

AN ABSTRACT THEORY OF PHYSICAL MEASUREMENTS

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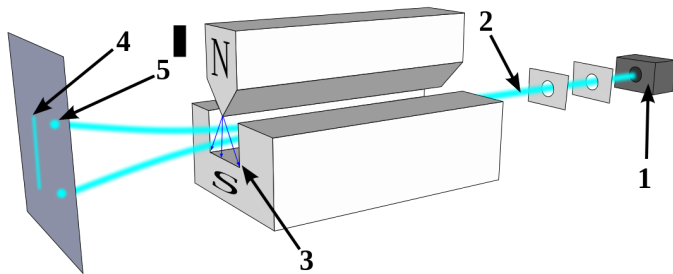
TQFT Seminar, IST, on March 19, 2021

Measurement z

One run yields *one bit* of *classical information*:

0 = down

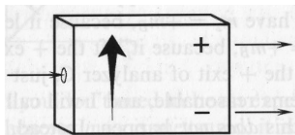
1 = up



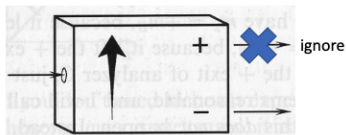
Stern-Gerlach analyzer

Information-induced order

z



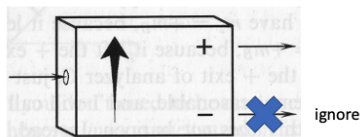
z^- (can only yield 0)



Disjunction

$$z = z^- \vee z^+$$

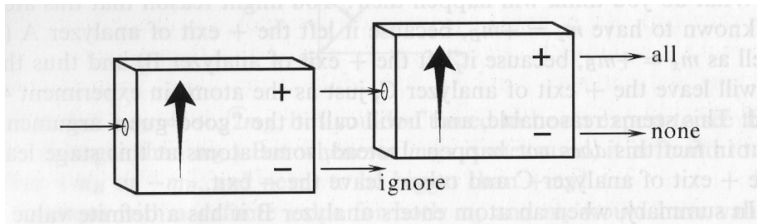
z^+ (can only yield 1)



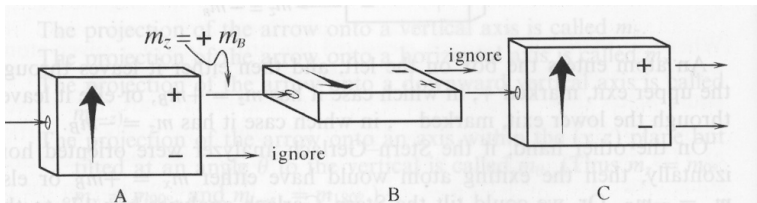
Temporal/causal order

Can have *sequentially composed* measurements:

$$zz^+ = z^+$$



$$zx^+z^+ = z^-x^+z^+ \vee z^+x^+z^+ \neq z$$



- ▶ Topological and algebraic structure of *spaces* of measurements.
- ▶ *Multiple runs* of z also yield *statistical information*:

0 occurs in M runs

1 occurs in N runs

Will not address this in this talk (no measure-theoretic structure).

Outline

1. The measurement problem
2. Rationale
3. Measurement spaces
4. Classical measurement spaces
5. Quantizations and observers

Part 1 — The measurement problem



“... the quantum postulate implies that any observation of atomic phenomena will involve an interaction with the agency of observation not to be neglected. Accordingly, an independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation.”

– Niels Bohr, *The quantum postulate and the recent development of quantum theory*, Supplement to “Nature,” April 14, 1928.



“...the description of the experimental arrangement and the recording of observations must be given in plain language, suitably refined by the usual physical terminology. This is a simple logical demand, since by the word “experiment” we can only mean a procedure regarding which we are able to communicate to others what we have done and what we have learnt.”

– Niels Bohr (1958), *Quantum physics and philosophy—causality and complementarity* (pp. 1–7) Woodbridge: Ox Bow Press (Reprinted in *The Philosophical writings of Niels Bohr, Essays 1958–1962 on atomic physics and human knowledge originally, Wiley 1963*).

- ▶ Two-state *system* (a qubit)
- ▶ Hilbert space $H_S = \mathbb{C}^2$
- ▶ Basis states: $|0\rangle$ and $|1\rangle$
- ▶ Experimental *apparatus*
- ▶ Hilbert space H_A
- ▶ Initial state: $|\text{Pointer}=?\rangle$

Time evolution in $H_S \otimes H_A$

$$e^{-i\hat{H}\Delta t}$$


$$\alpha |0\rangle \otimes |\text{Pointer}=?\rangle + \beta |1\rangle \otimes |\text{Pointer}=?\rangle$$

$$\alpha |0\rangle \otimes |\text{Pointer}=0\rangle + \beta |1\rangle \otimes |\text{Pointer}=1\rangle$$



"But in any case, no matter how far we calculate — to the mercury vessel, to the scale of the thermometer, to the retina, or into the brain, at some time we must say: and this is perceived by the observer. That is, we must always divide the world into two parts, the one being the observed system, the other the observer.

– John von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton Univ. Press, 1955 (translation of the 1932 german original).



"What exactly qualifies some physical systems to play the role of 'measurer'? Was the wavefunction of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system... with a PhD?"

– John S. Bell, *Against 'measurement'*, *Phys. World* 3 (1990).

Interpretations and variants

- ▶ *Realist*: decoherence, many-worlds, stochastic collapse, gravity-induced collapse, de Broglie–Bohm mechanics, contextual topos-based models...
- ▶ *Epistemic/subjective*: “Copenhagen” (partially), QBism...
- ▶ “New interpretations appear every year. None ever disappear.”
– David Mermin

Part 2 — Rationale

Classical mechanics:

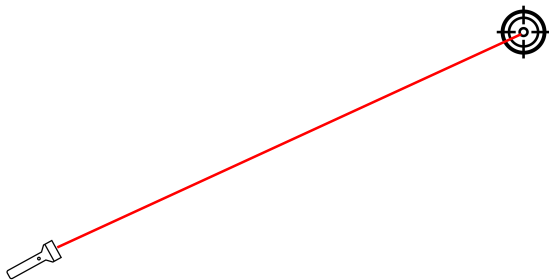
- ▶ Systems have *states*.
- ▶ States are defined by their *properties* (position, momentum...).
- ▶ States are *abstract* and state spaces are *geometric*.

Quantum mechanics:

- ▶ Systems do or do not have states, according to interpretation.
- ▶ Measurements condition the types of answers obtained from systems.
- ▶ If systems have states, measurements may change them.

Idea: define *spaces* of *abstract* measurements.

Classical physics



Copenhagen



Feynman



Measurement 1



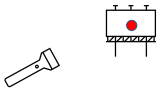
Measurement 2



Measurement 3



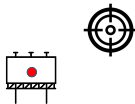
Measurement 4



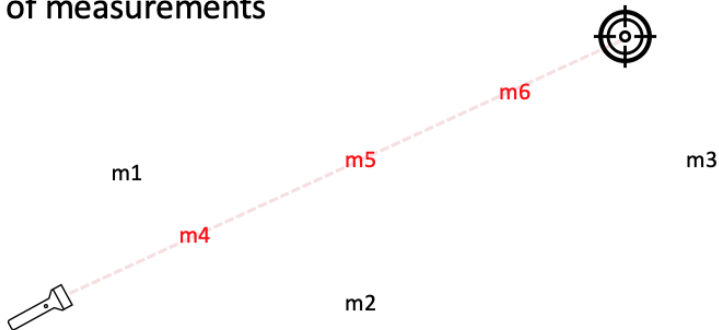
Measurement 5

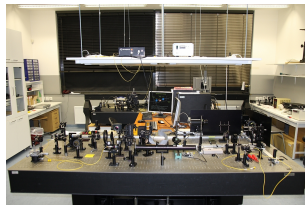
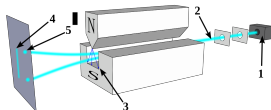
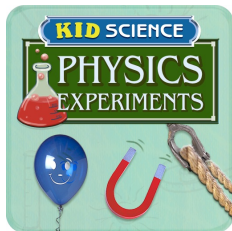


Measurement 6



Geometry in “space” of measurements





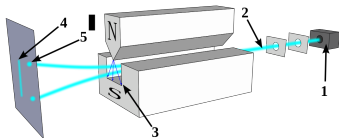
“Definition”

A *measurement* is a finite physical procedure, performed with an experimental apparatus, in the course of which a finite amount of *communicable classical information* is recorded.

Part 3 — Measurement spaces

(Based on arXiv:2102.01712)

- ▶ Set M of measurements.
- ▶ From each $m \in M$ *finite* quantities of *communicable classical information* can be obtained (\sim finite strings of 0s and 1s).
- ▶ These finite pieces of classical information are the *observable properties* associated with m .
- ▶ Each observable property can be identified with a *subset* $U \subset M$.
- ▶ $m \in U$ reads: *U can be recorded by performing m .*



$$\begin{aligned} z &\in U^+ \cap U^- \\ z^+ &\in U^+ \setminus U^- \\ z^- &\in U^- \setminus U^+ \end{aligned}$$

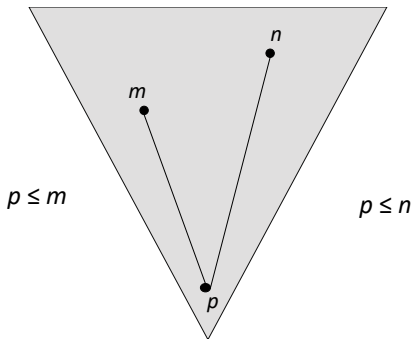
- ▶ $U \cap V$ logical *conjunction* of properties: observable
- ▶ $U \cup V$ logical *disjunction* of properties: observable
- ▶ Infinitary disjunction $\bigcup_i U_i$: observable
- ▶ Infinitary conjunction $\bigcap_i U_i$: not necessarily observable
- ▶ The observable properties are the open sets of a *topology* on M [cf. S.J. Vickers, *Topology Via Logic*, CUP, 1989].
- ▶ M *trivial* property
- ▶ \emptyset *impossible* property
- ▶ $m \sim n$ if m and n have the same neighborhoods.
- ▶ The equivalence classes $[m]$ are the *abstract measurements*.
- ▶ Quotient space of abstract measurements is T_0 .

- ▶ In fact M should be *sober*.
- ▶ Then the *specialization order*

$$m \leq n \iff m \in \overline{\{n\}}$$

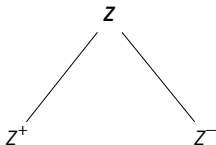
is a directed complete partial order, and the topology is contained in the Scott topology.

- ▶ The open sets are *upper-closed* in the specialization order (and also inaccessible by directed joins):



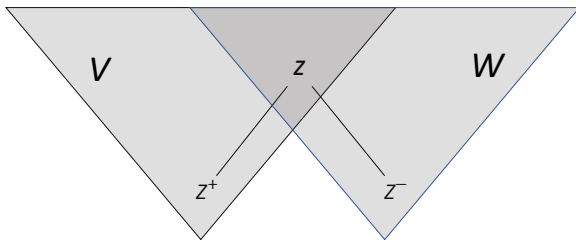
Disjunctions

Disjunctions are *joins* (suprema) in the specialization order:



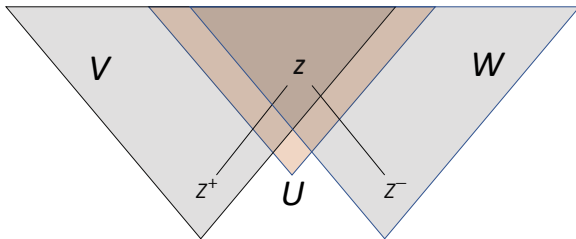
Disjunctions

Disjunctions are *joins* (suprema) in the specialization order:



Disjunctions

Disjunctions are *continuous* operations because $V \vee W = V \cap W$:



Definition

A *sober lattice* is a sober space L whose specialization order has a least element 0 and a continuous binary join operation $\vee : L \times L \rightarrow L$.

Fact: any sober lattice is a complete lattice (all joins exist).

Compositions

- ▶ mn is n *and then* m .
- ▶ If the composition is meaningless then $mn = 0$.
- ▶ $(mn)p = m(np)$
- ▶ $(m \vee n)p = mp \vee np$
- ▶ $0m = 0$
- ▶ Composition is *continuous*.

- ▶ m^* is *continuous formal reversal* of m .
- ▶ $m^{**} = m$
- ▶ $(mn)^* = n^* m^*$
- ▶ m is *reversible* if $mm^* m = m$.
- ▶ Fake reversibility is ruled out:

$$mm^* m \leq m \implies mm^* m = m$$

M is a *stably Gelfand quantale*.

Definition

A *measurement space* M is a topological involutive semigroup which is a sober lattice and for all $m, n, p \in M$ satisfies:

1. $0m = 0$;
2. $(m \vee n)p = mp \vee np$;
3. $mm^*m = m$ whenever $mm^*m \leq m$.

Shorter definition: a *measurement space* is a *sober involutive semiring* that satisfies condition 3.

Even shorter definition: a *measurement space* is a *sober stably Gelfand quantale*.

EXAMPLES

Theorem

Let A be a C^* -algebra. The involutive quantale $\text{Max } A$ [Mulvey 1989], with the lower Vietoris topology, is a measurement space.

(Sobriety in [R–Santos (2016)]; stably Gelfand condition in [R 2018a].)

- ▶ $P \in \text{Max } A \iff P$ is a closed linear subspace of A
- ▶ $PQ = \overline{\{ab \mid a \in P, b \in Q\}}$
- ▶ $P \vee Q = \overline{P + Q}$
- ▶ $P^* = \{a^* \mid a \in P\}$
- ▶ The lower Vietoris topology has a subbasis of open sets

$$\tilde{U} = \{P \in \text{Max } A \mid P \cap U \neq \emptyset\}$$

where U is open in A .

For the spin 1/2 example: $A = M_2(\mathbb{C})$.

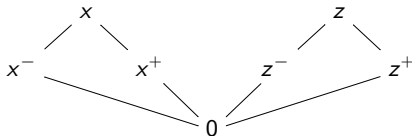
$$z^+ = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \quad z^- = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$$z = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

$$x^+ = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \quad x^- = \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle$$

$$x = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

Fragment of the specialization order:



Theorem (Kruml–R, 2004)

For unital C^ -algebras (even without the topology on the quantales):*

$$A \cong B \iff \text{Max } A \cong \text{Max } B.$$

- ▶ Automorphism of $\text{Max } \mathbb{C}^2$:

$$\begin{aligned}\alpha(\langle(z, w)\rangle) &= \langle(w, z)\rangle && \text{if } z \neq 0 \text{ and } w \neq 0 \\ \alpha(\langle(z, 0)\rangle) &= \langle(z, 0)\rangle \\ \alpha(\langle(0, w)\rangle) &= \langle(0, w)\rangle.\end{aligned}$$

α does not come from any $*$ -automorphism of \mathbb{C}^2 .

- ▶ But α is not continuous.

Part 4 — Classical measurement spaces

First question: what is a measurement space of *classical type*?

Example 1 — Lab wall

Let X be a locally compact space. The topology $\Omega(X)$, equipped with the Scott topology, is a measurement space:

$$UV = U \cap V$$

$$U \vee V = U \cup V$$

$$U^* = U$$

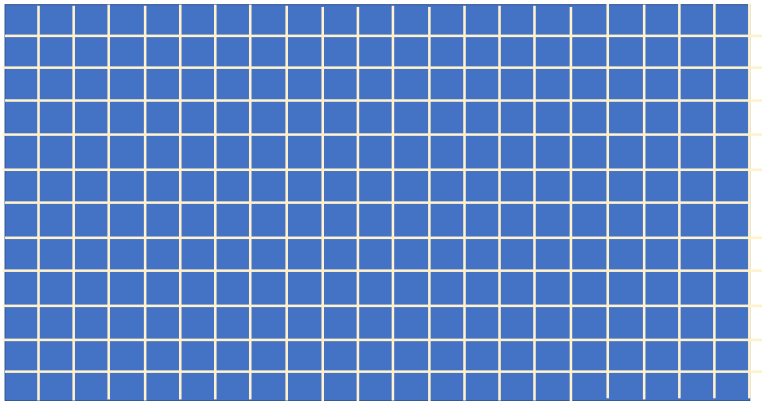
Corresponding physical situation: observe a wall of the lab by visual inspection, using light.



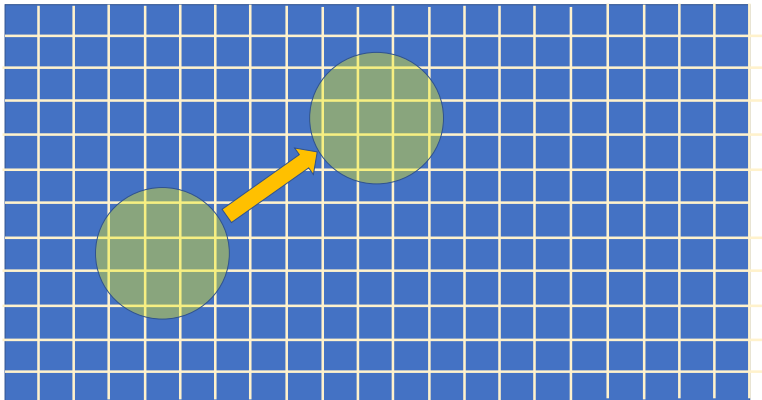
We never see *points* of the wall — each photon that hits our retina carries information about a *region* of the wall, no matter how small.

The “picture” of the wall as a *space with points* emerges from an integrated *mental image* that translates mathematically to a *geometric model*.

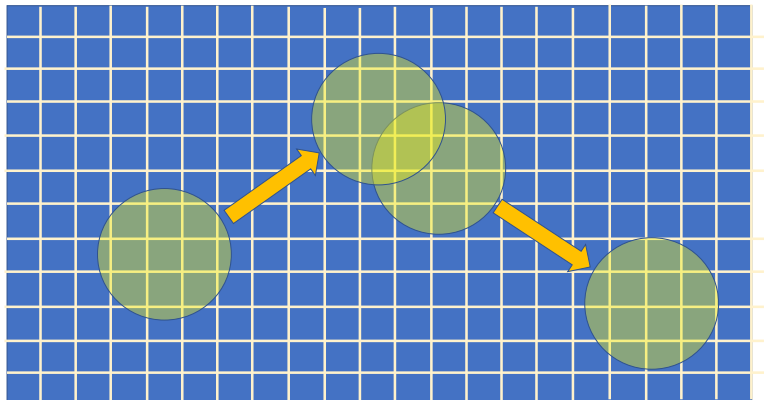
Example 2 — Adding partial symmetries



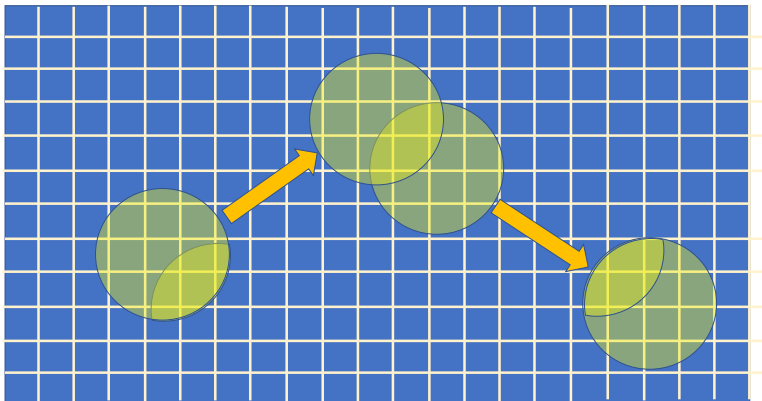
Example 2 — Adding partial symmetries



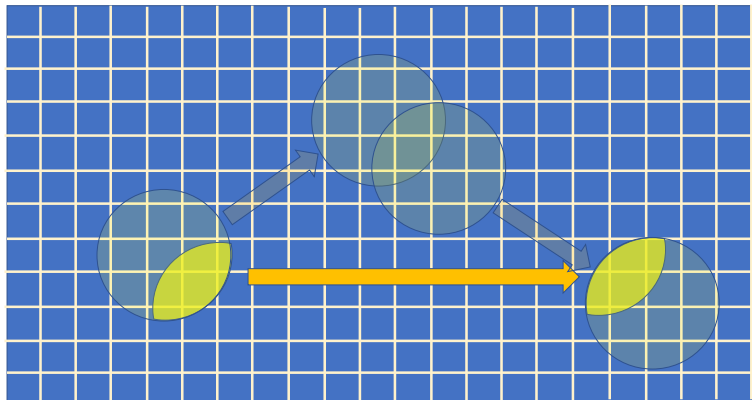
Example 2 — Adding partial symmetries



Example 2 — Adding partial symmetries



Example 2 — Adding partial symmetries



The composition is associative and defines a *pseudogroup* (a complete and infinitely distributive *inverse semigroup* [cf. Mark V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, WS, 2002]).

- ▶ Let G_0 be a locally compact Hausdorff space (the lab wall).
- ▶ The partial symmetries form a *pseudogroup* S whose idempotents correspond to the open sets of G_0 .
- ▶ For each symmetry $f \in S$ let $U_f \subset G_0 \times G_0$ be its graph, and let

$$G_1 = \bigcup_{f \in S} U_f.$$

- ▶ G_1 is an equivalence relation on G_0 , so it defines a *principal groupoid* G .
- ▶ The family $(U_f)_{f \in S}$ is a basis for a topology on G_1 that makes G a *locally compact Hausdorff étale groupoid* (second-countable if S is countable).

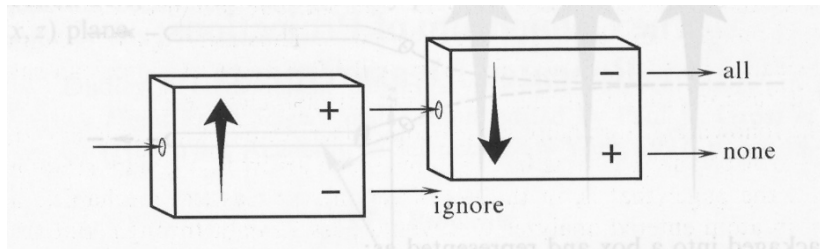
Measurement space of G

The topology $\Omega(G_1)$, equipped with the Scott topology, is a measurement space $\mathcal{O}(G)$:

$$\begin{aligned} UV &= \text{composition of } U \text{ and } V \text{ as binary relations} \\ U \vee V &= U \cup V \\ U^* &= \text{reversal of } U \text{ as a binary relation} \end{aligned}$$

Example

$$\text{flip } z^+ = z^- \text{ flip } z^+$$



Example 3 — Schwinger's selective measurements

- ▶ Physical quantity A with finitely many values a, a', a'', \dots
- ▶ $M(a)$ is the measurement that selects those systems whose value of A is a and rejects all others.

Example: $z^+ = M(+\hbar/2)$

- ▶ $M(a', a)$ selects the systems whose value of A is a , after which those systems emerge in a new state for which the value is a' .

Example: $M(-\hbar/2, +\hbar/2) = \text{flip } z^+ = z^- x z^+$

- ▶ $M(a)$ is identified with $M(a, a)$.
- ▶ Finite pair groupoid: $G_0 = \{a, a', a'', \dots\}$; $G_1 = G_0 \times G_0$.

- ▶ More generally: $G_0 = \text{space of values of } A$.
- ▶ $M(U)$ selects those systems whose value of A lies in open set U .
- ▶ $M(f)$ selects the systems whose value of A is some $a \in U$, after which those systems emerge in a new state for which the value is $f(a)$, where $f : U \rightarrow V$ is a partial symmetry of G_0 .
- ▶ Principal étale groupoid G and measurement space $\mathcal{O}(G)$ obtained as in the lab wall example.
- ▶ Or define G to be the (non-étale) pair groupoid $G_0 \times G_0$.

Part 5 — Quantizations and observers



What about “observers”?

Each observer has a *repertoire* of measurements $\mathcal{O} \subset M$.

Communication: each observer *approximates* measurements done by others:

$$\alpha : M \rightarrow \mathcal{O}$$

Definition

Let M be a measurement space. An *observer* of M is a pair (\mathcal{O}, α) in which \mathcal{O} is a classical subspace of measurements (e.g., $\cong \mathcal{O}(G)$) and $\alpha : M \rightarrow \mathcal{O}$ is a topological retraction onto \mathcal{O} such that for all $m, n \in M$ and $\omega \in \mathcal{O}$

$$\alpha(m \vee n) = \alpha(m) \vee \alpha(n)$$

$$\alpha(m^*) = \alpha(m)^*$$

$$\alpha(0) = 0$$

$$\alpha(\omega m) = \omega \alpha(m)$$

The observer is *persistent* if it further satisfies

$$\alpha(m\omega n) = \alpha(m)\omega\alpha(n),$$

and *localizable* if $m \leq \alpha(m)$ for all $m \in M$.

- ▶ $\alpha(m)$ is the best approximation of m from the point of view of the observer.
- ▶ If (\mathcal{O}', α') is another observer, the restriction $\alpha|_{\mathcal{O}'} : \mathcal{O}' \rightarrow \mathcal{O}$ translates measurements of an observer to the other.

Example

- ▶ G second countable locally compact Hausdorff étale groupoid.
- ▶ *"Quantization"*: $A = C_r^*(G)$ (or $A = C_r^*(G, E)$ for Fell line bundle).

Lemma (partially from [R 2018a])

The embedding $\iota : \mathcal{O}(G) \rightarrow \text{Max } A$ given by $U \mapsto \overline{C_c(U)}$ is continuous and preserves composition, involution and joins.

If $A = C_r^(G)$ the relative topology of the image $\mathcal{O} := \iota(\mathcal{O}(G))$ is the Scott topology.*

- ▶ Open support map $\text{supp}^\circ : \text{Max } A \rightarrow \mathcal{O}(G)$:

$$\text{supp}^\circ(P) = \{x \in G_1 \mid \exists a \in P \ a(x) \neq 0\}$$

- ▶ Define $\alpha : \text{Max } A \rightarrow \mathcal{O}$ by $\alpha = \iota \circ \text{supp}^\circ$.

Theorem (from [R 2018a])

The pair (\mathcal{O}, α) is an observer of $\text{Max } A$.

If G is compact: (1) the observer is localizable; (2) if G is principal with discrete orbits the observer is persistent; (3) if the observer is persistent then G is principal.

Multiple observers

- ▶ There are *many étale groupoids* associated to any measurement space M (at least one per projection $m = m^2 = m^* \in M$) [R 2018b].
- ▶ From commutative sub-C*-algebras of a C*-algebra A the groupoids are locally compact and locally Hausdorff — cf. [Renault 2008, R 2018a].
- ▶ Consider (\mathcal{O}, α) and (\mathcal{O}', α') with $\mathcal{O} \cong \mathcal{O}(G)$ and $\mathcal{O}' \cong \mathcal{O}(G')$. There are measurements in M corresponding to *partial symmetries*

$$f : U \rightarrow U'$$

where $U \subset G_0$ and $U' \subset G'_0$ are open sets. The totality of these symmetries defines a (partial) *Morita equivalence* [Lawson–R 2020, Quijano–R 2021] between the two observers.

- ▶ The partial symmetries are the *Stern–Gerlach measurements* in the terminology of Ciaglia et al.
- ▶ **Example:** in $M_2(\mathbb{C})$

$$\underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle}_{z^+} \underbrace{\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{x^+} \underbrace{\left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle}_{z^+ \leftarrow x^+} = \underbrace{\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{z^+ \leftarrow x^+}$$

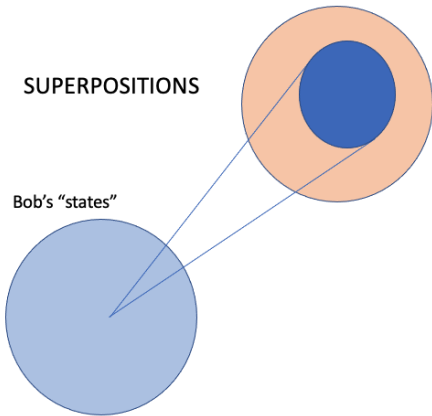
*THAT
MEASUREMENT
DESTROYED
MY STATE!*



SUPERPOSITIONS

Bob's "states"

Alice's "states"



Part 6 — Wrapping up

- ▶ Attempting to approach quantum mechanics geometrically (in a broad sense) and in a paradox free way, by taking measurements rather than states to be the points.
- ▶ The model is both realist and operational.
- ▶ The open sets correspond to the classical information that can be extracted from measurements.
- ▶ On the other hand, measurements are defined in terms of the classical information they yield.
- ▶ Neither information nor measurement takes precedence: measurement spaces bootstrap a definition of both.

- ▶ Observers provide a mathematical formulation of Bohr's classical/quantum divide, however without requiring observers in the definition of measurements in the first place: observers are derived "entities," so here the "shifty split" is not fundamental.
- ▶ A way to address Bell's qualms regarding information?



"Information? Whose information? Information about what?"

- ▶ Is classical information (and measurements) fundamental?



"... every physical quantity, every it, derives its ultimate significance from bits, binary yes-or-no indications, a conclusion which we epitomize in the phrase, *it from bit*."

– John A. Wheeler, 1989

- ▶ Open problems related to C^* -algebras and Fell bundles; statistical interpretation; dynamics; geometrization of observers...