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Probability and Stochastic Analysis Seminar (IST)

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Exact solution of an integrable non-equilibrium particle system

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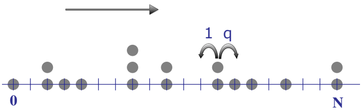
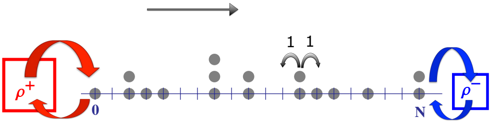
joint work (in progress) with Rouven Frassek (ENS Paris)

Introduction: non-equilibrium set-up in 1D

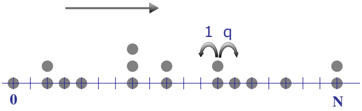
- asymmetry



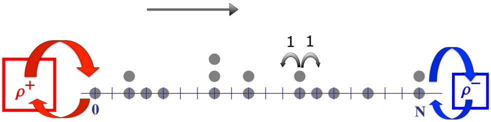
- density reservoirs



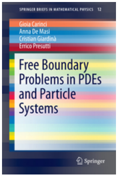
Introduction: non-equilibrium set-up in 1D

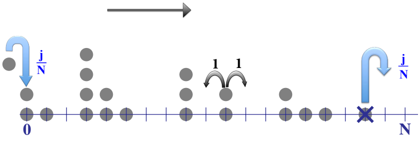


- asymmetry



- density reservoirs





- current reservoirs

Introduction: aim of the talk

- Solve the boundary-driven process introduced in [FGK (2020a)]

  Frassek, G., Kurchan, *Non-compact quantum spin chains as integrable stochastic particle processes*, J. Stat. Phys. 180, 366–397 (2020).

- See also [FGK (2020b)]:
  Frassek, G., Kurchan, *Duality and hidden equilibrium in transport models* SciPost Phys. 9, 054 (2020).

- Key idea: duality and integrability

Outline

1. Model
2. Results
   - absorbing dual
   - non-equilibrium steady state
3. Proof
   - algebraic description
   - mapping to equilibrium
4. Perspectives

1. Models

The basic

open symmetric ‘harmonic’ process

The basic model

Markov process $\{\eta(t), t \geq 0\}$ taking values on $\Omega_N = \mathbb{N}^N$ with generator

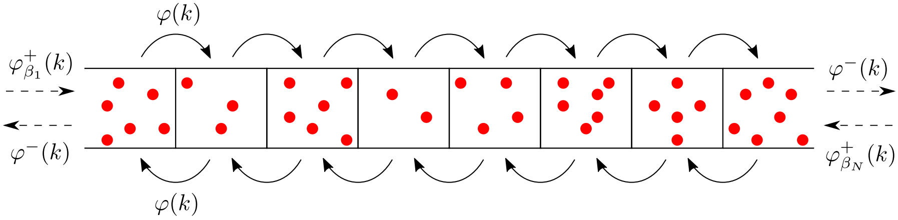
$$\mathscr{L} = \mathscr{L}_1 + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_N$$

$$\begin{aligned} \mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{k} \Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \frac{1}{k} \Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \Big] \end{aligned}$$

$$\begin{aligned} \mathscr{L}_1 f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{k} \Big[f(\eta - k\delta_1) - f(\eta) \Big] + \sum_{k=1}^{\infty} \frac{\beta_L^k}{k} \Big[f(\eta + k\delta_1) - f(\eta) \Big] \\ \mathscr{L}_N f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{k} \Big[f(\eta - k\delta_N) - f(\eta) \Big] + \sum_{k=1}^{\infty} \frac{\beta_R^k}{k} \Big[f(\eta + k\delta_N) - f(\eta) \Big] \end{aligned}$$

Remark: on the bulk diagonal $h(n) = \sum_{k=1}^{n} \frac{1}{k}$, harmonic numbers

The basic model



A chain of length $N = 8$, the i^{th} site corresponds to the i^{th} box.

$$\varphi(k) = \varphi^{-}(k) = \frac{1}{k} \qquad \varphi^{+}_{\beta_i}(k) = \frac{\beta_i^k}{k}$$

The basic model

- If $\beta_L = \beta_R = \beta$: *equilibrium* set-up. The product geometric distribution

$$\mu^{eq}(\eta) = \prod_{i=1}^{N} \left[\beta^{\eta_i}(1-\beta)\right] \qquad 0 < \beta < 1$$

is reversible, and thus stationary, with density

$$\rho(\beta) = \frac{\beta}{1-\beta}$$

- If $\beta_L \neq \beta_R$: *boundary driven non-equilibrium*

$$\mu(\eta) \ = ?$$

Remark: non-product law (cf. standard zero-range [Levine, Mukamel,Schütz])

The general

open symmetric ‘harmonic’ process

The general model (spin s)

$$\mathscr{L} = \mathscr{L}_1 + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_L$$

$$\begin{aligned} \mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \varphi_s(k,\eta_i) \Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi_s(k,\eta_{i+1}) \Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \Big] \end{aligned}$$

$$\mathscr{L}_i f(\eta) = \sum_{k=1}^{\eta_i} \varphi_s(k,\eta_i) \Big[f(\eta - k\delta_i) - f(\eta) \Big] + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} \Big[f(\eta + k\delta_i) - f(\eta) \Big]$$

$$\varphi_s(k,n) = \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n-k+1)\Gamma(n+2s)}$$

$$\psi(z) = \frac{\partial}{\partial z} \log \Gamma(z) \qquad\qquad \psi(z+1) = \psi(z) + \frac{1}{z}$$

$$h_s(n) = \sum_{k=1}^{n} \varphi_s(k,n) = \psi(n+2s) - \psi(2s) = \sum_{k=1}^{n} \frac{1}{k+2s-1}$$

The general model (spin s)

- If $s = \frac{1}{2}$ then we recover the basic model

- If $\beta_L = \beta_R = \beta$: *equilibrium* set-up. The product negative-binomial distribution

$$\mu^{eq}(\eta) = \prod_{i=1}^{N} \left[\frac{\beta^{\eta_i}}{\eta_i!} \frac{\Gamma(\eta_i + 2s)}{\Gamma(2s)} (1-\beta)^{2s} \right] \qquad 0 < \beta < 1$$

  is reversible.

- If $\beta_L \neq \beta_R$: *boundary driven* particle system

$$\mu(\eta) \;\; = ?$$

Relation to previous models

Relation to previous models

- The bulk part of the basic model is the $q \to 1$ limit of the MADM model [Sasamoto-Wadati]

$$\mathscr{L}^{MADM} = \sum_{i \in \mathbb{Z}} \mathscr{L}_{i,i+1}$$

$$\begin{aligned} \mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{[k]_q} \Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \frac{q^k}{[k]_q} \Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \Big] \end{aligned}$$

q-number $\qquad [k]_q = \dfrac{1 - q^k}{1 - q} \to k \quad \text{as} \quad q \to 1$

Relation to previous models

- The bulk part of the spin s model is the $q \to 1$ limit of the q-Hahn model

[Barraquand-Corwin], [Povolotsky]

$$\mathscr{L}^{q-Hahn} = \sum_{i\in\mathbb{Z}} \mathscr{L}_{i,i+1}$$

$$\begin{aligned}\mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \varphi^{r,q,\nu}(k,\eta_i)\Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta)\Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi^{\ell,q,\nu}(k,\eta_{i+1})\Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta)\Big]\end{aligned}$$

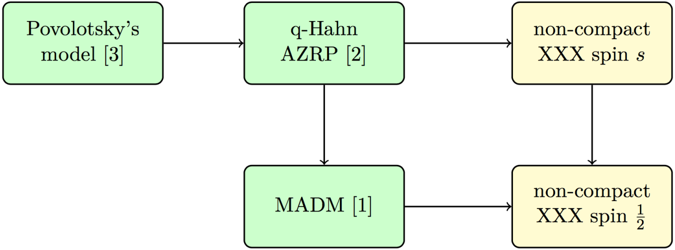
$$\varphi^{r,q,\mu}(k,n) = \frac{\nu^k(\nu;q)_{n-k}(q;q)_n}{[k]_q(\nu;q)_n(q,q)_{n-k}} \qquad \varphi^{\ell,q,\nu}(k,n) = \frac{(\nu;q)_{n-k}(q;q)_n}{[k]_q(\nu;q)_n(q,q)_{n-k}}$$

$$(\nu;q)_n = \prod_{j=0}^{n-1}(1-\nu q^j) \qquad \lim_{q\to 1}\frac{(q^{2s};q)_n}{1-q^n} = \frac{\Gamma(2s+n)}{\Gamma(2s)}$$

$$\lim_{q\to 1} \varphi^{r/\ell,q,q^{2s}}(k,n) = \varphi_s(k,n)$$

[See also Frassek '19]

Relation to previous models



2. Results

Preliminaries:

duality

Duality

Definition [Liggett]

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator $\mathscr{L}$,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator $\mathscr{L}^{dual}$

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $\mathscr{L}^{dual} = \mathscr{L}$.

In terms of generators:

$$\mathscr{L} D(\cdot, \xi)(\eta) = \mathscr{L}^{dual} D(\eta, \cdot)(\xi)$$

Example [Lévy]

$(X_t)_{t\geq 0}$ Brownian motion on $[0,\infty)$ started at $x>0$, reflected at the origin

$(Y_t)_{t\geq 0}$ Brownian motion on $[0,\infty)$ started at $y>0$, absorbed at the origin

$D(x,y) = 1_{\{x\leq y\}}$ ('Sigmund' duality fct)

$$\mathbb{E}_x(D(X_t,y)) = \mathbb{E}_y(D(x,Y_t))$$

$$\mathbb{P}_x[X_t \leq y] = \mathbb{P}_y[Y_t \geq x]$$

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$$\mathbb{E}_x(D(X_t,y)) = \mathbb{E}_y(D(x,Y_t))$$

$$\mathbb{P}_x[X_t \leq y] = \mathbb{P}_y[Y_t \geq x]$$

$$\int_0^y \left(e^{-\frac{(z-x)^2}{2t}} + e^{-\frac{(z+x)^2}{2t}}\right)\frac{dz}{\sqrt{2\pi t}} = \int_x^\infty \left(e^{-\frac{(z-y)^2}{2t}} - e^{-\frac{(z+y)^2}{2t}}\right)\frac{dz}{\sqrt{2\pi t}}$$

Duality: a useful tool

- ergodic theory [Liggett, ...]
- hydrodynamic limit [Presutti, De Masi, ...]
- KPZ scaling [Spohn, Borodin, Sasamoto, Corwin, ...]
- population genetics [Kingman, ...]
- martingale problem [Greven, ...]
- ...

the dual process is simpler: "from many to few"

Duality for Markov chains

Assume state spaces Ω, Ω_{dual} are countable sets, then the Markov generator L is a matrix $L(\eta, \eta')$ s.t.

$$L(\eta, \eta') \geq 0 \quad \text{if} \;\; \eta \neq \eta', \qquad \qquad \sum_{\eta' \in \Omega} L(\eta, \eta') = 0$$

$$\mathscr{L} D(\cdot, \xi)(\eta) = \mathscr{L}^{dual} D(\eta, \cdot)(\xi)$$

amounts to

$$LD = DL_{dual}^{T}$$

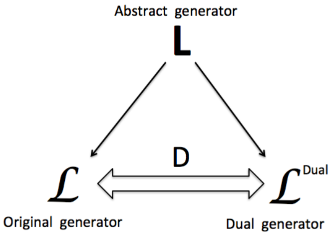
Indeed

$$\sum_{\eta'} L(\eta, \eta') D(\eta', \xi) = \sum_{\xi'} L^{dual}(\xi, \xi') D(\eta, \xi')$$

(Lie) Algebraic approach to duality

Overarching ideas:

- self-duality is derived from symmetries [Schütz]
- duality arises as a change of representation



Monograph in preparation with G. Carinci and F. Redig

'Duality of Markov processes: an algebraic approach'.

Preliminaries:

factorial moments

Factorial moments

The multivariate factorial moments of order $(\xi_1, \ldots, \xi_N) \in \mathbb{N}^N$ of an integer-valued random vector with distribution $\mu(\eta_1, \ldots, \eta_N)$ are defined as

$$F(\xi_1, \ldots, \xi_N) = \sum_{\eta \in \mathbb{N}^N} \left[\prod_{i=1}^{N} \eta_i(\eta_i - 1) \cdots (\eta_i - \xi_i + 1) \right] \mu(\eta_1, \ldots, \eta_N)$$

Inversion formula

$$\mu(\eta_1, \ldots, \eta_N) = \sum_{\xi \in \mathbb{N}^N} F(\xi_1, \ldots, \xi_N) \prod_{i=1}^{N} \binom{\xi_i}{\eta_i} \frac{(-1)^{\xi_i - \eta_i}}{\xi_i!}$$

Factorial moments

For us it will be convenient to consider

$$G(\xi_1,\ldots,\xi_N) = \sum_{\eta\in\mathbb{N}^N}\left[\prod_{i=1}^N \frac{\eta_i}{2s}\frac{(\eta_i-1)}{2s+1}\cdots\frac{(\eta_i-\xi_i+1)}{(2s+\xi_i-1)}\right]\mu(\eta_1,\ldots,\eta_N)$$

At equilibrium

$$\mu^{eq}(\eta) = \prod_{i=1}^N\left[\frac{\beta^{\eta_i}}{\eta_i!}\frac{\Gamma(\eta_i+2s)}{\Gamma(2s)}(1-\beta)^{2s}\right] \qquad 0<\beta<1$$

$$G^{eq}(\xi_1,\ldots,\xi_N) = \prod_{i=1}^N \rho^{\xi_i} \qquad \rho := \rho(\beta) = \frac{\beta}{1-\beta}$$

Results

The dual absorbing process

Markov process $\{\xi(t), t \geq 0\}$ taking values on $\Omega_{N+2} = \mathbb{N}^{N+2}$

Configurations: $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{N+2}$

Generator: $\mathscr{L}^{dual} = \mathscr{L}_{0,1} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_{N,N+1}$

$$\mathscr{L}_{i,i+1} f(\xi) = \begin{cases} \sum_{k=1}^{\xi_1} \varphi(k,\xi_1) \Big[f(\xi + k\delta_0 - k\delta_1) - f(\xi) \Big] & \text{if } i = 0 \\ \sum_{k=1}^{\xi_N} \varphi(k,\xi_N) \Big[f(\xi - k\delta_N + k\delta_{N+1}) - f(\xi) \Big] & \text{if } i = N \\ \sum_{k=1}^{\xi_i} \varphi(k,\xi_i) \Big[f(\xi - k\delta_i + k\delta_{i+1}) - f(\xi) \Big] & \text{otherwise} \\ + \sum_{k=1}^{\xi_{i+1}} \varphi(k,\xi_{i+1}) \Big[f(\xi + k\delta_i - k\delta_{i+1}) - f(\xi) \Big] & \end{cases}$$

Duality of the open symmetric harmonic process

Theorem [FGK (2020a)]

1. The open symmetric harmonic process $\{\eta(t), t \geq 0\}$ with generator $\mathscr{L}$ and the absorbing process $\{\xi(t), t \geq 0\}$ with generator $\mathscr{L}^{dual}$ are dual

$$D(\eta,\xi) = \rho_L^{\xi_0} \left[\prod_{i=1}^{N} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma(2s)}{\Gamma(\xi_i+2s)}\right] \rho_R^{\xi_{N+1}}$$

$$\rho_L = \frac{\beta_L}{1-\beta_L} \qquad \rho_R = \frac{\beta_R}{1-\beta_R}$$

Namely

$$\mathbb{E}_\eta\Big[D(\eta(t),\xi)\Big] = \mathbb{E}_\xi\Big[D(\eta,\xi(t))\Big]$$

Factorial moments and absorption probabilities

Theorem [FGK(2020a)]

2. Let $G(\xi_1, \ldots, \xi_N)$ be scaled factorial moments of non-equilibrium state μ.

   Consider dual configuration $\xi = (0, \xi_1, \ldots, \xi_N, 0)$ with $|\xi| = \sum_{i=0}^{N+1} \xi_i$ particles.

   Then

$$G(\xi_1, \ldots, \xi_N) = \sum_{k=0}^{|\xi|} \rho_L^k \, \rho_R^{|\xi|-k} \; p_\xi(k)$$

   where

$$p_\xi(k) = \mathbb{P}\left[\xi(\infty) = k\delta_0 + (|\xi| - k)\delta_{N+1} \,\middle|\, \xi(0) = \xi\right]$$

absorption probabilities

Mapping to equilibrium

Theorem [FGK(2020b)]

3. As a consequence of duality, there exists a matrix P such that

$$H^{eq} = P^{-1} H P$$

where

$$H^{eq} = (L^{eq})^T \qquad H = L^T$$

- Remark: The mapping P was observed macroscopically by [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim], [Tailleur, Kurchan, Lecomte]

Factorial moments: explicit expression I

Theorem [Frassek, G., (2021+)]

4. Let $G(\xi_1, \dots, \xi_N)$ be scaled factorial moments of non-equilibrium state μ. Then

$$G(\xi_1, \dots, \xi_N) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_\xi(n)$$

with

$$g_\xi(n) = \sum_{\substack{n_1, \dots, n_N \\ \sum_i n_i = n}} \prod_{i=1}^{N} \binom{\xi_i}{n_i} \prod_{j=1}^{2s} \frac{2s(N+2-i)-j}{2s(N+2-i)-j+\sum_{k=i}^{N} n_k}$$

In other words

$$p_\xi(k) = \sum_{n=k}^{|\xi|} (-1)^{n-k} \binom{n}{k} g_\xi(n)$$

Factorial moments: explicit expression II

Theorem [Frassek, G., (2021+)]

5. If we identify the dual configuration $\xi = (0, \xi_1, \ldots, \xi_N, 0)$ with the the ordered position vector $x = (x_1, x_2, \ldots, x_{|\xi|})$ with $1 \le x_1 \le x_2 \le \ldots \le x_{|\xi|} \le N$, then

$$G(\xi_1, \ldots, \xi_N) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_\xi(n)$$

with

$$g_\xi(n) = \sum_{1 \le i_1 < \ldots < i_n \le |\xi|} \prod_{\alpha=1}^{n} \frac{n - \alpha + 2s(N + 1 - x_{i_\alpha})}{n - \alpha + 2s(N+1)}$$

In other words

$$p_\xi(k) = \sum_{n=k}^{|\xi|} (-1)^{n-k} \binom{n}{k} \sum_{1 \le i_1 \le \ldots \le i_n \le |\xi|} \prod_{\alpha=1}^{n} \frac{n - \alpha + 2s(N + 1 - x_{i_\alpha})}{n - \alpha + 2s(N+1)}$$

Non-equilibrium steady state

Theorem [Frassek, G., (2021+)]

6. Using the inversion formula

$$\mu(\eta) = \sum_{\xi \in \mathbb{N}^N} \sum_{n \in \mathbb{N}^N} \rho_R^{|\xi|-|n|} (\rho_L - \rho_R)^{|n|} \varphi_\eta(\xi, n)$$

with

$$\varphi_\eta(\xi, n) = \prod_{i=1}^{N} \binom{\xi_i}{n_i} \binom{\xi_i}{\eta_i} \frac{(-1)^{\xi_i - \eta_i}}{\xi_i!} \frac{\Gamma(2s + \xi_i)}{\Gamma(2s)} \prod_{j=1}^{2s} \frac{2s(N+2-i) - j}{2s(N+2-i) - j + \sum_{k=i}^{N} n_k}$$

Correlation functions

1 dual particle : $\xi = \delta_{x_1}$

$$D(\eta, \delta_{x_1}) = \eta_{x_1}$$

$$\mathbb{E}[\eta_{x_1}] = \rho_L \, p_{x_1}(1) + \rho_R \, p_{x_1}(0)$$

$$p_{x_1}(1) = 1 - \frac{i}{N+1} \qquad\qquad p_{x_1}(0) = \frac{i}{N+1}$$

$$\rho_{x_1} := \langle \eta_{x_1} \rangle = \rho_L + \frac{\rho_R - \rho_L}{N+1} x_1$$ *Linear profile*

$$J := -\langle \eta_{x_1+1} - \eta_{x_1} \rangle = -\frac{\rho_R - \rho_L}{N+1}$$ *Fick's law*

2 dual particles : $\xi = \delta_{x_1} + \delta_{x_2}$

$$D(\eta, \delta_{x_1} + \delta_{x_2}) = \begin{cases} \eta_{x_1}\eta_{x_2} & \text{if} \quad x_1 \neq x_2 \\ \frac{1}{2}\eta_{x_1}(\eta_{x_1} - 1) & \text{if} \quad x_1 = x_2 \end{cases}$$

$$\mathbb{E}[D(\eta, \delta_{x_1} + \delta_{x_2})] = \rho_L^2\, p_{x_1,x_2}(2) + \rho_L\rho_R\, p_{x_1,x_2}(1) + \rho_R^2\, p_{x_1,x_2}(0)$$

$$p_{x_1,x_2}(2) = \left(1 - \frac{x_1}{N+2}\right)\left(1 - \frac{x_2}{N+1}\right)$$

$$p_{x_1,x_2}(1) = \frac{x_1 L + x_2(N+2) - 2x_1x_2}{(N+1)(N+2)}$$

$$p_{x_1,x_2}(0) = \frac{x_1(1+x_2)}{(N+1)(N+2)}$$

2 dual particles : $\xi = \delta_{x_1} + \delta_{x_2}$

$$\mathbb{C}ov(\eta_{x_1}, \eta_{x_2}) = \mathbb{E}[\eta_{x_1}\eta_{x_2}] - \mathbb{E}[\eta_{x_1}]\mathbb{E}[\eta_{x_2}]$$

$$G_{x_1,x_2} = \mathbb{E}[D(\eta, \delta_{x_1} + \delta_{x_2})] \qquad G_{x_1} = \mathbb{E}[D(\eta, \delta_{x_1})]$$

$$\mathbb{C}ov(\eta_{x_1}, \eta_{x_2}) = \begin{cases} G_{x_1,x_2} - G_{x_1}G_{x_2} & \text{if} \quad x_1 \neq x_2 \\ 2G_{x_1,x_1} + G_{x_1}[1 - G_{x_1}] & \text{if} \quad x_1 = x_2 \end{cases}$$

$$= \frac{x_1(N+1-x_2)}{(N+1)^2(N+2)}(\rho_L - \rho_R)^2 \qquad \text{if} \quad x_1 \neq x_2$$

Remark: Long range correlations

$$\lim_{N\to\infty} N\,\mathbb{C}ov(\eta_{y_1N}, \eta_{y_2N}) = y_1(1-y_2)(\rho_L - \rho_R)^2 \qquad 0 \leq y_1 < y_2 \leq 1$$

3 dual particles : $\xi = \delta_{x_1} + \delta_{x_2} + \delta_{x_3}$

$$\begin{aligned}\kappa(\eta_{x_1}, \eta_{x_2}, \eta_{x_3}) &= \mathbb{E}[\eta_{x_1}\eta_{x_2}\eta_{x_3}] + \mathbb{E}[\eta_{x_1}]\mathbb{E}[\eta_{x_2}]\mathbb{E}[\eta_{x_3}] \\ &- \mathbb{E}[\eta_{x_1}\eta_{x_2}]\mathbb{E}[\eta_{x_3}] - \mathbb{E}[\eta_{x_1}\eta_{x_3}]\mathbb{E}[\eta_{x_2}] - \mathbb{E}[\eta_{x_2}\eta_{x_3}]\mathbb{E}[\eta_{x_1}]\end{aligned}$$

Using three dual particles started at $1 \leq x_1 < x_2 < x_3 \leq N$

$$\kappa_3(\eta_{x_1}, \eta_{x_2}, \eta_{x_3}) = \frac{2x_1(N+1-2x_2)(N+1-x_3)}{(N+1)^3(N+2)(N+3)}(\rho_R - \rho_L)^3$$

Remark: for $0 \leq y_1 < y_2 < y_3 \leq 1$

$$\lim_{N\to\infty} N^2\, \kappa_3(\eta_{y_1 N}, \eta_{y_2 N}, \eta_{y_3 N}) = 2y_1(1-2y_2)(1-y_3)(\rho_R - \rho_L)^3$$

$N^{n-1}\kappa_n \sim (\rho_R - \rho_L)^n$ proved by orthogonal duality in [Floreani, Redig, Sau '20]

Comparison to

local equilibrium

Total mass $|\eta| = \sum_{x=1}^{N} \eta_x$

$\mathbb{E}[\cdot]$ non-equilibrium steady state

$\mathbb{E}_{loc}[\cdot]$ local equilibrium state

$\otimes_{x=1}^{N} NegBin(2s, \rho_x)$

- average

$$\mathbb{E}[|\eta|] = \mathbb{E}_{loc}[|\eta|] = \sum_{x=1}^{N} \rho_x = \sum_{x=1}^{N} \left(\rho_L + \frac{\rho_R - \rho_L}{N+1} x\right) = N\left(\frac{\rho_L + \rho_R}{2}\right)$$

- fluctuations

$$\mathbb{V}ar[|\eta|] = \sum_{x_1, x_2 = 1}^{N} \mathbb{C}ov[\eta_{x_1}, \eta_{x_2}]$$

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{V}ar[|\eta|] = \lim_{N\to\infty} \frac{1}{N} \mathbb{V}ar_{loc}[|\eta|] + \frac{s}{6}(\rho_L - \rho_R)^2$$

3. Proof

(Ideas)

Algebraic description

$\mathfrak{su}(1,1)$ Lie algebra

Non-compact Lie algebra with generators satisfying

$$[S^0, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = -2S^0$$

Representation with ∞-dimensional matrices (spin $s > 0$)

$$S^+|n\rangle = (2s+n)|n+1\rangle \qquad S^-|n\rangle = n|n-1\rangle \qquad S^0|n\rangle = (n+s)|n\rangle$$

$$|n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & & & \\ 2s & \ddots & & \\ & 2s+1 & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad S^0 = \begin{pmatrix} s & 0 & & \\ 0 & s+1 & \ddots & \\ & \ddots & s+2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Integrable Hamiltonian

$$H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

- Bulk: [Beisert, Faddeev, Korchemsky, ...]

$$H_{i,i+1} = 2\Big(\psi(\mathbb{S}_{i,i+1}) - \psi(2s)\Big)$$

$$\mathbb{S}_{i,i+1}(\mathbb{S}_{i,i+1} - 1) = S_i^0 S_{i+1}^0 - \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

- Boundary [FGK(2020a)]

$$H_1 = e^{-S_1^-}\, e^{\rho_L S_1^+}\left(\psi(S_1^0 + s) - \psi(2s)\right) e^{\rho_L S_1^+}\, e^{-S_1^-}$$
$$H_N = e^{-S_N^-}\, e^{\rho_R S_N^+}\left(\psi(S_N^0 + s) - \psi(2s)\right) e^{\rho_R S_N^+}\, e^{-S_N^-}$$

In the discrete representation

$$L = H^T$$

Duality explained

$$LD = DL_{dual}^{T}$$

$$\Updownarrow$$

$$H^{T}D = DH_{dual}$$

Bulk (self-)duality

- Symmetry bulk Hamiltonian

$$H_B = \sum_{i=1}^{N-1} H_{i,i+1}$$

$$[H_B, S^0] = [H_B, S^+] = [H_B, S^-] = 0$$

- Reversibility

$$H_B^T R = R H_B$$

$$R(\eta,\xi) = \prod_{i=1}^{N} \eta_i! \frac{\Gamma(2s)}{\Gamma(\eta_i + 2s)} \delta_{\eta,\xi}$$

- Bulk duality

$$H_B^T R e^{S^+} = R e^{S^+} H_B$$

$$Re^{S+}(\eta,\xi) = \prod_{i=1}^{N} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2s)}{\Gamma(\xi_i + 2s)}$$

Boundary duality

- Full Hamiltonian

$$(H_1 + H_B + H_N)^T Re^{S^+} = Re^{S^+}(\tilde{H}_1 + H_B + \tilde{H}_N)$$

$$\tilde{H}_1 = e^{-\rho_L S_1^-}\Big(\psi(S_1^0 + s) - \psi(2s)\Big)\, e^{\rho_L S_1^-}$$

$$\tilde{\mathscr{L}}_1 f(\eta) = \sum_{k=1}^{\eta_1} \varphi(k, \eta_1)\Big[\rho_L^k f(\eta - k\delta_1) - f(\eta)\Big]$$

- Change of representation

$$\mathcal{S}^0 = (\rho\partial_\rho + s)\,, \qquad \mathcal{S}^- = \partial_\rho\,, \qquad \mathcal{S}^+ = \rho(\rho\partial_\rho + 2s)$$

$$[\mathcal{S}^0, \mathcal{S}^\pm] = \pm\mathcal{S}^\pm \qquad [\mathcal{S}^+, \mathcal{S}^-] = -2\mathcal{S}^0$$

$$\rho_L = \mathcal{S}_0{}^+(\mathcal{S}_0^0 + s)^{-1}$$

$$(H_1 + H_B + H_N)^T Re^{S^+} = Re^{S^+}(H_{0,1}^{dual} + H_B + H_{N,N+1}^{dual})$$

$$\mathscr{L}_{0,1}^{dual} f(\xi) = \sum_{k=1}^{\xi_1} \varphi(k, \xi_1)\Big[f(\xi - k\delta_1 + k\delta_0) - f(\xi)\Big]$$

Isospectrality

- $H = H_1 + H_B + H_N$

$$H_i = e^{-S_i^-} \, e^{\rho_i S_i^+} \left(\psi(S_i^0 + s) - \psi(2s)\right) e^{\rho_i S_i^+} \, e^{-S_i^-}$$

- $H' = e^{S^-} \, H \, e^{-S^-} = H'_1 + H_B + H'_N$

$$H'_i = e^{\rho_i S_i^+} \left(\psi(S_i^0 + s) - \psi(2s)\right) e^{\rho_i S_i^+} \qquad \text{upper triangular}$$

$$\mathscr{L}'_i f(\eta) = \left[\sum_{k=1}^{\infty} \frac{\rho_i^k}{k} f(\eta + k\delta_i)\right] - h(\eta_i) f(\eta)$$

- $H^\circ = H^\circ_1 + H_B + H^\circ_N$

$$H^\circ_i = \psi(S_i^0 + s) - \psi(2s) \qquad \text{block diagonal}$$

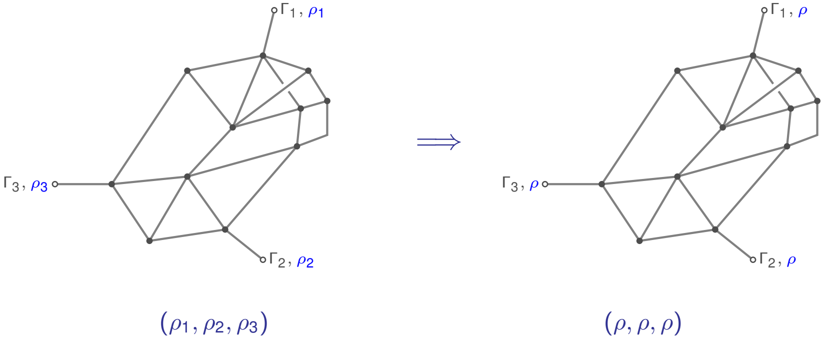
They all have the same spectrum (which is independent of ρ_L, ρ_R)!

As a consequence there exists P such that

$$H^{eq} = P^{-1} H P$$

Mapping to equilibrium

The mapping holds on any graph, with multiple reservoirs



$$H^{eq} = P^{-1} H P$$

Making diagonal the boundaries

- $H' = e^{S^-} H e^{-S^-}$

$$H'_1 = e^{\rho_L S_1^+} \left(\psi(S_1^0 + s) - \psi(2s)\right) e^{\rho_L S_1^+} \qquad H'_N = e^{\rho_R S_N^+} \left(\psi(S_N^0 + s) - \psi(2s)\right) e^{\rho_R S_N^+}$$

- $H'' = e^{-\rho_R S^+} H' e^{\rho_R S^+}$

$$H''_1 = e^{(\rho_L - \rho_R) S_1^+} \left(\psi(S_1^0 + s) - \psi(2s)\right) e^{(\rho_L - \rho_R) S_1^+} \qquad H''_N = \psi(S_N^0 + s) - \psi(2s)$$

- $H^\circ = W^{-1} H'' W$

$$H_1^\circ = \psi(S_1^0 + s) - \psi(2s) \qquad H_N^\circ = \psi(S_N^0 + s) - \psi(2s)$$

Non-trivial symmetries (QIS method)

- First non-local charge:

$$[H'', Q''] = 0 \qquad Q'' = Q^\circ - (\rho_L - \rho_R) Q^+$$

$$Q^\circ = S^0 \left(S^0 + 2s - 1 \right)$$

$$Q^+ = sS^+ + \sum_{i=1}^{N} S_i^+ \left(S_i^0 + 2 \sum_{j=i+1}^{N} S_j^0 \right)$$

- To find W we rather solve

$$W^{-1} Q'' W = Q^\circ$$

- Ansatz:

$$W = 1 + \sum_{k=1}^{\infty} (\rho_L - \rho_R)^k w_k \qquad \Longrightarrow \qquad [Q^\circ, w_k] = Q^+ w_{k-1}$$

- This difference equation can be solved

$$W = \sum_{k=0}^{\infty} (\rho_L - \rho_R)^k \frac{(Q^+)^k}{k!} \frac{\Gamma(2(S^0 + s))}{\Gamma(k + 2(S^0 + s))}$$

non-local!

Back tracking

- Trivially $H^\circ|\Omega\rangle = 0$ with $|\Omega\rangle = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$
- $H''|\psi''\rangle = 0$ defining $|\psi''\rangle = W|\Omega\rangle$ $\qquad H'' = WH^\circ W^{-1}$
- $H'|\psi'\rangle = 0$ defining $|\psi'\rangle = e^{\rho_R S^+}|\psi''\rangle$ $\qquad H' = e^{\rho_R S^+} H'' e^{-\rho_R S^+}$

$$G(\xi) = \langle \xi | R | \psi' \rangle = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_\xi(n)$$

- $H|\psi\rangle = 0$ defining $|\psi\rangle = e^{-S^-}|\psi'\rangle$ $\qquad H = e^{-S^-} H' e^{S^-}$

$$\mu(\eta) = \langle \eta | \psi \rangle = \sum_{\xi \in \mathbb{N}^N} \sum_{n \in \mathbb{N}^N} \rho_R^{|\xi|-|n|} (\rho_L - \rho_R)^{|n|} \varphi_\eta(\xi, n)$$

- Mapping to equilibrium

$$P = e^{-S_-} e^{\rho_R S_+} W e^{-\rho S_+} e^{S_-}$$

Other symmetries: consistency

- It is easy to see that

$$[H^{dual}, S^{random}] = 0 \qquad\qquad S^{random} = \sum_{i=0}^{N+1} S_i^-$$

- Removing a particle uniformly at random

$$S^{random}|\xi\rangle = \sum_{i=0}^{N+1} \xi_i |\xi - \delta_i\rangle$$

- Consistency property [Carinci, G., Redig, arXiv:1907.10583]
  - two particles

$$p_x(1) + p_y(1) = 2p_{xy}(2) + p_{xy}(1)$$

  - three particles

$$p_{yz}(2) + p_{xz}(2) + p_{xy}(2) = 3p_{xyz}(3) + p_{xyz}(2)$$

$$p_{yz}(1) + p_{xz}(1) + p_{xy}(1) = 2p_{xyz}(2) + 2p_{xyz}(1)$$

  - ....

4. Perspectives

The basic model in integral form

Lévy process $\{z(t), t \geq 0\}$ taking values on $\Omega_N = \mathbb{R}_+^N$

$z_i(t) \equiv$ energy at site $i \in \{1, 2, \ldots N\}$ at time $t \geq 0$

$$\mathscr{L} = \mathscr{L}_1 + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_N$$

$$\begin{aligned}
\mathscr{L}_{i,i+1} f(z) &= \int_0^{z_i} \frac{d\alpha}{\alpha} \Big[f(z - \alpha\delta_i + \alpha\delta_{i+1}) - f(z) \Big] \\
&+ \int_0^{z_{i+1}} \frac{d\alpha}{\alpha} \Big[f(z + \alpha\delta_i - \alpha\delta_{i+1}) - f(z) \Big]
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_i f(z) &= \int_0^{z_i} \frac{d\alpha}{\alpha} \Big[f(z - \alpha\delta_i) - f(z) \Big] \\
&+ \int_0^{\infty} d\alpha \frac{e^{-\lambda_i \alpha}}{\alpha} \Big[f(z + \alpha\delta_i) - f(z) \Big]
\end{aligned}$$

Conclusions

- Reasoning with Lie groups is useful for duality and its consequences.
- Integrability (on top of duality) opens up the possibility of explicit formulae for boundary-driven models.
- The algebraic approach to duality can be extended to quantum systems (e.g. FGK(2021), arXiv:2008.03476, quantum symmetric exclusion process).

Thank you for your attention