The Instanton -Torus Knot duality. Two topics on «perturbative» versus «nonperturbative»

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Part 1. Based on works, K.Bulycheva, S.Nechaev , A.G. 1409..
A.Milekhin , A.G. 1412..., A.Milekhin, N.Sopenko, A.G. 1506...to appear

Part 2. Based on works 1506.. , 1607... and to appear .

ın collaboration with V. Avetisov, M. Hovanessian,
S, Nechaev, M. Tamm and O. Valba

Outline of the talk

- Introduction. Where the questions comes from?

- Part I Condensates in 5D SQED and SU(2) SQCD from the torus knot invariants . Gluing the perturbative and nonperturbative contributions via torus knots. Instanton-torus knot duality.

- Part II Eigenvalue tunneling and the ground state of quenched random networks. New reincarnation of the eigenvalue instanton. Visualization of interplay between «perturbative versus nonperturbative»

- Conclusion

Part I

Explicit example of the following phenomena

$$< O > (a,t,q) = \sum_{n,m} (pert)^n (instanton)^m P_{n.m}(a,t,q)$$

The expansion coefficients — knot polynomials

$$P_{n,m}(a,q,t) = \sum_{k,l,s} a^k q^l t^s dim H^{kls}_{n,m}$$

Superpolynomial of the torus knot

Old and new physical questions

- What is the microscopic picture behind the condensate formation? Examples- squark and gluino condensates in SQCD due to the zero modes in the instanton ensemble(NSVZ 83). Topological sector in the nontopological theory
- Chiral condensate in QCD. How quasi-zero modes in the instanton-antiinstanton ensemble get collectivized?

Knot invariants in gauge theories

- $J(q,K) = \langle W(K) \rangle$ in SU(2) 3d Chern-Simons theory — Jones polynomial of knot K (Witten , 89). Can be generalized to all SU(N) groups-

  HOMFLY polynomial $H(a,q,K)$, $a=q^N$
- Generalization to superpolynomial P(a,q,t,K) (Dunfeld, Gukov,Rasmussen 04). Three gradings (a,q,t) in the Hilbert space
- All knot polynomials are particular indices

  $P(a,q,t.K)= \dim H_{ijk} a^i q^j t^k$ counts the multiplicities of the BPS states (Gukov, Scwartz, Vafa 04) in some theory

New tools

- New invariants of knots. Khovanov homologies and superpolynomials which generalize Jones and HOMFLY polynomials. Attempt to derive in the gauge theory(Witten10, Witten-Gaiotto)
- Seberg-Witten solution to N=2 SYM. Nekrasov partition sums. Explicit results for the instanton sums in the Omega-background
- Topological phases of matter. Classification via ground state degeneracy+ holonomy of Berry phase or via entanglement entropy

New math questions

- The superpolynomials of the torus knots are expressed via localization as very specific integrals over moduli space of points in C^2 (E.Gorsky-Negut , 13). Way to zero-size centered instantons in 5d theory.

$$P^{(n,m)} = \sum_{|\lambda|=n} \frac{q^{2\sum a} t^{2\sum l} \prod^{0,0}(1+Aq^{-a'}t^{-l'})(1-q^{a'}t^{l'})}{\prod(q^{a+1}-t^l)(t^{l+1}-q^a)} \times$$

$$\sum_{of shape\ \lambda}^{SYT} \frac{\prod_{i=1}^{n} \chi_i^{S_{m/n}(i)}(1-qt\chi_i)}{\prod_{i=1}^{n-1}(1-\chi_i)(1-qt\frac{\chi_2}{\chi_1})\ldots(1-qt\frac{\chi_n}{\chi_{n-1}})} \prod_{1\le i<j\le n} \frac{(\chi_j-q\chi_i)(\chi_j-t\chi_i)}{(\chi_j-\chi_i)(\chi_j-qt\chi_i)}$$

where

$$S_{m/n}(i) = \frac{im}{n} - \frac{(i-1)m}{n}$$

χ_i equals to $q^{a'}t^{l'}$.

Common algebraic DAHA structure behind the integrals over the instanton moduli and torus knot polynomials

(Okounkov-Maulik 11, Cherednik 11,E.Gorsky-Oblomkov-Rasmussen-Shende 12)

Torus knot polynomials --- twisted characters of the finite-dimensional representation of DAHA. This algebra unifies the permutation group and SL(2,R)

knots. The rational DAHA of type A_{n-1} with parameter c is generated by the $V = C^{n-1}, V^*$ and the permutation group. S_n with the following relations

$$\sigma x \sigma^{-1} = \sigma(x), \qquad \sigma y \sigma^{-1} = \sigma(y) \tag{93}$$

$$x_1 x_2 = x_2 x_1 \qquad y_1 y_2 = y_2 y_1 \tag{94}$$

$$yx - xy = < y, x > -c \sum_{s \in \mathfrak{S}} < \alpha_s, x >< y, \alpha_s^v > s \tag{95}$$

where $\mathfrak{S}$ is the set of all transpositions, and α_s, α_s^v are the corresponding roots and coroots.

c- coefficient in front of 5d CS term. Induces the instanton interaction

Torus knots can be drawn on the torus surface.
T(n,m) corresponds to two windings around cycles

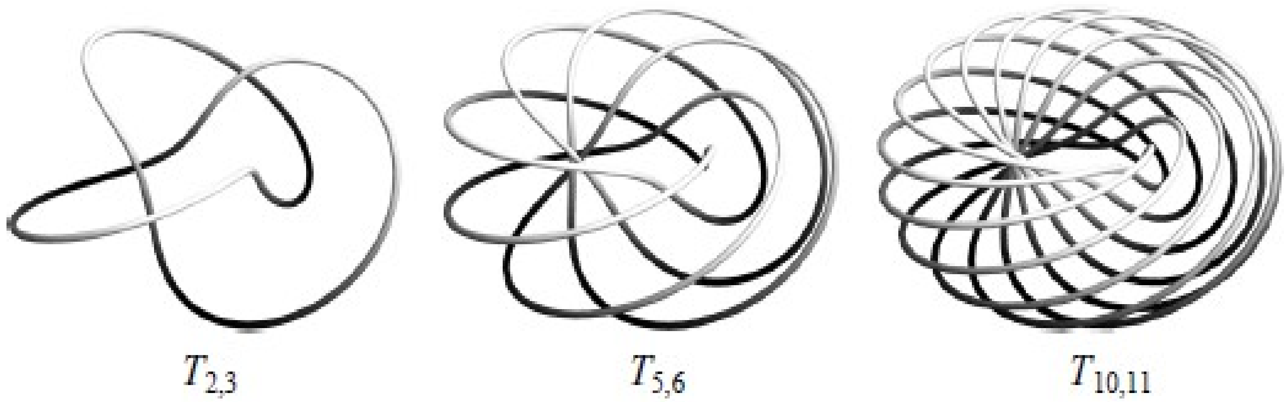


Figure 1: Few samples of torus knots from the series $T_{n,n+1}$: $T_{2,3}$, $T_{5,6}$, $T_{10,11}$.

5D SQED and SQCD

- Consider the 5d SUSY gauge theory with N_f=2 or N_f=3. One dimension is compact S^1. Add 5d CS term k A^F^F. Introduce Omega-deformation= two intependent rotations (angular velocities) in R^4.
- There is the explicit answer for the instanton partition function in this theory due to Nekrasov localization
- Surprise. The condensate of the massless flavor « is sum over the invariants of the T(n,m) torus knots in the momentum space»

Some facts on 5d SQCD

- The BPS particles in the theory are W-bosons, instantons. Due to the CS term the instanton charge induces the electric charge
- Complicated dyonic instantons(both charges). Even more complicated states with 3 charges(+flavor). Not fully classified. Monopoles are loops(monopole particles lifted tp 5d)
- One -loop effect of all BPS particles in 5d D with compact dimension reproduces all instanton partition sum in D=4 SYM theory(Nekrasov-Lawrence)

$$\Omega^m = \Omega^{mn} x_n, \qquad \Omega^{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & i\epsilon_1 & 0 & 0 \\ -i\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\epsilon_2 \\ 0 & 0 & i\epsilon_2 & 0 \end{pmatrix}.$$

Omega-deformation by external field

$$\mathcal{L}_m = -\frac{1}{4g^2} F_{mn}F^{mn} + \frac{1}{g^2}(\partial_m \phi + F_{mn}\Omega^n)(\partial^m \bar{\phi} - F^{mn}\Omega^n) +$$

$$\frac{1}{2}|D_m Q|^2 + \frac{1}{2}|D_m \tilde{Q}|^2 + \frac{2}{g^2}(i\partial_m(\Omega^m \bar{\phi} + \bar{\Omega}^m \phi) + g^2(\bar{Q}Q - \bar{\tilde{Q}}\tilde{Q}))^2 +$$

$$\frac{1}{2}|(\phi - m - i\Omega^m D_m)q|^2 + \frac{1}{2}|(\phi - \tilde{m} - i\Omega^m D_m)\tilde{q}|^2 + 2g^2|\tilde{q}q|^2$$

Spectrum of BPS particles

$$Z = \frac{1}{g^2} n_I + n_e a + \sum_i n_{f_i} m_{f_i}$$

String web (Aharony,Hanany, Kol)

Instanton-torus knot duality,

$$\frac{e^{\beta M}}{(1+a)\beta^2} \frac{d^2 Z_{nek}(q,t,m,M,m_a,Q)}{dM\,dm}\Bigg|_{m\to 0, M\to\infty} = \sum_n Q^n (tq)^{n/2} P_{n,nk+1}(q,t,a) \qquad (2)$$

where m_a, m, M are masses of three hypemultiplets and Q is the counting parameter for the instantons. The mapping between the parameters at the lhs and rhs goes as follows

$$t = \exp(-\beta\epsilon_1) \qquad (3)$$

$$q = \exp(-\beta\epsilon_2) \qquad (4)$$

$$a = -\exp(\beta m) \qquad (5)$$

$$Q = \exp(-\beta/g^2) \qquad (6)$$

Milekhin, A.G. 1412

Instanton-torus knot duality

The superpolynomial is the complicated product in terms
of the Young tableou

$$P(A,q,t)_{nk+1,n} =$$

$$\sum_{\lambda:|\lambda|=n} \frac{t^{(k+1)\sum l} q^{(k+1)\sum a}(1-t)(1-q)\prod^{0,0}(1-Aq^{-a'}t^{-l'})\prod^{0,0}(1-q^{a'}t^{l'})(\sum q^{a'}t^{l'})}{\prod(q^a-t^{l+1})\prod(t^l-q^{a+1})}$$

In this formula there is only one independent
index — n (instanton number)

Toric diagrams for 5d theories with matter. All evaluations are performed via topological vertex.

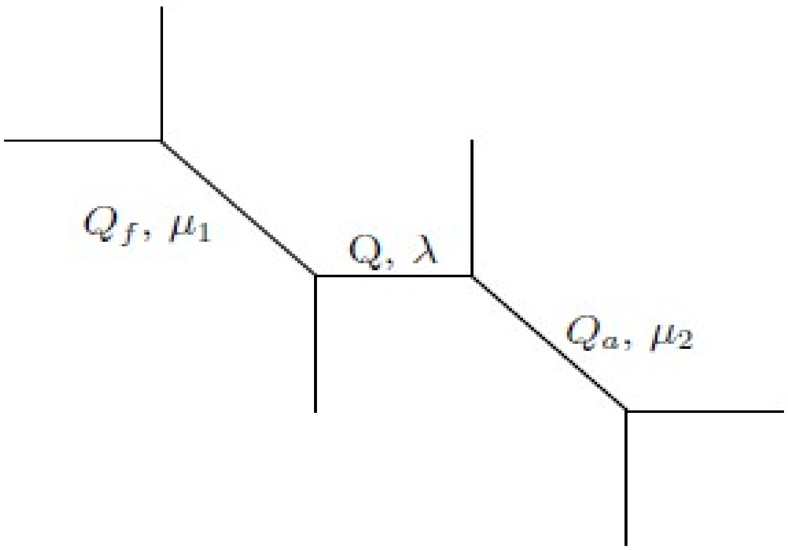


Figure 1: $\mathcal{O}(-1) \times \mathcal{O}(-1) \to \mathbb{P}^1$ with two blow-ups corresponding to the 5D SQED with two flavors and zero CS term

The condensate in this SQCD is generating function for the torus knots

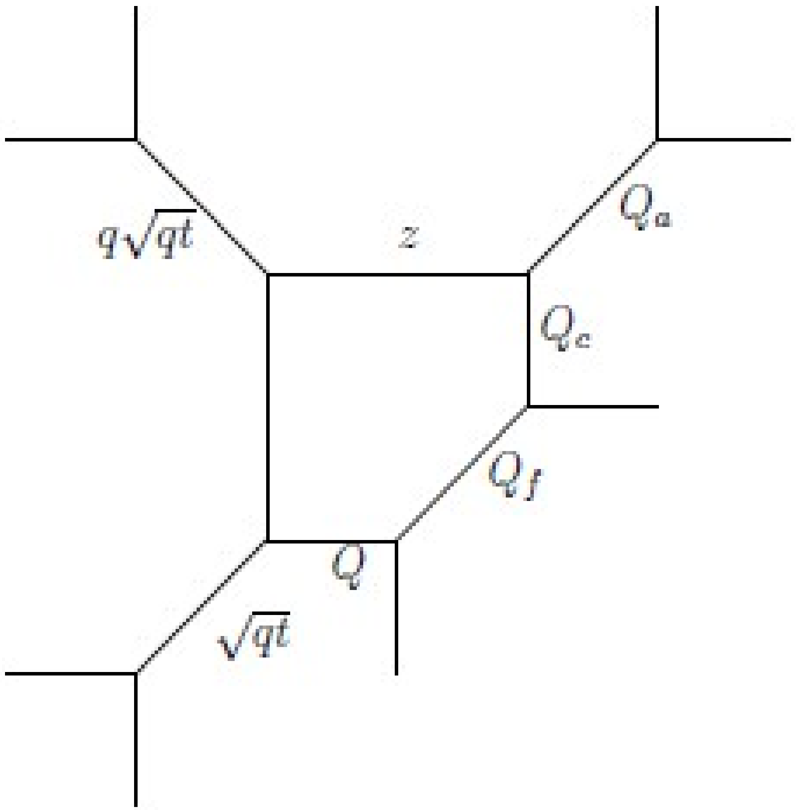


Figure 10: $SU(2)$ theory with four flavours

New findings

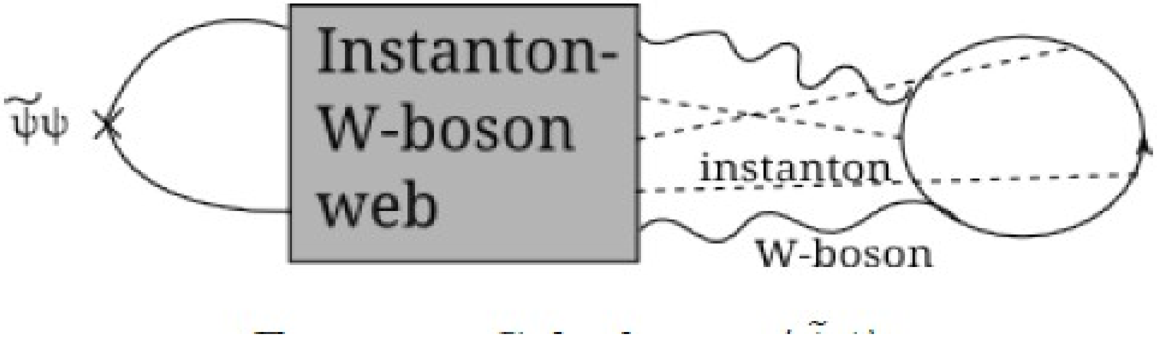
- The information about the knots is encoded in the condensate. Torus knots T(m,n) are important. The physical identifications of the numbers: n- instanton charge, m -electric charge
- The physical variable is expressed in terms of the double sum over the knots. The first example of such situation!
- The rank of the gauge group in the CS picture is the mass of the antifundamental(!)

Interpretation

- The knot invariants describe the multiplicity of BPS states at fixed 4d quantum numbers (n,m)
- Instantons are membranes in the internal space and draw the knots on the «flavor branes». Knots live in the internal Calabi-Yau («momentum») space
- Inverse geometrical transition explains the relation between the mass and the rank of the gauge group in CS picture

Knot invariant as entropic factor

- The place of the knots in the «diagrams»



HOMFLY for generic (n,m) knots

- Consider the N_f=2 theory with Lagrangian brane. Count the contribution of states with (n,m) quantum numbers into condensate
- Consider the SU(2) theory with N_f=4. Two masses fixed, one mass vanishes, one is arbitrary. Expand the condensate in series in two quantum numbers
- Consider N_f=2 U(1) with fractional 5d CS number k=m/n. Extract n- instanton contribution

Sum over the (n,m)

- Double series for the condensate(pert+nonpert)

$$\langle\tilde{\psi}\psi\rangle_{LB} = \frac{\partial Z_{inst}}{\partial m_f}\Big|_{m_f=0} = \sum_{n,m} Q_c^n z^m P_{n,nk+m}(A,q,t)$$

$$P(A,q,t)_{n,nk+m} =$$

$$\sum_{\lambda:|\lambda|=n} \frac{t^{(k+1)\sum l} q^{(k+1)\sum a}(1-t)\prod^{0,0}(1+Aq^{-a'}t^{-l'})\prod^{0,0}(1-q^{a'}t^{l'})}{\prod(q^a - t^{l+1})\prod(t^l - q^{a+1})} \times$$

$$Coef_{z^m} M(z)$$

where $M(z)$ is the contribution from the Lagrangian brane with zero framing:

$$M(z) = \prod_{j=1}^{l(\lambda)} \frac{1 - zt^{j-1}q^{\lambda_j}}{1 - zt^{j-1}}$$

Knot invariants in the dual systems

- Nekrasov partition function in 5d is related via AGT to the q-Liouville conformal block. The derivative of the conformal block- generating function for knot invariants
- The Nekrasov partition function — wave function of the holomorphic Hamiltonian system. Perturbative and nonperturbative effects in SYM= similar effects in QM(example-Toda, Basar-Dunne).Linking of pert and nonpert

  corrections via torus knot invariants (in progress)

Part II. Motivation

- Random matrix model — suitable playground for many nonperurbative and critical phenomena in SUSY gauge theory and 2d gravity
- Simplest nonperturbative phenomena- eigenvalue tunneling(selection of contour, Stokes lines etc) David 93, Marino-Shiappa-Vonk-Russo 20xx and many others
- Random matrix model- theory on unstable branes, tunneling - creation of stable brane(Verlinde-Mcgreevy, Klebanov-Maldacena-Seiberg, Gaiotto-Rastelli, Vafa et al)
- Can the random network be the laboratory for the investigation of nonperturbative effects?

Part II. Findings

- Random networks of Levin-Wen type(local+global constraints)
- New critical phenomena in the colorless networks. Cluster(=stable D-brane) formation in the ground state
- New critical phenomena in the multicolor networks. Plateau formation in the ground state
- Critical behavior via spectral density of the adjacency matrix. Matrix model approach and eigenvalue tunneling

Experimental data for one color

- The ER network or constant degree network
- The degree of the network is conserved-constraint (standard in biology and chemistry, instanton charge in instanton liquid)
- Chemical potential for the triangles. Different behavior for closed or open triangles
- Rich structure of the phase transition. Cliques formation(=almost complete graph)
- For ER network 1/p cliques emerge

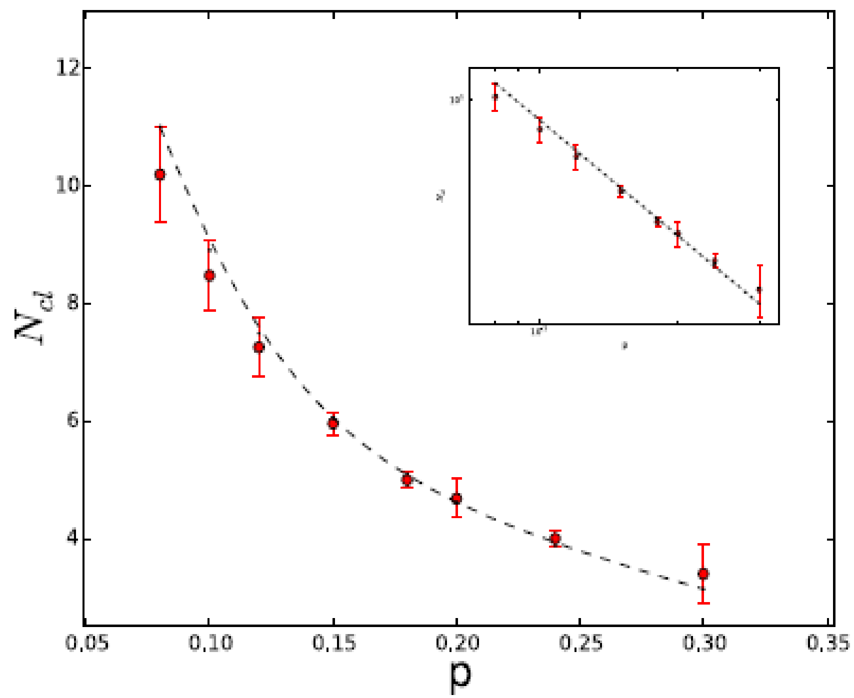


FIG. 1: The number of clusters N_{cl} as a function of the probability p in ER graph. The numerical data are obtained by averaging over 100 randomly generated graphs up to 512 vertices. Numerical values are fitted by the curve $p^{-0.95}$; the behavior in doubly logarithmic scale is shown in the insert.

The network is completely defragmented into the finite number of weakly coupled dense droplets above the critical point.

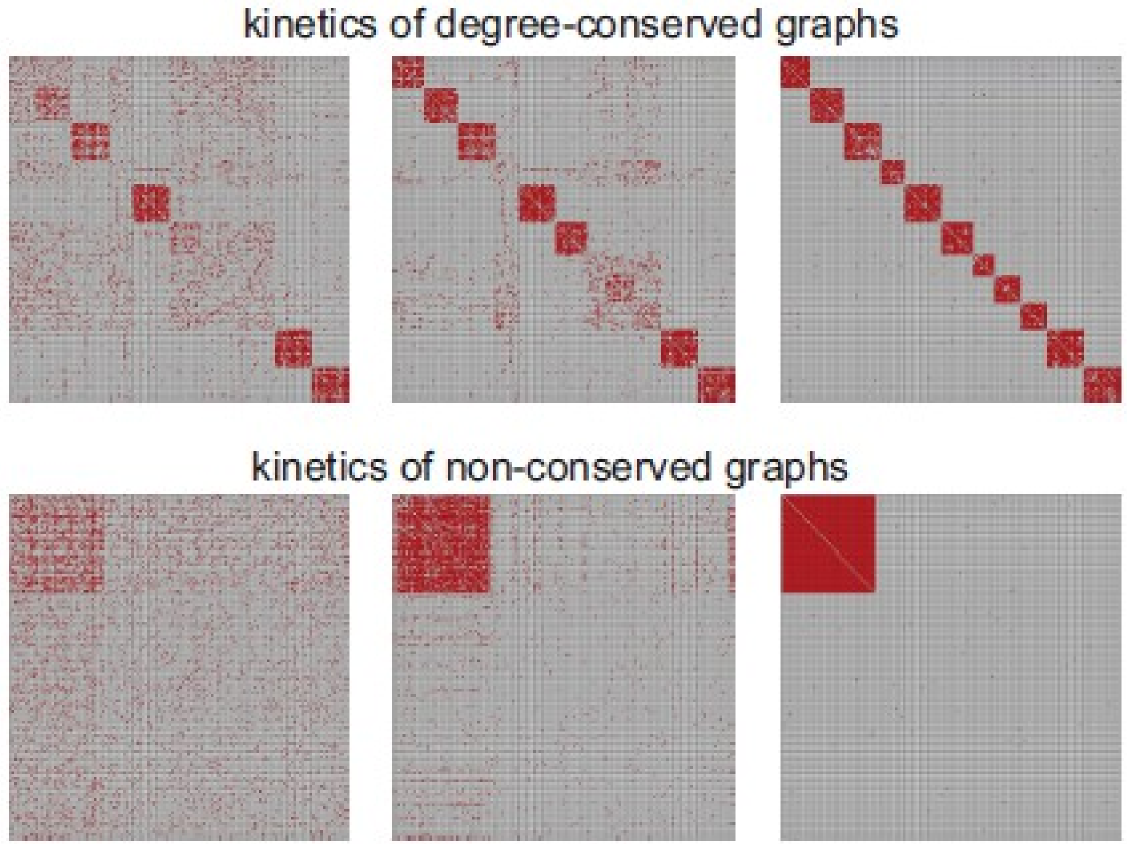


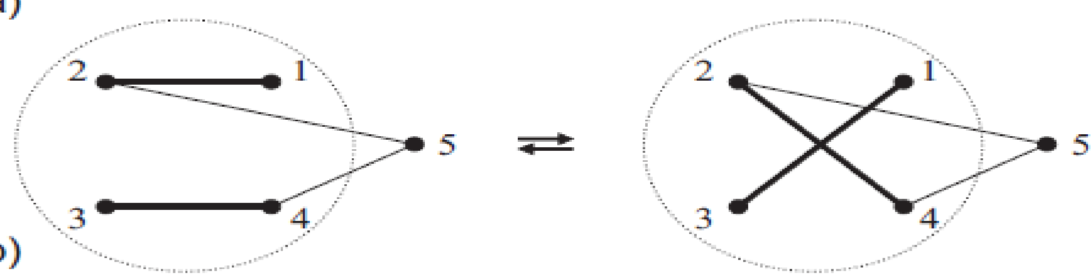
FIG. 5: Few typical samples of intermediate stages of the network evolution: upper panel – evolution with fixed vertex degree; lower panel – eolution with non-fixed vertex degree.

Model

- The possible moves in the network

undirected subgraphs- triads	[0]	[1]	[2]	[3]
concentration	c_0	c_1	c_2	c_3

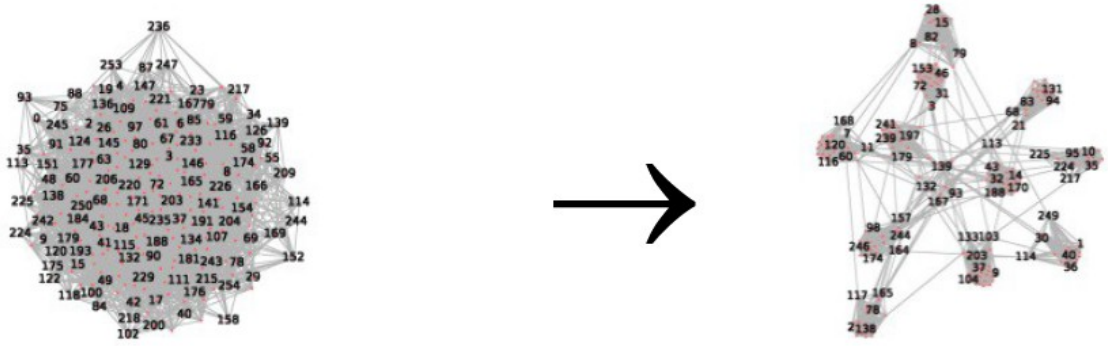
(a)



(b)

Figure 1: a) Possible triads in a non-directed network; b) Single link permutation: links (12) and (34) are removed, and links (13) and (24) are created. Triad {135} goes from type [0] to type [1], triads {125, 345} – from type [2] to type [1], and triad {245} – from type [2] to type [3]: three new triads of type [1] and one triad of type [3] are created instead of three triads of type [2] and one of type [0], compare to Eq.(1).

Typical evolution of the random initial network to the ground state



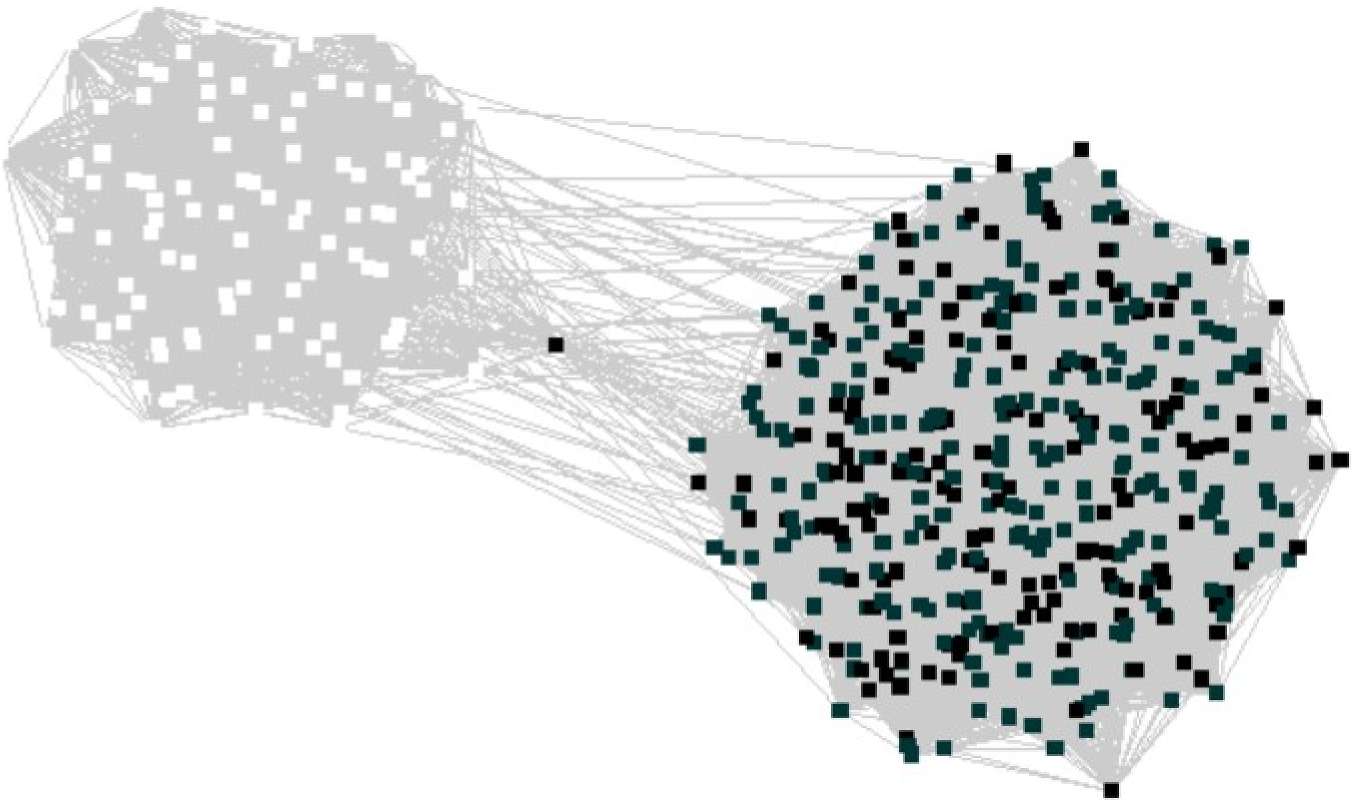
Number of the droplets in the ground state can be predicted!
All clusters are almost complete graphs.

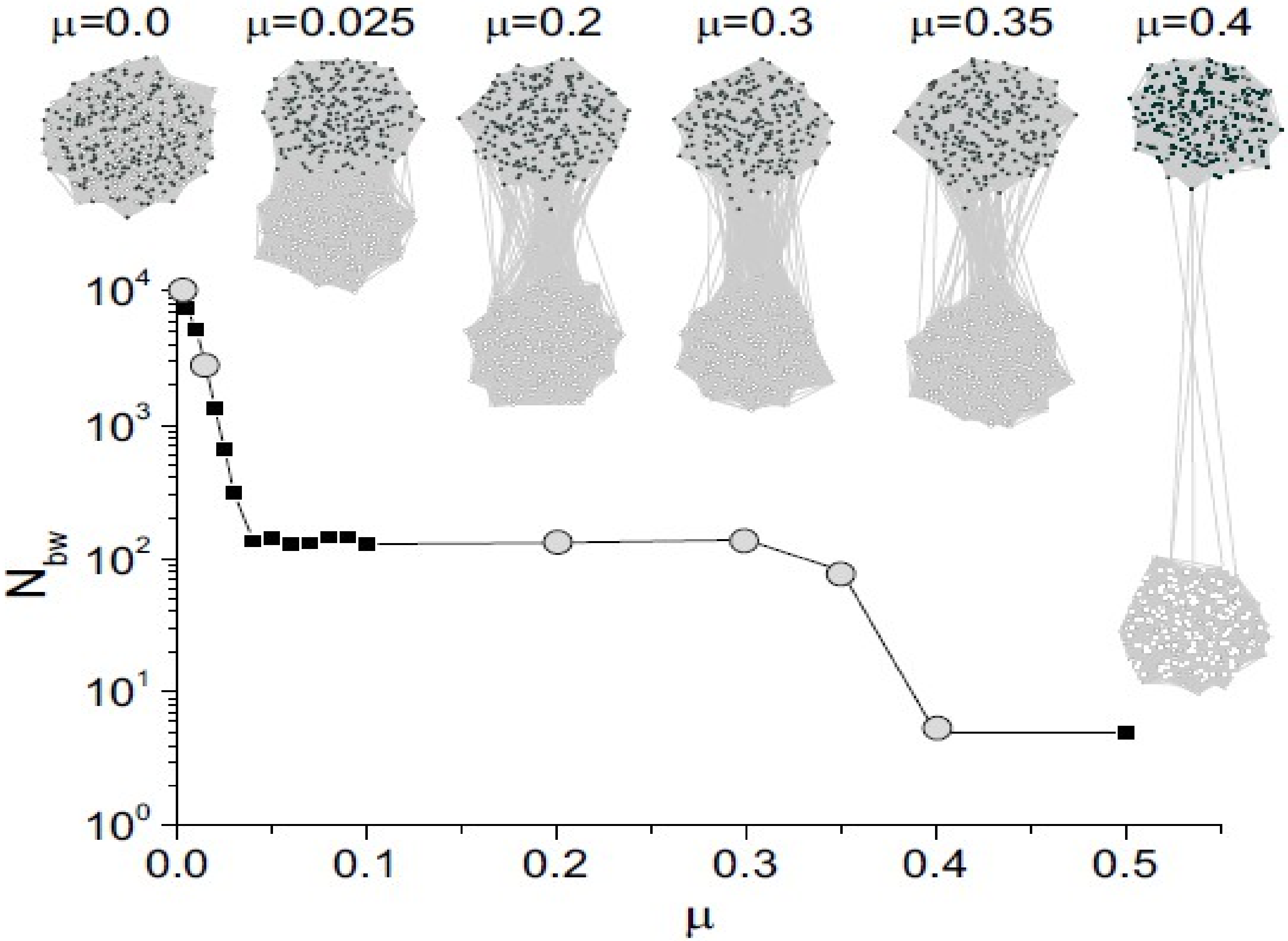
Experimental data for two colors

Chemical potential for all triangles(not necessary closed)

- Two possibilities. ER network and the network with the same degree at all vertices. The results for two cases are the same.
- There is critical behavior with the plateau formation for number of black-white links

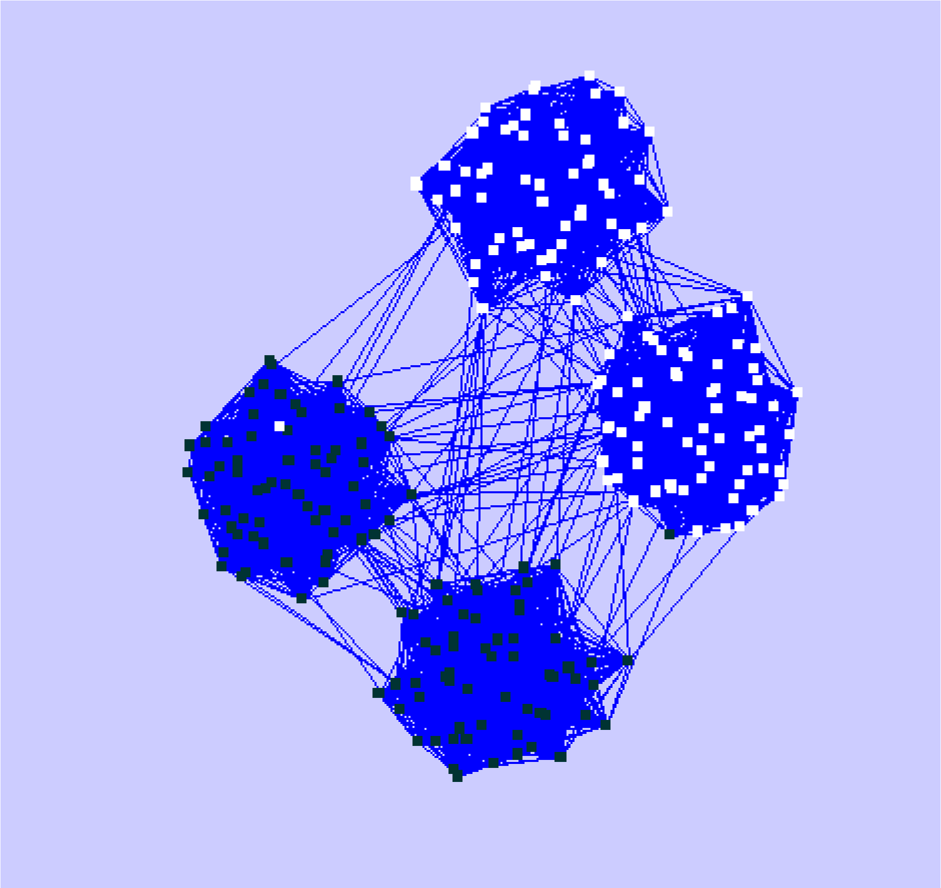
Immediate two color clusters formation from the homogeneous network





Dependence of the number of
white-black links on the chemical potential

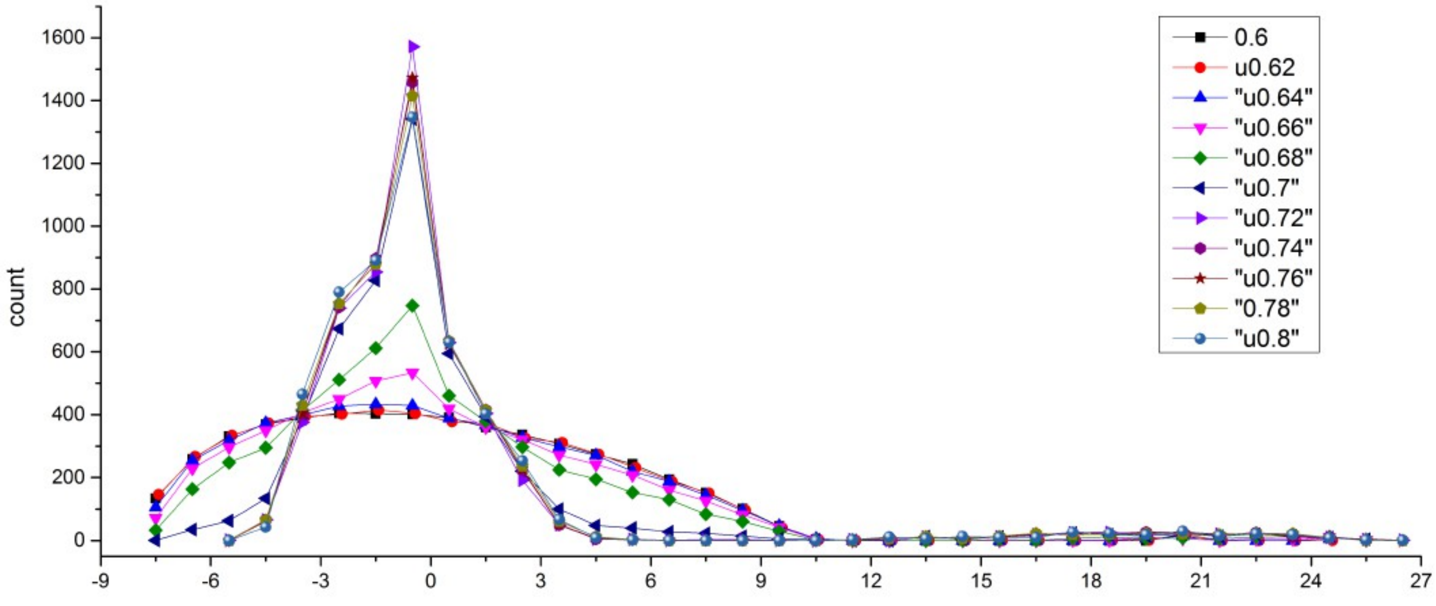
Ground state for network with chemical potential for closed triangles only .



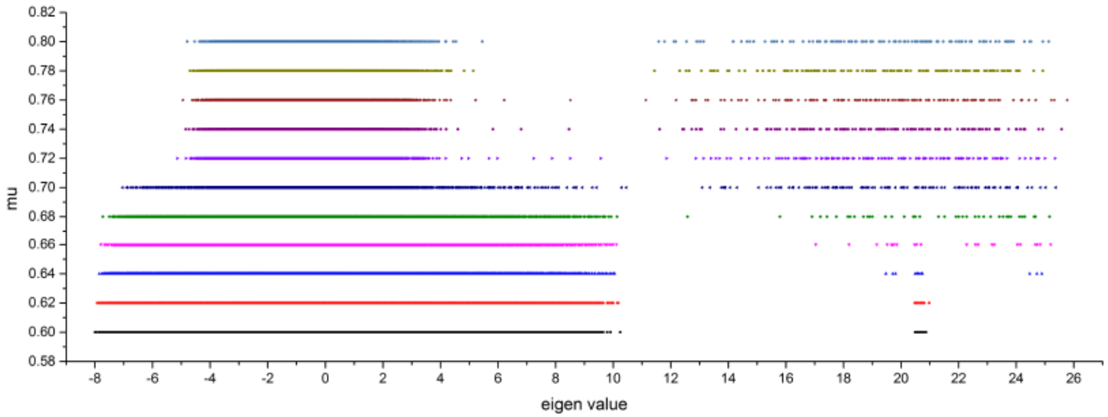
Spectral anatomy of transitions

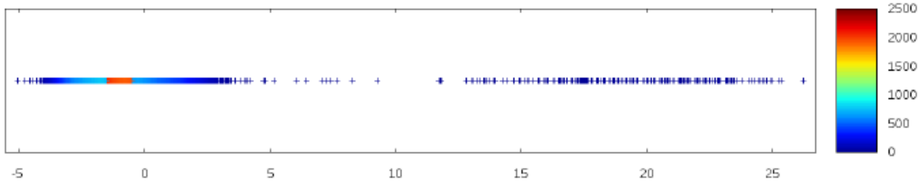
- Second zone formation from the separated eigenvalues moving from the central zone.
- Semicircle distribution before the phase transition. «Trianlge» density in the central zone after the phase transition + second zone. Triangle distribution — feature of the «conformal behavior»
- Collision of the individual eigenvalues in the two-color model at the point of plateau formation. Restoration of broken Z_2 symmetry!

Spectral density of adjacence matrix before and after phase transition

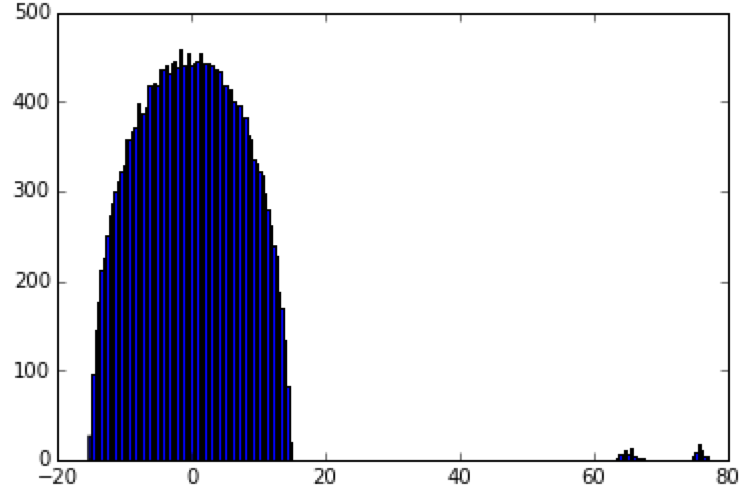


Second zone formation from clusters. Number of the Isolated eigenvalues of the adjacency matrix equals to the number of clusters(Newman et al ,13)

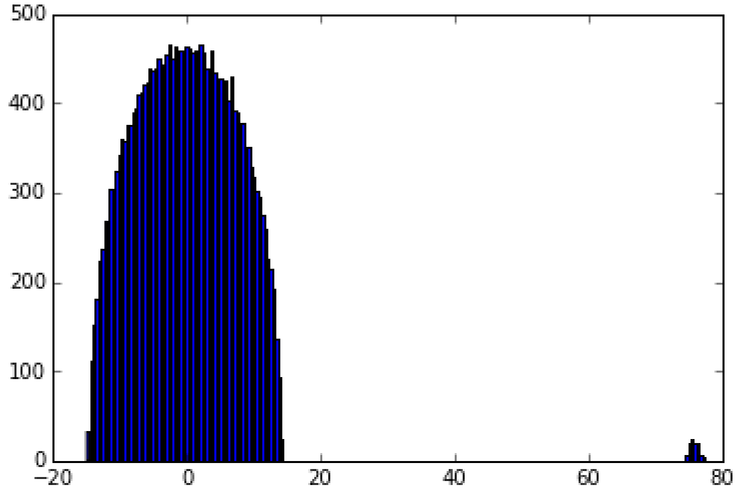




Spectrum before plateau formation



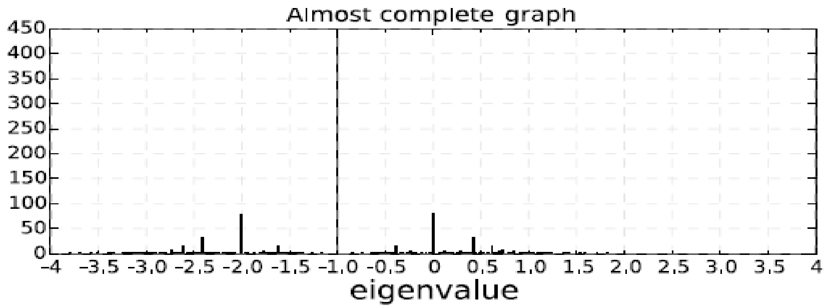
The point where plateau starts exactly at collision
of the separated eigenvalues



One separated eigenvalue-white cluster, the second — black cluster
Collision of eigenvalues — restoration of Z_2 symmetry

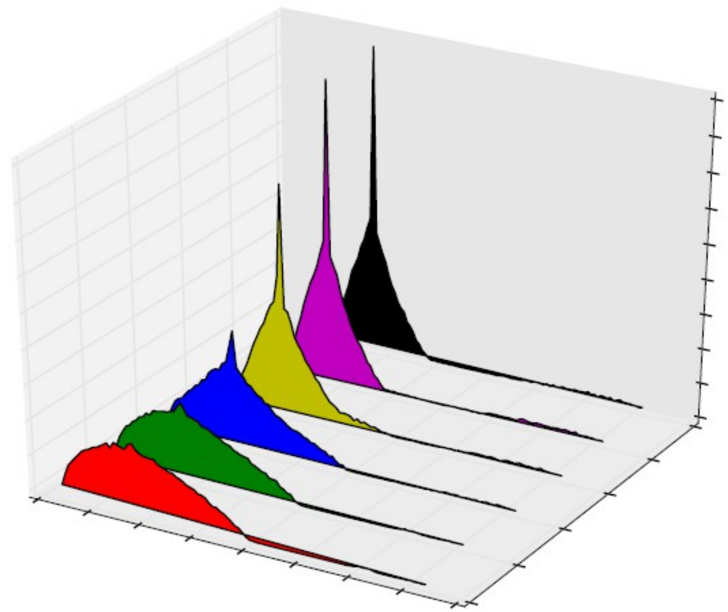
Summary on spectral density

- Perturbative modes only - Wigner semicircle
- Perturbative modes around single «instanton»



Hidden p-adic structure

Account of «instanton» interactions via perturbative modes



Surprizingly similar picture with the Dirac operator spectrum in QCD!

Matrix model description

Adjacency matrix is symmetric random matrix involving 1 and 0 only

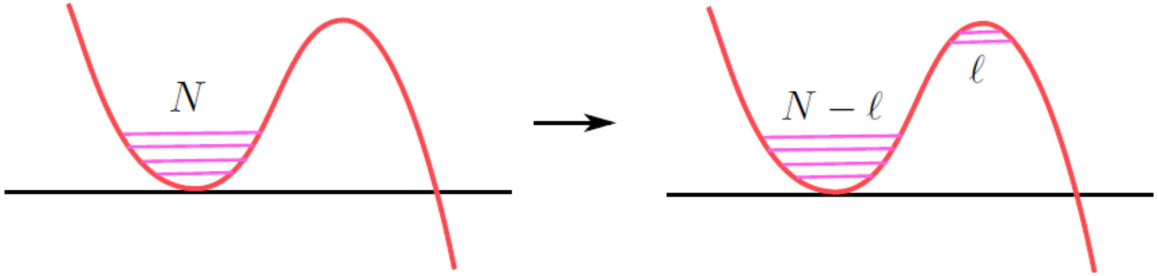
$$Z = \int dM exp(aTrM^2 - \mu TrM^3)$$

Additional constraint: sum of elements in each row and each column is fixed

a- parameter of the network . Chemical potential for the number of triangles yields «interaction term».

$$y^2 = (V')^2 + f(x)$$

The matrix model counterpart of the cluster formation. Eigenvalue tunneling — instanton nonperturbative phenomena in many physical situations. Formation of stable D-branes in the string theory from unstable D0,s.



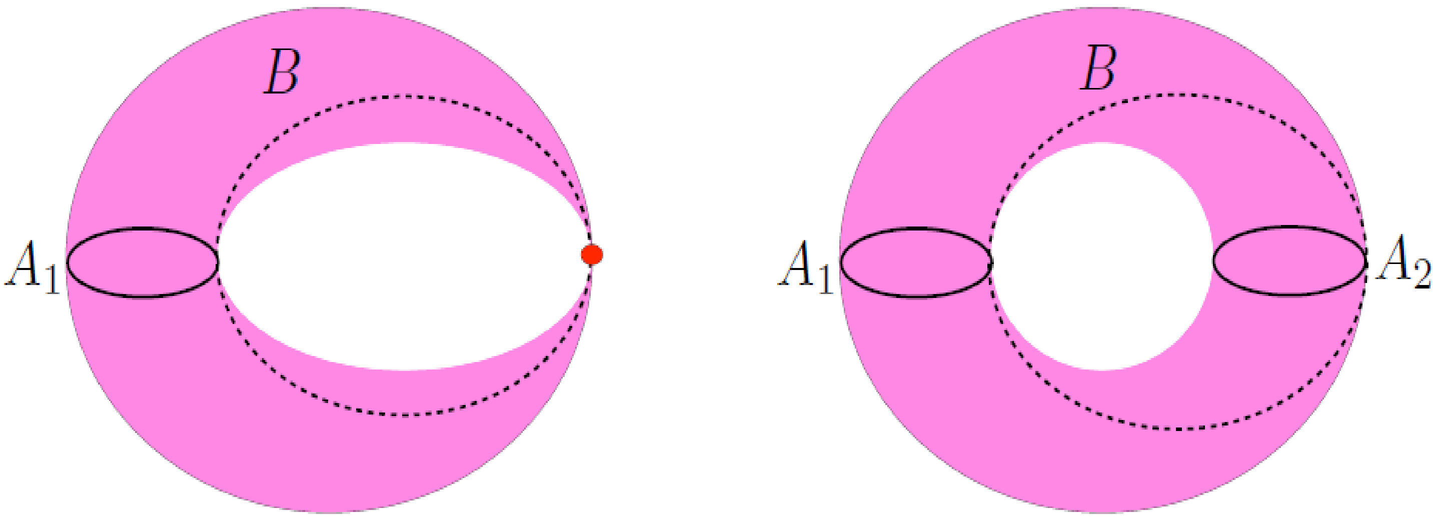


Figure 29: The left-hand side shows the spectral curve in the one-cut phase of the cubic matrix model. The instanton action relevant in the double-scaling limit is obtained by calculating the B-period of the one-form $y(x)\mathrm{d}x$, which goes from the filled cut A_1 to the pinched point. The two-cut phase, in which the pinched point becomes a filled interval, is shown on the right hand side. The instanton action is still given by the B-period integral.

Some Physics

- Random network — example of topological gravity without metric dependence. Similar potential in the matrix model
- Chemical potential for triangles= 2d cosmological constant
- Cluster creation= stable brane creation in the topological gravity. It is described as the eigenvalue tunneling
- Two color model. Top gravity+Ising model

Plateau formation

- Plateau formation=collision of the isolated eigenvalues of the adjacency matrix
- Example of the general phenomena — Argyres-Douglas strong coupling conformal point. Known in SUSY YM and topological strings
- Old question- what happens with the domain walls at AD point? Here we have seen the

  stability of the string connecting black and white clusters

Part I   and Part II

$$Z_{matr} = \int [dM] O(m_1) O(m_2) O(m_3) exp(t_2 Tr M^2 + t_3 Tr M^3)$$

In both cases the vev of some observable in matrix model with the cubic potential is evaluated

$$O(m) = det(M - m)$$

SQCD

«degree constraint»   in random network

Conclusion. Part I

- Just touch tip of the iceberg. Many surprises
- The «knotting» between electric degrees of freedom and instantons is important for the condensate formation. Knot invariants in the double expansion. Relation to resurgence?
- Unexpected appearance of knot invariants in Liouville conformal blocks. General linking and knotting of perturbative and nonperturbative contributions.

A lot of open questions...........

Conclusion. Part II

- Phase transition in the colorless network — nonperturbative formation of the ground state with multiple stable objects from unstable network (FZZT -branes from unstable ZZ branes)
- Phase transition in the two-color network- nonperturbative restoration of the broken Z_2 symmetry in the ground state
- Second zone formation in the spectrum. Genus one Riemann surface. Modular duality in network?
- Seems to be very general phenomena for any

  random network.