Quantum phase transition in supersymmetric QED$_3$

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- Outlook and open problems

- Progress on exact computations in supersymmetric quantum field theories. Localization: path integral $\Rightarrow$ finite dimensional integral.
- We study an Abelian theory, so our integral representation is one-dimensional (instead of matrix model).
- Localization rules in 3d: The coupling for the background vector multiplet responsible for FI terms contributes

$$S_{classical} = 2\pi i\eta \mathrm{Tr}\left(\sigma\right).$$

- For every $\mathcal{N} = 4$ hypermultiplet (matter) there is a factor

$$Z_{1-\mathrm{loop}} = \prod_{\rho} \frac{1}{\cosh\left(\pi\rho\left(\sigma\right)\right)}$$

- The conical or Mehler functions are associated Legendre functions $P_{\nu}^{\mu}(x)$ of complex index $\nu = -\frac{1}{2} + i\tau$.

$$P_{\nu}^{\mu}(x) = \frac{1}{\Gamma(1-\mu)} \left| \frac{1+x}{1-x} \right|^{\frac{\mu}{2}} {}_2F_1\Big(-\nu, \nu, 1-\mu; \frac{1}{2}(1-x) \Big).$$

- They are the kernel of the Mehler–Fock transform (relevant in hyperbolic geometry, disordered systems, ...) but we will not use all these properties today.
- Instead, it admits a number of equivalent integral representations, and one of them can be identified with a localization result.

- We focus on $\mathcal{N}=4$ SQED on S^3 with FI parameter. Thus, we consider an $\mathcal{N}=4$ $U(1)$ theory consisting of $2N$ $\mathcal{N}=4$ massive (flavor) hypermultiplets (N of mass m and N of mass $-m$), coupled to an $\mathcal{N}=4$ vector multiplet. Localization readily leads to:

$$
\begin{aligned}
Z_{\mathrm{QED}_3} &= \int_{-\infty}^{\infty} \frac{e^{i\eta x} dx}{\left[2\cosh\left(\frac{x+m}{2}\right) 2\cosh\left(\frac{x-m}{2}\right)\right]^N} \\
&= 2^{-N} \int_{-\infty}^{\infty} \frac{dx e^{i\eta x}}{\left[\cosh x + \cosh m\right]^N}.
\end{aligned}
$$

- This matter content but with a Chern-Simons term and in the $U(N)$ case was considered in Barranco-Russo (arXiv:1401.3672); Russo-Silva-MT CMP, 338, 1411 (2015); G. Giasemidis-MT, JHEP 1601 (2016) 068; MT, JHEP 1604 (2016) 168

- We then find

$$Z_{\mathrm{QED}_3} = \sqrt{2\pi}\,\frac{\Gamma\left(N+i\eta\right)\Gamma\left(N-i\eta\right)}{\Gamma\left(N\right)\left(\sinh(m)\right)^{N-\frac{1}{2}}}\;P^{\frac{1}{2}-N}_{-\frac{1}{2}+i\eta}(\cosh(m))\;. \qquad (1)$$

- Equivalently, in terms of an hypergeometric function ($z \equiv \cosh(m)$)

$$Z_{\mathrm{QED}_3} = \frac{\sqrt{2\pi}\;\Gamma\left(N+i\eta\right)\Gamma\left(N-i\eta\right)}{\Gamma\left(N\right)\Gamma\left(N+\frac{1}{2}\right)\left(1+z\right)^{N-\frac{1}{2}}} \qquad (2)$$
$$\times\;{}_2F_1\left(\frac{1}{2}-i\eta,\frac{1}{2}+i\eta,N+\frac{1}{2};\frac{1}{2}(1-z)\right).$$

- Particular cases, for two and four flavors:

$$Z^{N=1}_{\mathrm{QED}_3} = \frac{2\pi \sin(m\eta)}{\sinh(m)\sinh(\pi\eta)}, \tag{3}$$

$$Z^{N=2}_{\mathrm{QED}_3} = \frac{2\pi\left(\cosh m \sin(m\eta) - \eta \sinh m \cos(m\eta)\right)}{\sinh^3(m)\sinh(\pi\eta)}. \tag{4}$$

- These can also be obtained from residue integration (S. Benvenuti and S. Pasquetti, JHEP **1205**, 099 (2012)).
- Another offshots of the hyp. rep.: 1) a three-term recurrence relationship in N, used to generate explicit expressions. 2) A second-order differential equation (Schrodinger eq.). 3) A large m formula.

- The integrand above can be written as $e^{-NS(\lambda)}$ where the *action* S is

$$S(\lambda, x, z) = -i\lambda x + \log(\cosh x + \cosh m) \ .$$

- The saddle-point equation is then $(\lambda \equiv \eta / N)$

$$-i\lambda + \frac{\sinh x}{\cosh x + \cosh m} = 0,$$

which has as solutions,

$$x_{1,2} = \log\left(\frac{-\lambda \cosh m \pm i\Delta}{i + \lambda}\right) + 2\pi i n, \tag{5}$$

where $n \in \mathbb{Z}$ and $\Delta \equiv \sqrt{1 - \lambda^2 \sinh^2 m}$.

- We show that the theory undergoes a large N phase transition at $\lambda_c \equiv 1/\sinh m$, or, more generally, at the critical line $\lambda \sinh(m) = 1$ in the (λ, m) space, where $\Delta = 0$.
- *Subcritical phase* $(\lambda \sinh(m) < 1)$. All saddle points lie on *Im* axis. The saddle point x_1 with $n = 0$ is the relevant one and, to leading order for large N, the Z becomes

$$Z_{\mathrm{QED}_3} \approx \frac{\sqrt{2\pi}}{\sqrt{NS''(x_1)}} \exp\left(-NS\left(x_1\right)\right). \tag{6}$$

where $S''(z,x) = \left(z\cosh\left(x\right)+1\right) / \left(\cosh\left(x\right)+z\right)^2$.

- The two saddle points move to the complex plane, with $x_2 = -x_1^*$. Action is complex, with

$$Re\left(S\left(x_1\right)\right) = Re\left(S\left(x_2\right)\right), \quad Im\left(S\left(x_1\right)\right) = -Im\left(S\left(x_2\right)\right).$$

- Both saddle points contribute with equal weights and need to be taken into account. Then:

$$Z \approx \sqrt{\frac{2\pi}{N}} e^{-NRe(S(x_1))} \left(\frac{e^{-iNIm(S(x_1))}}{\sqrt{S''\left(x_1\right)}} + c.c. \right).$$

- For $N \gg 1$, this expression agrees with the exact analytic expression. For large m, it is also in precise agreement with a large mass formula in the paper, based on the hyp. representation.

Susy QED_3

Second order phase transition

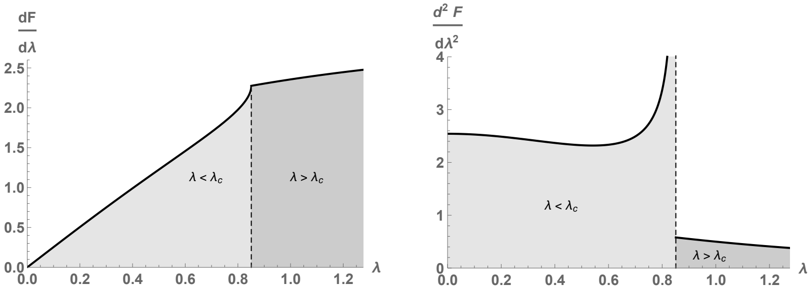

Figure: (a) Behavior of $dF/d\lambda$. (b) Discontinuity of $d^2F/d\lambda^2$ at the transition point $(m = 1)$.

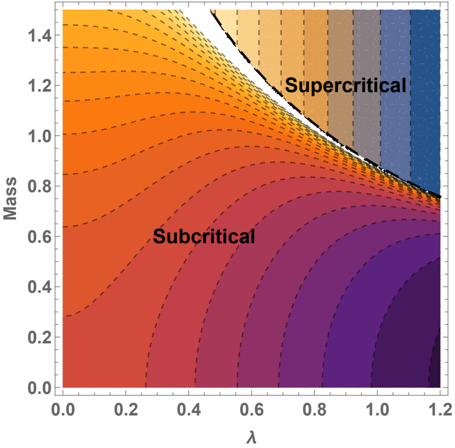

Figure: Phase diagram. The critical line (dashed) $\lambda \sinh(m) = 1$ separates the two phases. The plot also shows the contour lines of $d^2F/d\lambda^2$ (increasing from dark to light).

Correlation functions of gauge invariant mass operators

See also Dedushenko, Pufu and Yacoby [arXiv:1610.00740]

- By diff. the F w.r.t. m, one generates correlators of the gauge invariant mass operator

$$J_3 = \frac{1}{2N}\left(\widetilde{Q}_{1,i}Q_1^i - \widetilde{Q}_{2,i}Q_2^i\right),$$

where Q_1 are the chiral multiplets of mass m and Q_2 the chiral multiplets of mass $-m$.

- For example, for the simple $N = 1$ case, we have that

$$\langle J_3 \rangle \propto \frac{dF}{dm} = \eta \cot\left(m\eta\right) - \coth\left(m\right),$$

$$\langle J_3 J_3 \rangle - \langle J_3 \rangle \langle J_3 \rangle \propto \frac{d^2F}{dm^2} = -\frac{\eta^2}{\sin^2\left(m\eta\right)} + \frac{1}{\sinh^2\left(m\right)}.$$

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- Returning to the large N free energy, we find that $\langle J_3 \rangle$ is continuous, whereas

$$\left(\frac{d^2F}{dm^2}\right)_{\lambda<\lambda_c} = \frac{1}{N\sinh^2 m}\left(1 - \frac{\cosh m}{\sqrt{1-\lambda^2\sinh^2 m}}\right)$$

$$\left(\frac{d^2F}{dm^2}\right)_{\lambda>\lambda_c} = \frac{1}{N\sinh^2 m}\ .$$

- Thus d^2F/dm^2 is discontinuous, implying a discontinuity in the two-point function of operator J_3. The 2-point correlation function diverges as the critical line is approached from the subcritical phase.

Summary and Outlook

- The saddle-point analysis suggests that the mechanism that triggers the phase transition is quite generic. Also, the non-Abelian theory ($\mathcal{N} = 4$ SYM theory) admits a determinantal representation whose elements are conical functions (in progress).

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- The saddle-point analysis suggests that the mechanism that triggers the phase transition is quite generic. Also, the non-Abelian theory ($\mathcal{N} = 4$ SYM theory) admits a determinantal representation whose elements are conical functions (in progress).
- The theory is dual to a A_{N-1} quiver gauge theory $\Rightarrow$ quiver gauge theory also has a novel type of phase transition in the limit when the number of quiver nodes goes to infinity.
- The Z_{QED_3} with FI term is given in terms of the conical function. This simple formula encapsulates rich physical phenomena: large N phase transitions, asymptotic 1/N expansion, emergence of complex saddle points, non-perturbative effects and aspects of mirror symmetry.