Superfields and superschemes

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Superspaces and superfields

A *superspace* is constructed from a certain topological space $|X|$ by putting on it the *structural sheaf*, a sheaf of superalgebras

$$U \subset_{\text{open}} |X| \to \mathcal{F}(U)$$

in such way that there are a restriction maps for $V \subset_{\text{open}} U$

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

which glue together correctly.

There is a *stalk* at each point $a \in |X|$, $\mathcal{F}_a$, that is a *local superalgebra* (it has a unique maximal ideal I_a)

$$xI_a \subset I_a, \qquad \forall x \in \mathcal{F}_a.$$

Examples of sheaves: Continuous functions (topological space); smooth functions (differentiable manifolds); polynomials (algebraic varieties). The stalk in these cases are *germs* of functions. For example, $C^\infty(\mathbb{R})$: at each a the maximal ideal is $\langle x - a\rangle$.

Toy model of superspace. $\mathbb{R}^{1|1}$. To each open set we associate $\mathcal{O}^{1|1}(U) := C^\infty(U) \otimes \wedge(\theta) = C^\infty(U)[\theta]$.

If x is the global coordinate in $\mathbb{R}$, we say that (x, θ) are *global coordinates* on $\mathbb{R}^{1|1}$.

When $U = \mathbb{R}$ we speak about *global sections*

$$\tilde{\Phi} = \tilde{A} + \tilde{G}\,\theta\,,$$

with $\tilde{A}, \tilde{G} \in C^\infty(\mathbb{R})$. They are both **even** !!
$\tilde{\Phi}$ is the defining object of the superspace, still it is not a *superfield*, as understood in physics.

One could multiply $\tilde{G}$ by some odd quantity $\psi = \xi\tilde{G}$ but this presents problems:
1. Is it possible to recover the multiplication of the structural sheaf?
2. What do we do with quantities like

$$\psi(x)\psi(x') = 0??$$
$$\psi(x)\dot{\psi}(x) = 0??$$

Volkov-Akulov multiplet. $\mathbb{C}^{4|2}$. Chiral superfield in $D=2$

$$\Phi = A + \theta^\alpha \chi_\alpha + \theta^\alpha \theta_\alpha F, \qquad \alpha = 1,2\,,$$

satisfying $\Phi^2 = 0$.

$$A^2 = 0, \qquad A\chi_\alpha = 0, \qquad 4AF - \chi^\alpha \chi_\alpha = 0$$

If F is invertible;

$$A = \frac{\chi_1 \chi_2}{4F}\,.$$

If one imposes the constraint on elements of the structural sheaf

$$\tilde{\Phi} = \tilde{A} + \theta^\alpha \tilde{G}_\alpha + \theta^\alpha \theta_\alpha \tilde{F}, \qquad \alpha = 1,2\,,$$

$\tilde{\Phi}^2 = 0$ implies

$$\tilde{A}^2 = 0, \qquad 2\tilde{A}\tilde{G}_\alpha = 0, \qquad \tilde{A}\tilde{F} = 0\,,$$

whose solution is, for invertible $\tilde{F}$, $\tilde{A} = 0$.

Some work of interpretation is needed here.

The even rules principle

Theorem. (Deligne and Morgan). Let $\{V_i\}_{i\in I}$, $I = 1, \ldots, n$ be a family of super vector spaces, V another super vector space and $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ a commutative superalgebra. We denote $V_{i\,0}(\mathcal{B}) = (\mathcal{B} \otimes V_i)_0 = \mathcal{B}_0 \otimes V_0 + \mathcal{B}_1 \otimes V_1$ and $V_0(\mathcal{B}) = (\mathcal{B} \otimes V)_0$.

Any family of $\mathcal{B}_0$-multilinear maps

$$V_{1\,0}(\mathcal{B}) \times \cdots \times V_{n\,0}(\mathcal{B}) \xrightarrow{f_{\mathcal{B}}} V_0(\mathcal{B})$$

which is *functorial* in $\mathcal{B}$ comes from a unique morphism

$$V_1 \otimes \cdots \otimes V_n \xrightarrow{f} V$$

that is,

$$f_{\mathcal{B}}(b_1 \otimes v_1, b_2 \otimes v_2, \ldots, b_n \otimes v_n) = \\ (-1)^p b_1 \cdots b_n \, f(v_1 \otimes \cdots \otimes v_n)\,,$$

where p is the number of pairs (i, j) with $i < j$ and v_i, v_j odd.

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Definition. Let V and W be two superspaces. We say that a family of morphisms

$$\left\{ f_{\mathcal{B}} : V(\mathcal{B}) \to W(\mathcal{B}), \quad \mathcal{B} \in (\text{c salgebras}) \right\}$$

is *functorial* in $\mathcal{B}$ if given a superalgebra morphism

$$\mathcal{B} \xrightarrow{h} \mathcal{B}'$$

the diagram

$$\begin{array}{ccc} V(\mathcal{B}) & \xrightarrow{f_{\mathcal{B}}} & W(\mathcal{B}) \\ {\scriptstyle V(h)}\downarrow & & \downarrow{\scriptstyle W(h)} \\ V(\mathcal{B}') & \xrightarrow{f_{\mathcal{B}'}} & W(\mathcal{B}') \,. \end{array}$$

commutes.

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Definition. A *superfield* in $\mathbb{R}^{1|1}$ is a functorial family

$$\Phi = \{\, \Phi_{\mathcal{B}} \in (\mathcal{B} \otimes C^{\infty}(\mathbb{R})[\theta])_0\,,\; \mathcal{B} \in (\text{c salgebras})\,\}\;.$$

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Clearly, superfields can be multiplied. By the even rules principle, this family of products defines the standard product in $C^{\infty}(\mathbb{R})[\theta]$. (First question answered).

Let us consider $\mathcal{B} = C^{\infty}(\mathbb{R}) \otimes \wedge[\xi^1, \xi^2]$. Then

$$\begin{aligned} A(x) &= A_0(x) + A_{12}(x)\xi^1\xi^2, \\ \phi(x) &= \phi_1(x)\xi^1 + \phi_2(x)\xi^2\,. \end{aligned}$$

$$\begin{aligned} \phi(x)\phi(x') &= (\phi_1(x)\phi_2(x') - \phi_2(x)\phi_1(x'))\xi^1\xi^2, \\ \phi(x)\dot{\phi}(x) &= (\phi_1(x)\dot{\phi}_2(x) - \phi_2(x)\dot{\phi}_1(x))\xi^1\xi^2\,, \end{aligned}$$

which, generically, are different from zero.

We will need as many odd variables in $\mathcal{B}$ as different points or number of derivatives we want to consider.

But we have them all! Provided a functorial behaviour is assumed. Moreover: it is enough to consider Grassmann algebras.

The odd variables ξ_i are not physical quantities; they serve only to reproduce the correct algebraic behaviour of superfields. (We will talk later about observables and nilpotent variables)

(Second question answered).

Remark. We haven't introduced at all the super Poincaré group, so no supersymmetry is assumed here. These considerations are previous to the introduction of a transformation supergroup.

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Superchemes

We can still see the superfield in another way. We fix $\mathbb{C}^{4|2}$.

$$\Phi = A + \theta^\alpha \chi_\alpha + \theta^\alpha \theta_\alpha F, \qquad \alpha = 1, 2\,,$$

A, $F \in \mathcal{B}_0 \otimes C^\infty(\mathbb{C}^4)$ and $\chi_\alpha, \in \mathcal{B}_1 \otimes C^\infty(\mathbb{C}^4)$.

A morphism of superspaces $\mathbb{C}^{4|n} \to \mathbb{C}^{2|2}$ is a homeomorphism of the topological spaces together with a map of algebras

$$C^\infty(\mathbb{R}^2)[\theta^1, \theta^2] \to C^\infty(\mathbb{R}^4)[\xi^1, \dots, \xi^n]\,.$$

This is determined once we provide two even sections and two odd sections of $\mathbb{C}^{4|n}$, which are the images of the global coordinates in $\mathbb{C}^{2|2}$. Each superfield, as above, provides with its component fields (A, F, χ_1, χ_2) a morphism of the corresponding superspaces.

We want to study how to impose constraints in the component fields. We need some definitions.

We consider first *affine algebras*: commutative, over $\mathbb{C}$, finitely generated, with no nilpotents. Then, there is an equivalence of categories with *affine algebraic varieties*, commonly seen as the zero locus of some polynomials.

Points of an affine algebraic variety are in correspondence with *maximal ideals*. For example

$$a \in \mathbb{C} \leftrightarrows \langle x - a \rangle \subset C^{\infty}(\mathbb{R})\,.$$

One step further is to consider all the *prime ideals*. Then we recover the points of the variety and all its irreducible subvarieties. This space has a topology (Zariski) where closed sets are sets of prime ideals that contain a certain ideal and it is called the *spectrum of* A, Spec(A). (Essentially, algebraic subsets).

Localization of the affine algebra over a point $\mathfrak{p}$ gives the stalk of a the structural sheaf

$$\left\{ \frac{f}{g}, \quad f, g \in A,\ g \in F - \mathfrak{p} \right\}\,.$$

We have then defined

$$\underline{\text{Spec}}(F) := (\text{Spec}(F), \mathcal{O}_F) \quad \approx \quad X := (|X|, \mathcal{O}_X)\,.$$

What if now we relax some of the properties of the affine algebras? Suppose they are over $\mathbb{R}$ or that they contain nilpotents. What geometrical objects correspond to general algebras? It is the category of *affine schemes* (affine?).

Example. $\mathbb{C}[x, y]$ and the ideal $\langle x^2 \rangle$. The quotient, $F = \mathbb{C}[x, y]/\langle x^2 \rangle$, has a nilpotent x.

$$f_0 + x\, f_1 \in F, \qquad f_0, f_1 \in \mathbb{C}[y]\,.$$

The solution of the polynomial equation $x^2 = 0$ over $\mathbb{C}$ is $x = 0$. Quotienting by the ideal of all nilpotents, $F_{\text{red}} \cong \mathbb{C}[y]$. But F 'reminds' the double multiplicity of the solution. Points in Spec(F) are $\langle y - a, x^2 \rangle$, with stalk

$$F_a = \left\{ \frac{f_0 + x\, f_1}{g_0 + x\, g_1} \quad \middle| \quad f_i, g_i \in \mathbb{C}[y], \quad g_0(a) \neq 0 \right\}.$$

$\square$

Definition. An *affine scheme* X is a topological space $|X|$ together with a sheaf of algebras $\mathcal{O}_X$ which is isomorphic to $\underline{\text{Spec}}(F)$ for some algebra F. $\square$

Affine superalgebras are commutative superalgebras whose reduced algebra (quotient by the ideal of the odd nilpotents) is affine. Also, one requires finite generation of A_0 and A_1. $\underline{\text{Spec}}(\mathcal{A})$ defines an *affine supervariety*.

Definition. An *affine superscheme* S is a topological space $|S|$ together with a sheaf of algebras $\mathcal{O}_S$ which is isomorphic to $\underline{\text{Spec}}(\mathcal{A})$ for some superalgebra $\mathcal{A}$. $\square$

We can now go back to the Volkov-Akulov multiplet.

$$\Phi = A + \theta^\alpha \chi_\alpha + \theta^\alpha \theta_\alpha F, \quad \alpha = 1, 2\,,$$

Forcing $\Phi^2 = 0$ implies to take quotient by the ideal in $C^\infty(\mathbb{C}^2)[\chi_1, \chi_2]$.

$$\langle A^2,\, A\chi_\alpha,\, 2AF - \chi_1\chi_2 \rangle\,.$$

We study first the *reduced scheme* (setting the fermions to 0)

$$\langle A^2,\, AF \rangle\,.$$

A is a nilpotent, so the algebra is not affine. There is an open set where the scheme is isomorphic to an affine one: the points where F is invertible.

$$C^\infty(\mathbb{C}^2)[F^{-1}] \Big/ \Big\langle A,\, FF^{-1} - 1 \rangle \simeq C^\infty(\mathbb{C}^\times)\,.$$

This is the regular or smooth part of the scheme, represented by the object $\mathbb{C}^\times = \mathbb{C} - \{0\}$.

The same construction can be carried over the full superscheme by a change of variables

$$A' = 4AF - \chi^\alpha \chi_\alpha, \qquad F' = F \qquad \chi'_\alpha = \chi_\alpha\,.$$

The regular part of the scheme

$$C^\infty(\mathbb{C}^2)[\chi_1,\chi_2][F^{-1}]\Big/\Big\langle A', FF^{-1}-1\Big\rangle \simeq$$
$$C^\infty(\mathbb{C}^\times)[\chi_1,\chi_2]\,.$$

Let us consider now $\Phi^n = 0$, $n > 2$. The ideal is now

$$\Big\langle A^n, \quad A^{n-1}\chi_\alpha, \quad A^{n-2}\left(4AF-(n-1)\chi^\alpha\chi_\alpha\right)\Big\rangle\ .$$

The reduced scheme is given by the ideal

$$\Big\langle A^n,\, A^{n-1}F\Big\rangle\,.$$

One can still invert F, but this will give $A^{n-1} = 0$, so we cannot get rid of the nilpotents. This scheme **does not** have regular part.

Devil's trick (by physicists). Impose an **extra relation**

$$A = a\,\chi^{\alpha}\chi_{\alpha}\,.$$

(a could be function of F). This solves trivially all the equations.

For $n = 2$, a is determined; for $n = 3$ it is not, so there are extra solutions.

The drawback: SUSY is broken (not in $n = 2$).

In order to have SUSY invariance, even nilpotents must be kept

A comment on observables and nilpotent variables

Let F be an algebra, consider Spec(F) and $\mathfrak{p}$, a prime in F. We consider the integral domain $F/\mathfrak{p}$ (the product of two $\neq 0$ elements is $\neq 0$).

Let $F_{\mathfrak{p}}$ the localization and consider $\kappa(\mathfrak{p}) := F_{\mathfrak{p}}/\mathfrak{p}$. This is a field, called the *residue field*. It can be different from point to point.

Example. $F = \mathbb{C}[x]$. At $\langle x-a \rangle$ the residue field is $\mathbb{C}$. At $\langle 0 \rangle$ it is the field of rational functions.

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For every $f \in F$ we can define a 'function' on Spec(F) with values in the residue field via the canonical maps

$$\begin{array}{ccccc} F & \longrightarrow & F_{\mathfrak{p}} & \longrightarrow & \kappa(\mathfrak{p}) \\ f & \longrightarrow & f & \longrightarrow & f(\mathfrak{p}) \, . \end{array}$$

This is the way of recovering the interpretation of 'algebra of functions' for topological spaces, algebraic varieties, differential manifolds...

If F contains a nilpotent n then $n \in \mathfrak{p}$ for all prime ideals, so $n(\mathfrak{p}) = 0$. F **cannot** be reproduced from an algebra of functions on Spec(F).

Classically, observables are functions on a phase space (or a space of fields) and the possible results of measurements are numbers, the values of these functions. In this interpretation, odd degrees of freedom (or odd fields, like electrons) cannot be seen classically.

Of course in quantum physics things change. The commutative algebra of observables is deformed to a non commutative one (operators on a Hilbert space).

Example. Let us consider a Grassmann algebra $\wedge(\theta, \pi)$ with canonical super Poisson bracket

$$\{\theta, \pi\}_+ = 1\,.$$

The Grassmann algebra is deformed to a *noncommutative superalgebra* freely generated by indeterminates Θ, Π satisfying the commutation rules

$$[\Theta, \Pi]_+ = \mathrm{i}\hbar\,\mathrm{id}\,,$$

and the rest 0. By a change of variables, this can be seen isomorphic to the Clifford algebra $C(1, 1)$. All the Clifford algebras are noncommutative superalgberas.

From here one constructs a Hilbert space and hermitian observables as in the even case. Odd variables acquire ‘visibility’ in the quantum realm.

Conlusions

The translation between the physics understanding of superfields and the structural sheaf of a superspace is done through the functorial mechanism of the ‘even rules’.

We considered here only scalar superfields, but one could take sheaves of modules for different types of superfields.

Systems that realize non linearly supersymmetry may need the introduction of even nilpotent variables.

The classical limit of odd degrees of freedom is only mathematical, not related to observables.

Quantum, even nilpotent variables? (No idea).