### Skein Lasagna Modules for 2-handlebodies

#### Ikshu Neithalath

UCLA

#### March 12, 2021

Joint with C. Manolescu

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Skein Lasagna Modules for 2-handlebodies

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# MWW Invariant

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 $\mathcal{S}_{*,i,i}^{N}(W; L)$ , triply graded abelian group

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Generalization of Khovanov-Rozansky  $\mathfrak{gl}_N$  link homology

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Generalization of Khovanov-Rozansky  $\mathfrak{gl}_N$  link homology

\* = blob degree, i = homological degree, j = quantum degree  $S_{*,i,i}^{N}(B^{4}; L) = KhR_{N}^{i,j}(L)$ , supported in \* = 0

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# Main Objects

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The skein lasagna module,  $S_0^N(W; L)$ .

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"Free abelian group generated by embedded surfaces in W, modulo local relations coming from cobordism maps in Khovanov-Rozansky homology."

The cabled Khovanov-Rozansky homology of a framed link  $K \subset S^3$ , <u>KhR</u><sub>N</sub>(K).

"Direct sum of the Khovanov-Rozansky homology groups of an infinite family of cables of K, modulo relations coming from cobordism maps between these cables."

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#### Theorem

Let W be the 2-handlebody associated to K. Then we have an isomorphism,

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where  $\underline{KhR}_{N,\alpha}(K)$  is the cabled Khovanov-Rozansky homology of K.

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When K is the 0-framed unknot, so  $W = S^2 \times D^2$ , the invariant is supported in homological degree 0 and is given by:

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$$\sum_{j=0}^{\infty} \mathsf{rk}\,\mathcal{S}^{\mathsf{N}}_{0,0,-j}(\mathcal{S}^2 \times D^2; \emptyset, \alpha) x^j = \prod_{k=1}^{\mathsf{N}-1} \tfrac{1}{1-x^{2k}}$$

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p > 0 :  $\mathcal{S}^2_{0,0,j}(W; \emptyset, 0) = 0$ , for all j

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$$egin{aligned} &p>0:\mathcal{S}^2_{0,0,j}(W;\emptyset,0)=0, ext{ for all } j\ &p<0:\mathcal{S}^2_{0,0,0}(W;\emptyset,0)\cong\mathbb{Z}, \end{aligned}$$

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$$p > 0 : S_{0,0,j}^2(W; \emptyset, 0) = 0$$
, for all  $j$   
 $p < 0 : S_{0,0,0}^2(W; \emptyset, 0) \cong \mathbb{Z}$ , 0 in other quantum degrees.

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• Connected sum formula:  $\mathcal{S}_0^N(W_1 
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- Putting non-empty links in the boundary:  $\mathcal{S}_0^N(W; L; \Bbbk) \cong \mathcal{S}_0^N(W; \emptyset; \Bbbk) \otimes_{\Bbbk} \operatorname{KhR}_N(L; \Bbbk)$  for  $L \subset B^3 \subset \partial W$

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Adding a 4-handle to get a closed 4-manifold:  $\mathcal{S}_0^N(W; \emptyset) \cong \mathcal{S}_0^N(W \setminus B^4; \emptyset)$  for W closed

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 $\mathcal{S}^2_{0,0,0}(\mathbb{CP}^2; \emptyset, 0) = 0$ 

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$$\mathcal{S}^2_{0,0,0}(\mathbb{CP}^2; \emptyset, 0) = 0$$
  
 $\mathcal{S}^2_{0,0,0}(\overline{\mathbb{CP}}^2; \emptyset, 0) \cong \mathbb{Z}$ 

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# Lasagna Fillings

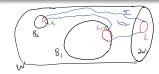
#### Definition

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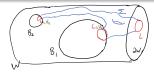
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#### Definition

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### A lasagna filling F of (W; L) is $F = (\Sigma, \{B_i, L_i, v_i\})$ ,



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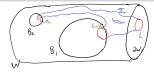
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  - a finite set of disjoint input balls  $B_i \subset int(W)$





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  - a finite set of disjoint input balls  $B_i \subset int(W)$
  - a framed, oriented surface  $\Sigma$  embedded in  $W \setminus \bigcup \operatorname{int}(B_i)$  with  $\partial \Sigma = L \cup_i L_i$ ,  $L_i$  a link in  $\partial B_i$



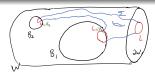
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  - a finite set of disjoint input balls  $B_i \subset int(W)$
  - a framed, oriented surface  $\Sigma$  embedded in  $W \setminus \bigcup$  int $(B_i)$  with  $\partial \Sigma = L \cup_i L_i, L_i$  a link in  $\partial B_i$
  - $v_i$  a homogeneous element in KhR<sub>N</sub>( $L_i$ )



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  - $v_i$  a homogeneous element in  $KhR_N(L_i)$



If  $W = B^4$ , can define map  $\operatorname{KhR}_N(\Sigma) : \otimes \operatorname{KhR}_N(L_i) \to \operatorname{KhR}_N(L)$  and an evaluation  $\operatorname{KhR}_N(F) = \operatorname{KhR}_N(\Sigma)(\otimes v_i) \in \operatorname{KhR}_N(L)$ .

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 $\mathcal{S}_0^N(W;L) = \mathbb{Z}\{ \text{ lasagna fillings of } (W;L) \}/ \sim$ 

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 $\mathcal{S}_0^{\sf N}({\sf W};{\sf L})=\mathbb{Z}\{ ext{ lasagna fillings of }({\sf W};{\sf L})\}/\sim$ 

 $\sim$  is generated by:

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lks Ski multilinearity in input labels v<sub>i</sub>

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- isotopies

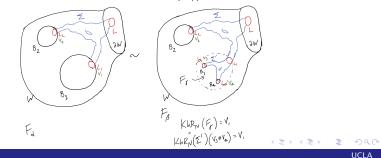
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isotopies

•  $F_{\alpha} \sim F_{\beta}$  if  $F_{\beta}$  is obtained from  $F_{\alpha}$  by inserting a filling  $F_{\gamma}$  of  $(B^4; L_1)$  into an input ball  $B_1$  in  $F_{\alpha}$  and  $KhR_N(F_{\gamma}) = v_1$ 



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 $K \subset S^3$  *p*-framed, oriented knot.

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 $K \subset S^3$  *p*-framed, oriented knot. For  $\ell^-, \ell^+ \ge 0$ , let  $K(\ell^-, \ell^+)$  be  $\ell^-$  and  $\ell^+$  negatively/positively oriented strands parallel to *K*.  $(p(\ell^- + \ell^+), \ell^- + \ell^+)$ -cable of *K*.  $K(1, 1) = \partial R$ 

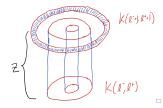
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Skein Lasagna Modules for 2-handlebodies

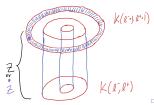
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 $Z = R \cup \text{cylinder}(K(\ell^-, \ell^+)) \text{ is a cobordism}$  $K(\ell^-, \ell^+) \rightarrow K(\ell^- + 1, \ell^+ + 1).$ 



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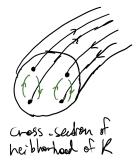
 $Z = R \cup \text{cylinder}(K(\ell^-, \ell^+))$  is a cobordism  $K(\ell^-, \ell^+) \rightarrow K(\ell^- + 1, \ell^+ + 1)$ . Also have  $\dot{Z}$ , the same cobordism decorated by a dot.



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Let  $B_{\ell^-,\ell^+}$  be the subgroup of the braid group on  $\ell^- + \ell^+$  strands that permutes the first  $\ell^-$  and last  $\ell^+$  among themselves.

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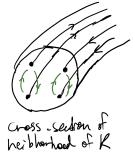
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 $\beta: \mathcal{B}_{\ell^-,\ell^+} \to \mathsf{Aut}(\mathsf{KhR}_N(\mathcal{K}(\ell^-,\ell^+)))$ 

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The cabled Khovanov homology at level  $\alpha \in \mathbb{Z}$  is

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The cabled Khovanov homology at level  $\alpha \in \mathbb{Z}$  is  $\underline{\mathsf{Kh}}_{2,\alpha}(\mathsf{K}) := \bigoplus_{r \ge 0} \mathsf{Kh} \left( \mathsf{K}(r - \alpha^{-}, r + \alpha^{+}) \right) \left\{ -(2r + |\alpha|) \right\} / \sim$ 

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• 
$$\beta(b)v \sim v$$
  
•  $\operatorname{Kh}(Z)v \sim 0$ 

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$$\beta(b)v \sim v Kh(Z)v \sim 0 Kh(\dot{Z})v \sim v$$

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#### Theorem

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Let W be the 2-handlebody associated to K. Then we have an isomorphism,

 $\Phi:\underline{\mathsf{KhR}}_{N,\alpha}(K)\cong\mathcal{S}_0^N(W;\emptyset,\alpha)$ 

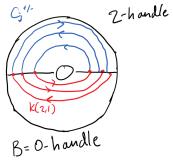
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Take  $|K| = 1, N = 2, \alpha = 0$  for simplicity.

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Take 
$$|\mathcal{K}| = 1, \mathcal{N} = 2, \alpha = 0$$
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 $\widetilde{\Phi} : \bigoplus_{r \ge 0} \operatorname{Kh}(\mathcal{K}(r, r))\{-\} \twoheadrightarrow S_0^2(\mathcal{W}; \emptyset, 0):$ 



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with:  
 $C_{r \ge 0} \to C_{r  

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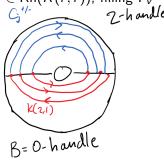
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with:  
• input ball the 0-handle  $B$   
 $B = 0 - h$  and  $e$ 

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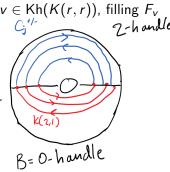
lks Sk • decorated with K(r, r), labelled with v



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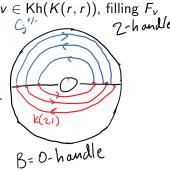
- input ball the 0-handle B
- decorated with K(r, r), labelled with v
- surface = core-parallel discs  $C_j^{\pm}$ ,  $1 \le j \le r$



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- decorated with K(r, r), labelled with v
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Set  $\widetilde{\Phi}(v) = [F_v]$ 



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Claim:  $\widetilde{\Phi}$  factors through <u>Kh<sub>2,0</sub>(K)</u>.

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Claim:  $\widetilde{\Phi}$  factors through <u>Kh<sub>2,0</sub>(K)</u>. Braid group action permutes the discs, giving isotopic fillings.

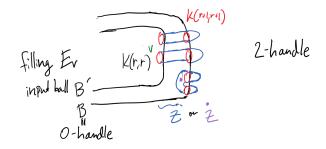
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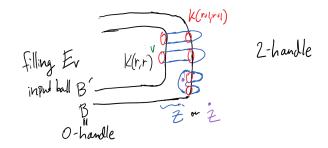
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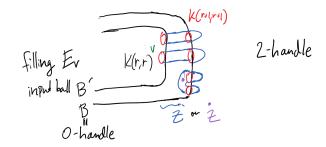
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 $E_v \sim F_{\mathrm{Kh}(\mathrm{Z})(\mathrm{v})}$ 

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$$\begin{split} E_v &\sim F_{\mathrm{Kh}(\mathrm{Z})(v)} \\ \dot{E}_v &\sim F_{\mathrm{Kh}(\dot{\mathrm{Z}})(v)} \end{split}$$

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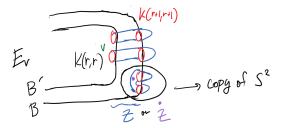
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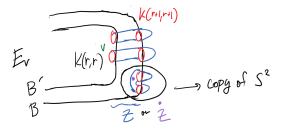
Ikshu Neithalath Skein Lasagna Modules for 2-handlebodies

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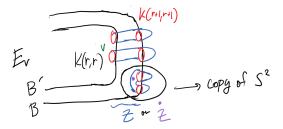


$$egin{aligned} E_{
m v} &\sim F_{
m Kh(Z)(v)} &\sim 0 \ \mbox{because} \ S^2 &\sim 0 \ \dot{E}_{
m v} &\sim F_{
m Kh(\dot{Z})(v)} \end{aligned}$$

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$$E_{v} \sim F_{Kh(Z)(v)} \sim 0$$
 because  $S^{2} \sim 0$   
 $\dot{E}_{v} \sim F_{Kh(\dot{Z})(v)} \sim F_{v}$  because dotted  $S^{2} \sim 1$ 

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 $\mathsf{Claim}: \ \Phi^{-1}: \mathcal{S}^2_0(W; \emptyset, 0) \to \underline{\mathsf{Kh}}_{2,0}(K) \text{ is well-defined}$ 

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Claim:  $\Phi^{-1}: S_0^2(W; \emptyset, 0) \to \underline{Kh}_{2,0}(K)$  is well-defined

Exhibit  $[F] = \widetilde{\Phi}(v)$  by an isotopy and evaluation. Set  $\Phi^{-1}([F]) = [v]$ 

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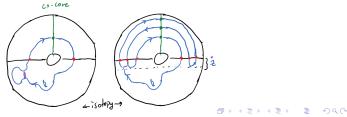
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# Thank you!

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