Resurgence in Gauge and String Theories
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Resummation Problems and Nonperturbative Corrections

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Based on

1505.07460 w/ K. Okuyama

1511.02860 w/ M. Marino

1606.01894 w/ H. Katsura and Y. Tachikawa

In this talk, I consider a resummation problem of a two-parameter expansion

$$\boldsymbol{F(g,Q) = \sum_{d,n=1}^{\infty} f_{n,d}\, g^n Q^d}$$

This kind of expansions typically appears in string theory or in gauge theories

See e.g. Couso-Santamaria's talk

Main Massage

$$F(g,Q) = \sum_{d,n=1}^{\infty} f_{n,d}\, g^n Q^d$$

$$\sum_{n=1}^{\infty} F_n(Q) g^n \qquad\qquad \sum_{d=1}^{\infty} \widetilde{F}_d(g) Q^d$$

Different Results

- This is of course because these two infinite sums are non-commutative, in general
- I show it in a few examples
- It turns out that interesting "nonperturbative" corrections appear
- These "nonperturbative" corrections look different from those appearing in usual transseries expansions

Plan of the Talk

1. A Simple Example: the Faddeev Quantum Dilogarithm

2. Exact Quantization Conditions for the Relativistic Toda Lattice

Apology: In my talk, no resurgent analysis appears, but such an analysis would be important for deeper understanding in the future

1. A Simple Example:

the Faddeev Quantum Dilogarithm

Faddeev Quantum Dilogarithm

- Definition

Faddeev '94

$$\boxed{\Phi_b(z) := \exp\left[\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}s}{s} \frac{\mathrm{e}^{-2\mathrm{i}sz}}{4\sinh(bs)\sinh(b^{-1}s)}\right]}$$

- This function often appears in theoretical physics (2d CFTs, supersymmetric gauge theories, topological string theory, etc)
- It has two parameters b and z
- There is an obvious symmetry under $b \to b^{-1}$

“Semiclassical” Expansion

- In the limit $b \to 0$, the quantum dilog has the following expansion

$$\mathrm{i}\log\Phi_b(z) = -\pi z^2 - \frac{\pi}{12}(b^2 + b^{-2})$$

$$-\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}(1/2)}{(2n)!} \mathrm{Li}_{2-2n}(-\mathrm{e}^{-2\pi b z})(2\pi b^2)^{2n-1}$$

- In the following, I slightly change the notation by

$$\hbar = 2\pi b^2, \qquad Q = \mathrm{e}^{-t} = \mathrm{e}^{-2\pi b z}$$

$$F(\hbar, Q) = -\mathrm{i}\log\Phi_b(z) = \frac{t^2}{2\hbar} + \frac{1}{24}\left(\hbar + \frac{4\pi^2}{\hbar}\right) + f(\hbar, Q)$$

$$f(\hbar, Q) = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}(1/2)}{(2n)!} \mathrm{Li}_{2-2n}(-Q)\hbar^{2n-1}$$

- In the classical limit, it reduces to the standard dilogarithm

- One-parameter deformation of the dilogarithm $\rightarrow$ Quantum dilogarithm

- In this notation, the symmetry structure is translated into

$$(\hbar, t) \mapsto (\widetilde{\hbar}, \widetilde{t}) = \left(\frac{4\pi^2}{\hbar}, \frac{2\pi t}{\hbar}\right) \quad \text{S-dual transform}$$

- After expanding the polylogarithm, we finally obtain the two-parameter expansion

$$\boxed{f(\hbar, Q) = \frac{1}{\hbar} \sum_{n=0}^{\infty} \sum_{d=1}^{\infty} \frac{(-1)^{n+d} B_{2n}(1/2)}{(2n)!} d^{2n-2} \hbar^{2n} Q^{d}}$$

- In the following, I want to discuss two resummations in this expansion

- I assume $\hbar > 0$ and $t > 0$, for simplicity

Resummation in hbar

- Let us first consider the resummation of the semiclassical expansion

$$f(\hbar, Q) = \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}(1/2)}{(2n)!} \mathrm{Li}_{2-2n}(-Q) \hbar^{2n-1}$$

- This sum turns out to be a divergent series

$$B_{2n}(1/2) \sim (2^{1-2n} - 1)(-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}}$$

$$\mathrm{Li}_{2-2n}(-Q) \sim (2n-2)! \left[\frac{1}{(t+\pi \mathrm{i})^{2n-1}} + \frac{1}{(t-\pi \mathrm{i})^{2n-1}} \right]$$

Borel Sum

- The standard way to resum a factorially divergent series is the Borel sum
- Let us review it briefly
- Consider the following formal divergent series

$$\boldsymbol{h(z) = \sum_{n=0}^{\infty} h_n z^n, \qquad h_n \sim n!}$$

- Borel transform

$$\mathcal{B}[h](\zeta) = \sum_{n=0}^{\infty} \frac{h_n}{n!} \zeta^n$$ Convergent series!

- Borel sum

$$\mathcal{S}[h](z) = \int_0^{\infty} \mathrm{d}\zeta \, \mathrm{e}^{-\zeta} \mathcal{B}[h](\zeta z)$$

- The Borel sum gives a meaning to formal divergent series
- I do not discuss Borel summability here

- Roughly, this can be viewed as

$$h(z) = \sum_{n=0}^{\infty} n! \cdot \frac{h_n}{n!} z^n$$

$$n! = \int_0^\infty \mathrm{d}\zeta \, \mathrm{e}^{-\zeta} \zeta^n$$

- This idea of resummations can be used for more complicated situations

- In our case, it is not easy to compute the Borel transform analytically
- There is a smart way to do an exact resummation
- Here we use an integral representation of the Bernoulli polynomial

$$B_{2n}(1/2) = (-1)^n \, 4n \int_0^\infty \mathrm{d}x \, \frac{x^{2n-1}}{\mathrm{e}^{2\pi x} + 1} \qquad (n \geq 1)$$

- Plug this representation into the asymptotic expansion, and exchange the sum and the integral

Results

- The fianl result takes a simple form

$$\boldsymbol{f}^{\mathbf{resum}}(\hbar, Q) = \frac{1}{\hbar}\mathbf{Li_2}(-Q) + \int_0^\infty \mathbf{d}x \frac{1}{\mathrm{e}^{2\pi x}+1}\log\left(\frac{1+Q\mathrm{e}^{-\hbar x}}{1+Q\mathrm{e}^{\hbar x}}\right)$$

- The result recovers the S-dual invariance! This invariance is not manifest in the above representation

$$\boldsymbol{f}^{\mathbf{resum}}(\hbar, Q) = \boldsymbol{f}^{\mathbf{resum}}(\widetilde{\hbar}, \widetilde{Q})$$

$$\widetilde{\hbar} = \frac{4\pi^2}{\hbar}, \quad \widetilde{Q} = \mathrm{e}^{-2\pi t/\hbar}$$

- It turns out that this resummation reproduces the original exact answer (but I have no proofs)

$$\frac{t^2}{2\hbar} + \frac{1}{24}\left(\hbar + \frac{4\pi^2}{\hbar}\right) + f^{\mathrm{resum}}(\hbar, Q)$$

$$= -\mathrm{i}\int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}s}{s} \frac{\mathrm{e}^{-2\mathrm{i}st}}{4\sinh(2\pi s)\sinh(\hbar s)}$$

Another Resummation

- One can first do the sum in hbar

$$\boldsymbol{f}(\hbar, \boldsymbol{Q}) = \frac{\mathbf{1}}{\hbar} \sum_{\boldsymbol{n}=\mathbf{0}}^{\infty} \sum_{\boldsymbol{d}=\mathbf{1}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{n}+\boldsymbol{d}} \boldsymbol{B}_{\mathbf{2}\boldsymbol{n}}(\mathbf{1}/\mathbf{2})}{(\mathbf{2}\boldsymbol{n})!} \boldsymbol{d}^{\mathbf{2}\boldsymbol{n}-\mathbf{2}} \hbar^{\mathbf{2}\boldsymbol{n}} \boldsymbol{Q}^{\boldsymbol{d}}$$

$$= \sum_{\boldsymbol{d}=\mathbf{1}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{d}}}{\mathbf{2}\boldsymbol{d} \sin \frac{\boldsymbol{d}\hbar}{\mathbf{2}}} \boldsymbol{Q}^{\boldsymbol{d}}$$

- This result is, however, problematic because each coefficient diverges at some particular values of hbar

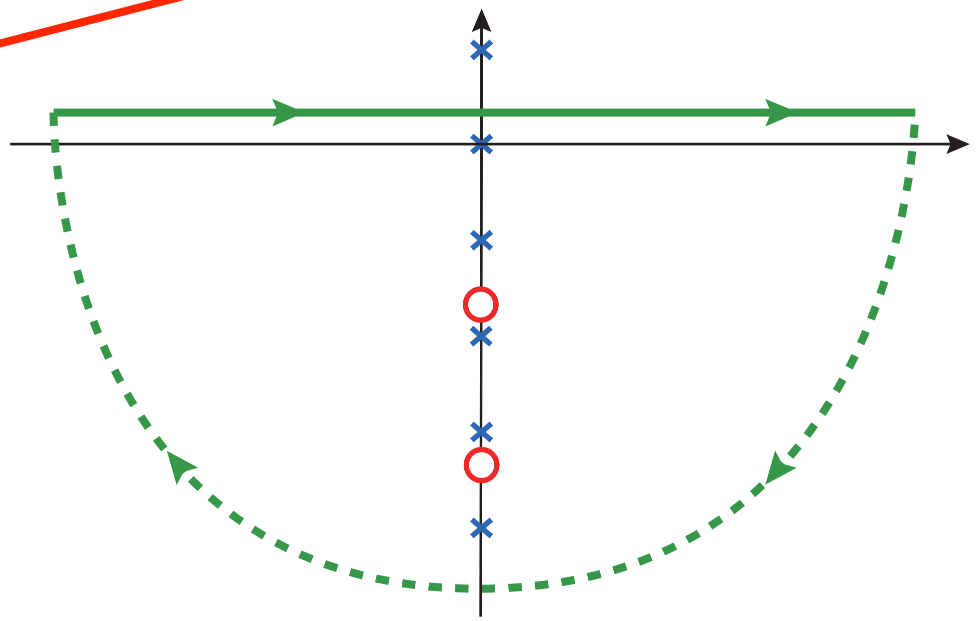
What is happening in this way?

Resolution

- There is an additional contribution

$$\boldsymbol{F(\hbar, Q) = -\mathrm{i} \int_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}s}{s} \frac{\mathrm{e}^{-2\mathrm{i}st}}{4\sinh(2\pi s)\sinh(\hbar s)}}$$

Two kinds of poles



$$\boldsymbol{= \underline{\frac{t^2}{2\hbar} + \frac{1}{24}\left(\hbar + \frac{4\pi^2}{\hbar}\right)}}$$

$$\boldsymbol{+ \underline{\sum_{d=1}^{\infty} \frac{(-1)^d}{2d\sin\frac{d\hbar}{2}} \mathrm{e}^{-dt}} + \underline{\sum_{d=1}^{\infty} \frac{(-1)^d}{2d\sin\frac{2\pi^2 d}{\hbar}} \mathrm{e}^{-\frac{2\pi d t}{\hbar}}}}$$

Remarks

- This result is manifestly invariant under the S-dual transform

$$f(\hbar, Q) = \sum_{d=1}^{\infty} \frac{(-1)^d}{2d \sin \frac{d\hbar}{2}} Q^d + \sum_{d=1}^{\infty} \frac{(-1)^d}{2d \sin \frac{d\widetilde{\hbar}}{2}} \widetilde{Q}^d$$

symmetric

- The naive sum in hbar leads to only the former part

- The latter must be added to reproduce the exact result

- The latter contribution is nonperturbative in hbar

$$\widetilde{Q} = \mathrm{e}^{-\frac{2\pi t}{\hbar}}$$

- The coefficients in the nonperturbative part admit expansions in 1/hbar rather than in hbar unlike transseries

- The poles in the perturbative part are precisely cancelled by those in the nonperturbative part

Summary So Far

$$f(\hbar, Q) = \frac{1}{\hbar} \sum_{n=0}^{\infty} \sum_{d=1}^{\infty} \frac{(-1)^{n+d} B_{2n}(1/2)}{(2n)!} d^{2n-2} \hbar^{2n} Q^d$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}(1/2)}{(2n)!} \mathrm{Li}_{2-2n}(-Q) \hbar^{2n-1}$$

$$\sum_{d=1}^{\infty} \frac{(-1)^d}{2d \sin \frac{d\hbar}{2}} Q^d$$

$$\frac{1}{\hbar} \mathrm{Li}_2(-Q) + \int_0^{\infty} \mathrm{d}x \frac{1}{\mathrm{e}^{2\pi x} + 1} \log\left(\frac{1 + Q\mathrm{e}^{-\hbar x}}{1 + Q\mathrm{e}^{\hbar x}}\right)$$

$$+ \sum_{d=1}^{\infty} \frac{(-1)^d}{2d \sin \frac{d\widetilde{\hbar}}{2}} \widetilde{Q}^d$$

Exact Result

Summary So Far

The latter resummation is insufficient to reproduce the exact result, but this kind of resums often appears in (topological) string theory

2. Exact Quantization Conditions for the Relativistic Toda Lattice

Relativistic Toda Lattice

Ruijsenaars '90

- A generalization of the Toda lattice
- It is still integrable
- In the "non-relativistic" limit, it reduces to the standard Toda lattice

Hamiltonian

$$\boldsymbol{H_1 = \sum_{n=1}^{N}\left(1 + q^{-1/2}R^2 \mathrm{e}^{x_n - x_{n+1}}\right)\mathrm{e}^{Rp_n}}$$

$$[x_n, p_m] = \mathrm{i}\hbar\delta_{nm} \qquad q = \mathrm{e}^{\mathrm{i}R\hbar}$$

$$x_{N+1} = x_1$$

$$\Big\downarrow\ R \to 0$$

Toda lattice

$$H_1 = N + R\sum_{n=1}^{N} p_n$$

$$+R^2\sum_{n=1}^{N}\left(\frac{p_n^2}{2} + \mathrm{e}^{x_n - x_{n+1}}\right) + \mathcal{O}(R^3)$$

Commuting Hamiltonians

$$H_1 = \sum_{n=1}^{N} \left(1 + q^{-1/2} R^2 \mathrm{e}^{x_n - x_{n+1}}\right) \mathrm{e}^{R p_n}$$

$$\vdots$$

$$H_{N-1} = \sum_{n=1}^{N} \left(1 + q^{-1/2} R^2 \mathrm{e}^{x_{n-1} - x_n}\right) \mathrm{e}^{-R p_n}$$

$$H_N = \exp\left(\sum_{n=1}^{N} p_n\right)$$

$$[H_n, H_m] = 0$$

Eigenvalue Problem

- The eigenvalue problem

$$H_k \Psi(x_1, \ldots, x_N) = E_k \Psi(x_1, \ldots, x_N)$$

- In the non-relativistic case, this eigenvalue problem was solved by Gutzwiller in 1980
- Nekrasov and Shatashvili proposed another solution in the gauge theory language
- These two results turned out to be completely equivalent Kozlowski & Teschner '10

Here I want to show that topological string theory can be used to solve the eigenvalue problem for the relativistic Toda lattice

The Simplest Case

- For simplicity, I show the result for N=2
- In the center of mass frame, the (first) Hamiltonian is reduced to

$$\boldsymbol{H} = \mathrm{e}^{\boldsymbol{Rp}} + \mathrm{e}^{-\boldsymbol{Rp}} + \boldsymbol{R}^{2}(\mathrm{e}^{\boldsymbol{x}} + \mathrm{e}^{-\boldsymbol{x}}) \qquad [\boldsymbol{x}, \boldsymbol{p}] = \mathbf{i}\hbar$$

- Then, the eigenvalue problem leads to the following difference equation

$$\boxed{\psi(\boldsymbol{x} + \mathbf{i}\boldsymbol{R}\hbar) + \psi(\boldsymbol{x} - \mathbf{i}\boldsymbol{R}\hbar) + \boldsymbol{R}^{2}(\mathrm{e}^{\boldsymbol{x}} + \mathrm{e}^{-\boldsymbol{x}})\psi(\boldsymbol{x}) = \boldsymbol{E}\psi(\boldsymbol{x})}$$

Remarks

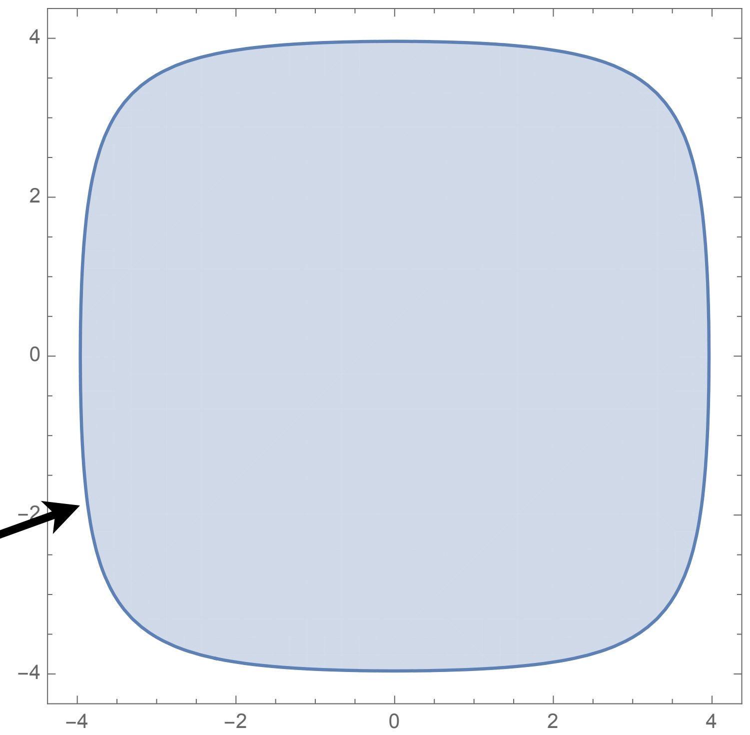
- In the non-relativistic limit, this difference equation reduces to the Schrödinger equation with cosh potential
- By requiring the square integrability, the Hamiltonian has an infinite number of discrete eigenvalues
- Our goal is to find out an equation to determine these eigenvalues exactly

BS Quantization Condition

- The standard way to get approximated eigenvalues is the Bohr-Sommerfeld quantization condition

$$\oint_B \mathrm{d}x\, p(x) = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

$$\mathrm{e}^{Rp} + \mathrm{e}^{-Rp} + R^2(\mathrm{e}^{x} + \mathrm{e}^{-x}) = E$$



- This is a good approximation for $\hbar \to 0$ or $n \to \infty$

Relation to Calabi-Yau Geometry

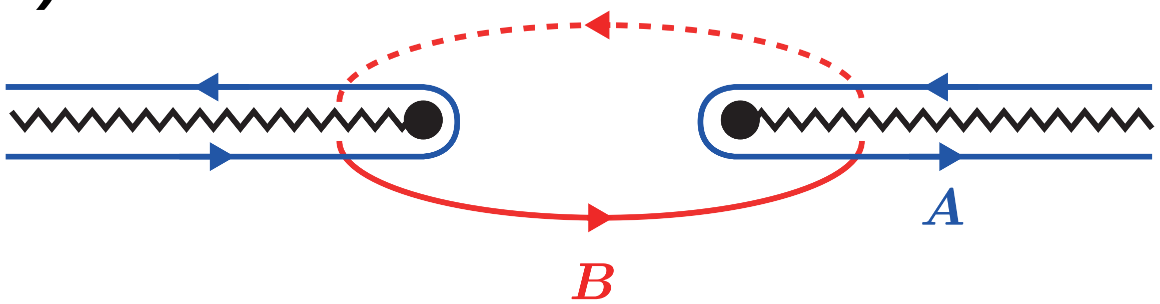
- The spectral curve of the relativistic Toda lattice is viewed as an algebraic curve that describes certain (mirror) Calabi-Yau threefold

$$\mathbf{e}^{\boldsymbol{Rp}} + \mathbf{e}^{-\boldsymbol{Rp}} + \boldsymbol{R}^{\mathbf{2}}(\mathbf{e}^{\boldsymbol{x}} + \mathbf{e}^{-\boldsymbol{x}}) = \boldsymbol{E}$$

Mirror curve corresponding to local Hirzebruch surface $\mathbb{F}_{\mathbf{0}}$

- One can rewrite the BS quantization condition by the topological string free energy (I set R=1 below)

$$t = \oint_A \mathrm{d}x\, p(x)$$



$$\frac{\partial F_0}{\partial t} = \oint_B \mathrm{d}x\, p(x)$$

t : Kähler modulus

F_0 : Prepotential

- The latter is nothing but the LHS in the BS condition

- The former relates t to E

Quantum Corrections

- The BS quantization condition is the first approximation in the semiclassical limit
- Dunham proposed a systematic way to compute the quantum correction to the BS condition

Dunham '32

$$\psi(x) = \exp\left[\frac{\mathrm{i}}{\hbar}\int^{x} \mathrm{d}x' P(x';\hbar)\right]$$

$$P(x;\hbar) = P_0(x) + \hbar P_1(x) + \cdots$$

$$\oint_B \mathrm{d}x\, P(x;\hbar) = 2\pi\hbar n$$

NS Quantization Condition

- Nekrasov and Shatashvili proposed a smart way to resume the quantum correction

Nekrasov & Shatashvili '09

$$\frac{\partial F_{\mathrm{NS}}}{\partial t} = 2\pi\left(n + \frac{1}{2}\right)$$

- The NS free energy is a one-parameter deformation of the prepotential (or the genus zero free energy)

NS Free Energy

- The NS free energy is obtained by the special limit of the refined topological string free energy

$$\boldsymbol{F_{\mathrm{NS}}(t;\hbar) := \lim_{\epsilon_2 \to 0} \epsilon_1 \epsilon_2 F_{\mathrm{ref}}(t;\epsilon_1,\epsilon_2)\big|_{\epsilon_1=\hbar}}$$

- This is explicitly given by

$$\boldsymbol{F_{\mathrm{NS}}(t;\hbar) = \frac{t^3}{3\hbar} - \left(\frac{\hbar}{6} + \frac{2\pi}{3\hbar}\right) t + F_{\mathrm{NS}}^{\mathrm{inst}}(t;\hbar)}$$

$$\boldsymbol{F_{\mathrm{NS}}^{\mathrm{inst}}(t;\hbar) = \sum_{j_L,j_R} \sum_{w,d_1,d_2} \frac{1}{w^2} N_{j_L,j_R}^{d_1,d_2} \frac{\sin\frac{\hbar w}{2}(2j_L+1)\sin\frac{\hbar w}{2}(2j_R+1)}{\sin^3\frac{\hbar w}{2}} \mathrm{e}^{-w(d_1+d_2)t}}$$

Integers

Problem

- As in the quantum dilog, this expression is problematic

$$F_{\mathrm{NS}}^{\mathrm{inst}}(t;\hbar)=\sum_{j_L,j_R}\sum_{w,d_1,d_2}\frac{1}{w^2}N^{d_1,d_2}_{j_L,j_R}\frac{\sin\frac{\hbar w}{2}(2j_L+1)\sin\frac{\hbar w}{2}(2j_R+1)}{\sin^3\frac{\hbar w}{2}}e^{-w(d_1+d_2)t}$$

This factor has an infinite number of poles

- Therefore we cannot use the NS quantization condition for some particular values of hbar even though the eigenvalue problem itself is well-defined for any hbar

Resolution

- The resolution to this problem is almost same as the case of the quantum dilog
- There is a “nonperturbative” correction to the NS free energy
- This was first remarked by Kallen and Marino, and then a prescription to compute the complete correction was conjectured by Grassi, Marino and myself

Kallen & Marino ’13; Grassi, YH & Marino ’14

Exact Quantization Condition

- The final result is remarkably beautiful

Grassi, YH & Marino '14; Wang, Zhang & Huang '15

$$\boxed{\begin{aligned}\frac{t^2}{\hbar} - \frac{1}{6}\left(\hbar + \frac{4\pi^2}{\hbar}\right) + \frac{\partial}{\partial t} F_{\mathrm{NS}}^{\mathrm{inst}}(t;\hbar) \\ + \frac{\partial}{\partial \widetilde{t}} F_{\mathrm{NS}}^{\mathrm{inst}}(\widetilde{t};\widetilde{\hbar}) = 2\pi\left(n + \frac{1}{2}\right)\end{aligned}}$$

$$\widetilde{t} = \frac{2\pi t}{\hbar}, \quad \widetilde{\hbar} = \frac{4\pi^2}{\hbar}$$

$$t = \oint_A \mathrm{d}x\, P(x;\hbar) = 2\log E + \sum_{\ell=1}^{\infty} \frac{a_\ell(\hbar)}{E^{2\ell}}$$

Remarks

- The exact quantization condition is symmetric in $(\boldsymbol{t}, \hbar)$ and $(\widetilde{\boldsymbol{t}}, \widetilde{\hbar})$
- By the S-transform, the perturbative part and the nonperturbative part are exchanged
- These properties are completely the same as those in the Faddeev quantum dilogarithm
- Though we have no rigorous proof for the exact quantization condition, we have a lot of numerical evidence

More on S-duality

- The S-dual transform implicitly relates the spectra for $\hbar$ and $\widetilde{\hbar}$

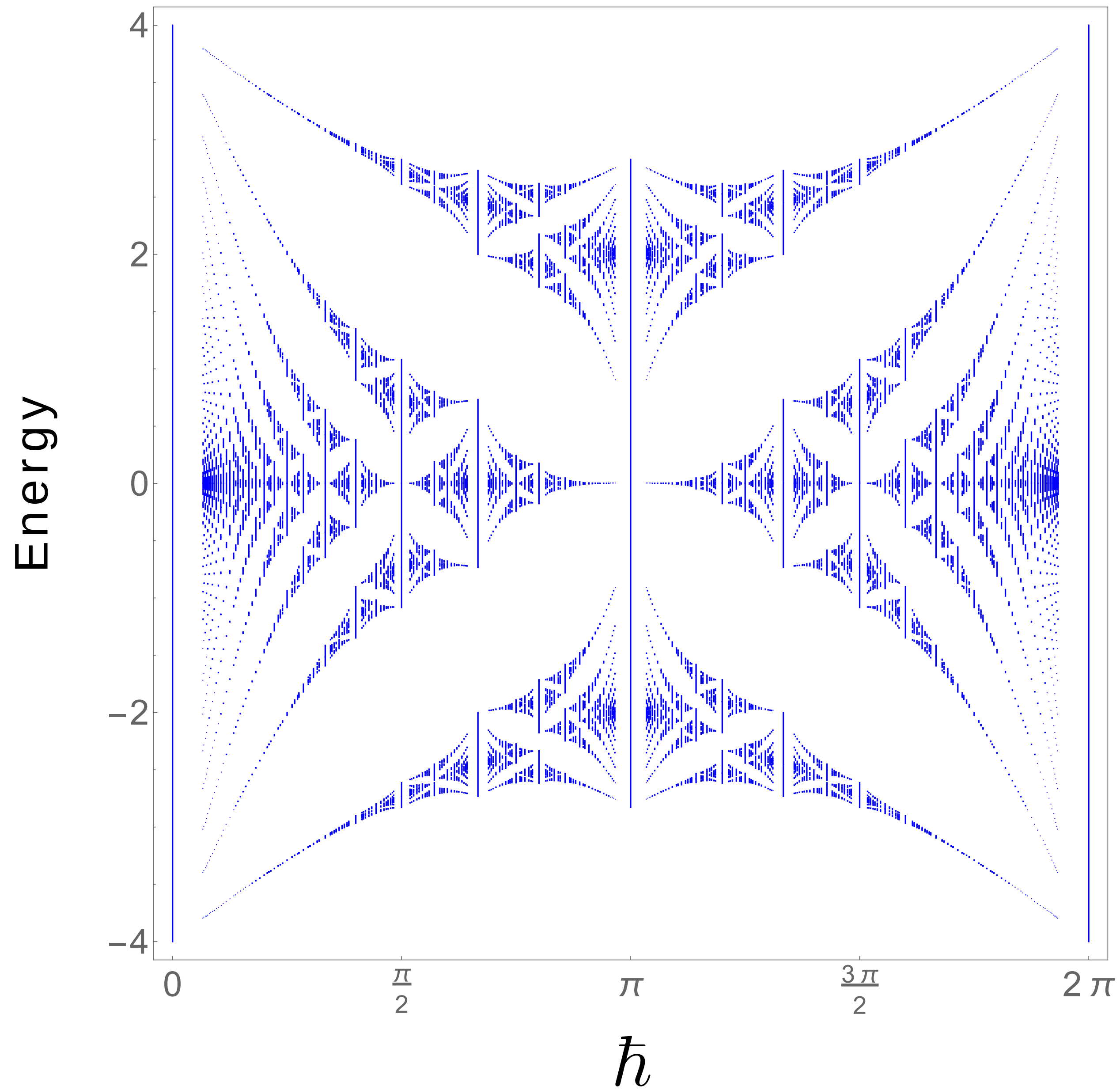
$$\widetilde{\boldsymbol{t}}(\widetilde{\boldsymbol{E}}, \widetilde{\hbar}) = \frac{\boldsymbol{2\pi}}{\hbar} \boldsymbol{t}(\boldsymbol{E}, \hbar) \longrightarrow \widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{E}}(\boldsymbol{E}, \hbar)$$

- In fact, there are simple algebraic relations if q is a root of unity YH, Katsura & Tachikawa '16

n	Relation between $\hbar = 2\pi/n$ and $\widetilde{\hbar} = 2\pi n$
2	$\widetilde{E} = E^2 - 4$
3	$\widetilde{E} = E(E^2 - 6)$
4	$\widetilde{E} = (E^2 + 2E - 2)(E^2 - 2E - 2)$
5	$\widetilde{E} = E(E^4 - 10E^2 + \frac{35-5\sqrt{5}}{2})$

- Recently, the branch cut structure of the quantum Kähler modulus t was identified with Hofstadter's butterfly, well-known in a 2d electron system with a uniform magnetic flux

YH, Katsura & Tachikawa '16



Generalization

- Marino and I generalized this result to the relativistic Toda with arbitrary N

  YH & Marino '15

- The corresponding Calabi-Yau geometry is much more complicated (but known)

  Iqbal & Kashani-Poor '03; Taki '07

- The exact quantization conditions take the universal form

- This result was further generalized to the Goncharov-Kenyon integrable systems (or cluster integrable systems)

  Franco, YH & Marino '15

Summary

- In 2-parameter expansions, resummation problems have a rich structure
- I reviewed some consequences of the different resummations
- Sometimes, "S-dual nonperturbative" corrections appear
- Some quantum integrable systems are solved by using Calabi-Yau geometries

Interesting Directions

- Rigorous derivation of the exact quantization conditions
- Complexify the parameters x, E or hbar
- Construct the eigenfunctions $\rightarrow$ open string sector

Marino & Zakany '16

- A nice appraoch for these purposes is the exact WKB analysis

See Takei's and Kashani-Poor's talks

- The resurgent analysis is also important

See Couso-Santamaria's talk

Thank you