2,5} + c_{3,4}\right)}{64 \mathscr{J}^7} + \ldots$$

Semiclassical Bethe Equations

Charges $J_2 = J_2 \gg 1$ implies half the spins are flipped, on a long chain.

Many Bethe roots ($\sim \sqrt{\lambda}$) but all in a line — one-cut resolvent:

$$G(\tilde{x}) = \sum_k \frac{1}{\tilde{x} - \tilde{x}_k} \frac{\tilde{x}_k^2}{\tilde{x}_k^2 - \tilde{g}^2} = 2\pi m - \frac{\sqrt{1 + (4\pi m \tilde{x})^2} - \sqrt{1 + (4\pi m \tilde{g})^2}}{2(\tilde{x} - \tilde{g}^2 / \tilde{x})}$$

Energy correction due to arbitrary one-loop dressing phase is

$$\delta E = \frac{m^4 c_{1,2}}{4 \mathcal{J}^3} + \frac{m^6 \left(-4c_{1,2} - c_{1,4} + c_{2,3}\right)}{16 \mathcal{J}^5} + \frac{m^8 \left(15c_{1,2} + 5c_{1,4} + 2c_{1,6} - 5c_{2,3} - 2c_{2,5} + c_{3,4}\right)}{64 \mathcal{J}^7} + \ldots$$

To match the string theory, you want [Hernandez & Lopez, 2006]

$$c_{r,s}^{\mathrm{HL}} = -8 \frac{(r-1)(s-1)}{(r+s-2)(s-r)}, \quad r,s \geq 2, \quad r+s \text{ odd}$$

Semiclassical Bethe Equations

Charges $J_2 = J_2 \gg 1$ implies half the spins are flipped, on a long chain.

Many Bethe roots ($\sim \sqrt{\lambda}$) but all in a line — one-cut resolvent:

$$G(\tilde{x}) = \sum_k \frac{1}{\tilde{x} - \tilde{x}_k} \frac{\tilde{x}_k^2}{\tilde{x}_k^2 - \tilde{g}^2} = 2\pi m - \frac{\sqrt{1 + (4\pi m \tilde{x})^2} - \sqrt{1 + (4\pi m \tilde{g})^2}}{2(\tilde{x} - \tilde{g}^2/\tilde{x})}$$

Energy correction due to arbitrary one-loop dressing phase is

$$\delta E = \frac{m^4 c_{1,2}}{4\mathcal{J}^3} + \frac{m^6 \left(-4c_{1,2} - c_{1,4} + c_{2,3}\right)}{16\mathcal{J}^5} + \frac{m^8 \left(15c_{1,2} + 5c_{1,4} + 2c_{1,6} - 5c_{2,3} - 2c_{2,5} + c_{3,4}\right)}{64\mathcal{J}^7} + \ldots$$

To match the string theory, you want [Hernandez & Lopez, 2006]

$$c_{r,s}^{\mathrm{HL}} = -8 \frac{(r-1)(s-1)}{(r+s-2)(s-r)}, \quad r, s \geq 2, \quad r + s \text{ odd}$$

Many other checks of this phase, all perfect
including derivations from crossing, or from magnon scattering.

$AdS_3 \times S^3 \times T^4$ Integrability

This time around we construct all-λ results from $\lambda \gg 1$ side only.

[Babichenko, Stefanski, Zarembo, 2010]

$$AdS_5 \times S^5 \to \underbrace{AdS_3 \times S^3}_{\text{massive}} \times \underbrace{T^4}_{\text{massless}}$$

$AdS_3 \times S^3 \times T^4$ Integrability

This time around we construct all-λ results from $\lambda \gg 1$ side only.

[Babichenko, Stefanski, Zarembo, 2010]

$$AdS_5 \times S^5 \to \underbrace{AdS_3 \times S^3}_{\text{massive}} \times \underbrace{T^4}_{\text{massless}}$$

Best studied part is the massive sector,
where mostly you can ignore the massless modes: 2012-2015

- Centrally extended symm → S-matrix $S^{\bullet\bullet}$ [Borsato et. al.] and Bethe equations. (No changes in $su(2)$ sector.)
- Unitarity methods for S-matrix [Bianchi, Forini, Hoare]
- Near-BMN scattering [Roiban, Sundin, Tseytlin, Wulff] (all $AdS_n \times S^n$!)
- Calculation of the dressing phase $\sigma^{\bullet\bullet}$ from crossing [Borsato et. al.] or semiclassical magnon scattering [MCA].

$AdS_3 \times S^3 \times T^4$ Integrability

This time around we construct all-λ results from $\lambda \gg 1$ side only.

[Babichenko, Stefanski, Zarembo, 2010]

$$AdS_5 \times S^5 \to \underset{\text{massive}}{AdS_3 \times S^3} \times \underset{\text{massless}}{T^4}$$

Best studied part is the massive sector,
where mostly you can ignore the massless modes: 2012-2015

- Centrally extended symm → S-matrix $S^{\bullet\bullet}$ [Borsato et. al.] and Bethe equations. (No changes in $su(2)$ sector.)
- Unitarity methods for S-matrix [Bianchi, Forini, Hoare]
- Near-BMN scattering [Roiban, Sundin, Tseytlin, Wulff] (all $AdS_n \times S^n$!)
- Calculation of the dressing phase $\sigma^{\bullet\bullet}$ from crossing [Borsato et. al.] or semiclassical magnon scattering [MCA].

In addition, the full S-matrix $\begin{bmatrix} S^{\bullet\bullet} & S^{\bullet\circ} \\ S^{\bullet\circ} & S^{\circ\circ} \end{bmatrix}$ is known up to dressing phases.

[Borsato, Ohlsson Sax, Sfondrini, Stefanski]

Spinning Strings in $AdS_3 \times S^3 \times T^4$

Look at the same comparison as before:

Easy to adapt the string calculation (just explores S^3)

$$\delta E_{\text{string}} = \frac{m^4}{2\mathcal{J}^3} - \frac{7m^6}{12\mathcal{J}^5} + \frac{29m^8}{48\mathcal{J}^7} - \frac{97m^{10}}{160\mathcal{J}^9} + \frac{2309m^{12}}{3840\mathcal{J}^{11}} + \dots$$

The $su(2)$ Bethe equations are identical, but the one loop phase is

$$c_{r,s} = \left[2\frac{s-r}{r+s-2} - \delta_{r,1} + \delta_{1,s} \right], \qquad r+s \text{ odd}, \ r,s \geq 1$$

[Borsato, Ohlsson Sax, Sfondrini, Stefanski, Torrielli, 2013] [MCA, 2013]

Spinning Strings in $AdS_3 \times S^3 \times T^4$

Look at the same comparison as before:

Easy to adapt the string calculation (just explores S^3)

$$\delta E_{\text{string}} = \frac{m^4}{2\mathcal{J}^3} - \frac{7m^6}{12\mathcal{J}^5} + \frac{29m^8}{48\mathcal{J}^7} - \frac{97m^{10}}{160\mathcal{J}^9} + \frac{2309m^{12}}{3840\mathcal{J}^{11}} + \dots$$

The $su(2)$ Bethe equations are identical, but the one loop phase is

$$c_{r,s} = \left[2\frac{s-r}{r+s-2} - \delta_{r,1} + \delta_{1,s}\right], \qquad r+s \text{ odd}, \ r,s \geq 1$$

which gives the wrong answer: [Borsato, Ohlsson Sax, Sfondrini, Stefanski, Torrielli, 2013] [MCA, 2013]

$$\begin{aligned}\delta E_{\text{BAE}} &= +\frac{m^4}{4\mathcal{J}^3} - \frac{13m^6}{48\mathcal{J}^5} + \frac{25m^8}{96\mathcal{J}^7} - \frac{311m^{10}}{1280\mathcal{J}^9} + \frac{1723m^{12}}{7680\mathcal{J}^{11}} + \dots \\ &= \delta E_{\text{string}} - \frac{m^4}{4\mathcal{J}^3} + \frac{5m^6}{16\mathcal{J}^5} - \frac{11m^8}{32\mathcal{J}^7} + \frac{93m^{10}}{256\mathcal{J}^9} - \frac{193m^{12}}{512\mathcal{J}^{11}} + \dots \\ &= \delta E_{\text{string}} + \frac{m^2(\mathcal{J}-\kappa)}{2\kappa^2}\end{aligned}$$

recall $\kappa = \sqrt{m^2 + \mathcal{J}^2}$

Regime Change

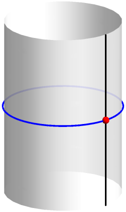
In $AdS_5 \times S^5$ we can distinguish:

1. Infinite L, with free spectrum $\sum_i E(p_i)$
2. Bethe regime: $e^{ip_iL} = \prod_{j\neq i} S(p_i, p_j)$ quantises p_i giving $1/L$ corrections
3. Lüscher regime, exponential corrections $\sim e^{-L/h}$
4. Small L, studied using TBA.

Regime Change

In $AdS_5 \times S^5$ we can distinguish:

1. Infinite L, with free spectrum $\sum_i E(p_i)$
2. Bethe regime: $e^{ip_iL} = \prod_{j\neq i} S(p_i, p_j)$ quantises p_i giving $1/L$ corrections
3. Lüscher regime, exponential corrections $\sim e^{-L/h}$
4. Small L, studied using TBA.



Lüscher corrections can be derived [Lüscher, 1986] from the new Feynman diagram possible wrapping a finite box. Result is

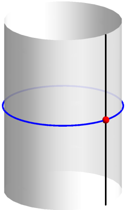
$$\delta E^F = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_b^{4+4} (-1)^{F_b} e^{-iq_\star L} S_{ba}^{ba}(p, q_\star)$$

with physical $(\epsilon(p), p)$ and virtual $(iq, q_\star)$,
choose contour $q \in \mathbb{R}$.

Regime Change

In $AdS_5 \times S^5$ we can distinguish:

1. Infinite L, with free spectrum $\sum_i E(p_i)$
2. Bethe regime: $e^{ip_i L} = \prod_{j\neq i} S(p_i, p_j)$ quantises p_i giving $1/L$ corrections
3. Lüscher regime, exponential corrections $\sim e^{-L/h}$
4. Small L, studied using TBA.

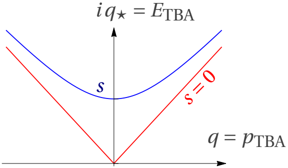


Lüscher corrections can be derived [Lüscher, 1986] from the new Feynman diagram possible wrapping a finite box. Result is

$$\delta E^F = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_b^{4+4} (-1)^{F_b} e^{-iq_\star L} S_{ba}^{ba}(p, q_\star)$$

with physical $(\epsilon(p), p)$ and virtual $(iq, q_\star)$, choose contour $q \in \mathbb{R}$.

This exponent is very different at mass $s = 0$:

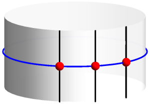


Applied to Circular Strings

For spinning strings, not one but many (order $\sqrt{\lambda}$) physical particles.

So we need a multiparticle Lüscher formula derived from TBA: [Bajnok & Janik, 2008]

$$\delta E = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{b}^{4+4} (-1)^{F_b} e^{-iq_\star L} \prod_{k=1}^{K} S_{ba}^{ba}(p_k, q_\star) + \text{zero}$$

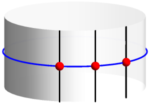


Applied to Circular Strings

For spinning strings, not one but many (order $\sqrt{\lambda}$) physical particles.

So we need a multiparticle Lüscher formula derived from TBA: [Bajnok & Janik, 2008]

$$\delta E = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{b}^{4+4} (-1)^{F_b} e^{-i q_\star L} \prod_{k=1}^{K} S_{ba}^{ba}(p_k, q_\star) + \text{zero}$$



Using $S^{\bullet\circ}$ from [Borsato, Ohlsson Sax, Sfondrini, Stefanski, 2015] for massless virtual particle:

$$\delta E = \int dq\, e^{-|q|L\#} \text{stuff} = \frac{-m^4}{2\mathscr{J}^3} + \frac{15m^6 + \pi^2 m^8}{24\mathscr{J}^5} - \frac{990m^8 + 135\pi^2 m^{10} + 2\pi^4 m^{12}}{1440\mathscr{J}^7} + \mathcal{O}\left(\frac{1}{\mathscr{J}^9}\right)$$

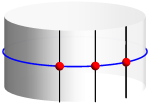
Right order, wrong coefficients...

Applied to Circular Strings

For spinning strings, not one but many (order $\sqrt{\lambda}$) physical particles.

So we need a multiparticle Lüscher formula derived from TBA: [Bajnok & Janik, 2008]

$$\delta E = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{b}^{4+4} (-1)^{F_b} e^{-i q_\star L} \prod_{k=1}^{K} S_{ba}^{ba}(p_k, q_\star) + \text{zero}$$



Using $S^{\bullet\circ}$ from [Borsato, Ohlsson Sax, Sfondrini, Stefanski, 2015] for massless virtual particle:

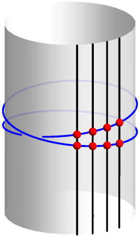
$$\delta E = \int dq\, e^{-|q| L \#} \text{stuff} = \frac{-m^4}{2\mathcal{J}^3} + \frac{15 m^6 + \pi^2 m^8}{24 \mathcal{J}^5} - \frac{990 m^8 + 135\pi^2 m^{10} + 2\pi^4 m^{12}}{1440 \mathcal{J}^7} + \mathcal{O}\left(\frac{1}{\mathcal{J}^9}\right)$$

Right order, wrong coefficients...

... but certainly enough to show that we cannot ignore such corrections:

Regimes #2 and #3 overlap.

Massless Multiple Wrapping

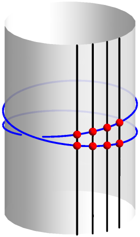


Lüscher formula above is usually leading exponential e^{-L}, and wrapping twice would give e^{-2L}, and so on.

However with a massless particle, all of these are order $1/L^3$ so we must sum:

$$\delta E = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-inq_\star L} \sum_{b}^{4+4} (-1)^{F_b} \left[\prod_{k=1}^{K} S_{ba}^{ba}(q_\star, p_k) \right]^n$$

Massless Multiple Wrapping



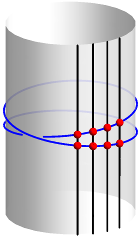
Lüscher formula above is usually leading exponential e^{-L}, and wrapping twice would give e^{-2L}, and so on.

However with a massless particle, all of these are order $1/L^3$ so we must sum:

$$\begin{aligned}
\delta E &= \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-inq_\star L} \sum_b^{4+4} (-1)^{F_b} \left[\prod_{k=1}^{K} S_{ba}^{ba}(q_\star, p_k) \right]^n \\
&= \frac{2}{\phi} \left[-\frac{\pi}{3} + \frac{1}{\pi} \mathrm{Li}_2\left(e^{2i\theta}\right) + \frac{1}{\pi} \mathrm{Li}_2\left(e^{-2i\theta}\right) \right] \\
&= -\frac{m^2(\mathcal{J}-\kappa)}{\kappa^2} \quad = -2\left[-\frac{m^4}{4\mathcal{J}^3} + \frac{5m^6}{16\mathcal{J}^5} - \frac{11m^8}{32\mathcal{J}^7} + \frac{93m^8}{256\mathcal{J}^9} + \ldots \right]
\end{aligned}$$

Almost perfect!

Massless Multiple Wrapping



Lüscher formula above is usually leading exponential e^{-L}, and wrapping twice would give e^{-2L}, and so on.

However with a massless particle, all of these are order $1/L^3$ so we must sum:

$$\begin{aligned}\delta E &= \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-inq_\star L} \sum_{b}^{4+4} (-1)^{F_b} \left[\prod_{k=1}^{K} S_{ba}^{ba}(q_\star, p_k) \right]^n \\ &= \frac{2}{\phi} \left[-\frac{\pi}{3} + \frac{1}{\pi} \mathrm{Li}_2\left(e^{2i\theta}\right) + \frac{1}{\pi} \mathrm{Li}_2\left(e^{-2i\theta}\right) \right] \\ &= -\frac{m^2(\mathscr{J} - \kappa)}{\kappa^2} \quad = -2\left[-\frac{m^4}{4\mathscr{J}^3} + \frac{5m^6}{16\mathscr{J}^5} - \frac{11m^8}{32\mathscr{J}^7} + \frac{93m^8}{256\mathscr{J}^9} + \ldots \right]\end{aligned}$$

Almost perfect!

The mixed-mass dressing phase of $S^{\bullet\circ}$ isn't very well fixed; here we assume ordinary AFS phase as in [B O S S, 2015].

Zooming Out

What this means:

- Perhaps there is no Bethe regime, without wrapping.
- Clearly we need a TBA including the massless modes.

Some puzzles:

- $SL(2)$ sector i.e. $AdS_3 \times S^1$:
  - Circular strings — stricter test than $su(2)$, one more parameter. [H&L 2006]
    Lüscher gives correct order δE but wrong coefficients? [2017?]
  - Short folded strings — calculated to $\mathcal{O}(S^2)$ and agrees with Bethe.
    [Gromov & Valatka, 2011] [Beccaria & Macorini 2013]
- Two-loop disagreement in dispersion relation for massless particles
  [Sundin & Wulff 2015]

Muito Obrigado, Bardzo Dziękuję!