Mathematics of magic angles for bilayer graphene

Lisbon IST QM3

Simon Becker

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Joint work with

Mark Embree Jens Wittsten Maciej Zworski

The spectrum close to the Dirac points in twisted bilayer graphene is described by the following Hamiltonian acting on $L^2(\mathbb{C};\mathbb{C}^4)$

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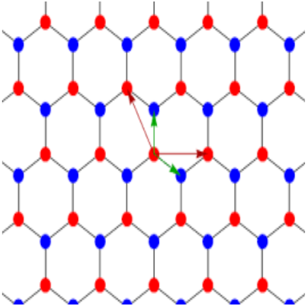

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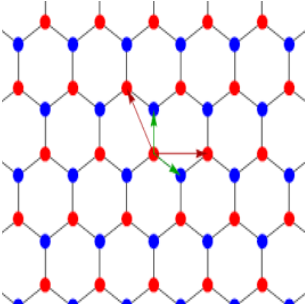

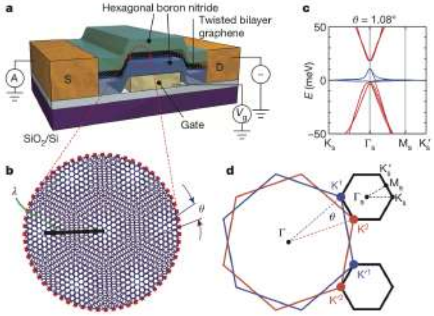

Cao et al '18, Yankovitz et al '18: superconductivity at $\theta \simeq 1.08^\circ$

The operator of today

$$D(\alpha) = \begin{pmatrix} 2D_{\bar z} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar z} \end{pmatrix} \quad \text{on } \mathbb{C}/\Gamma, \quad D_{\bar z} = \frac{1}{2i}(\partial_{x_1} + i\partial_{x_2})$$

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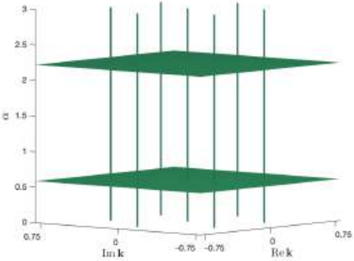

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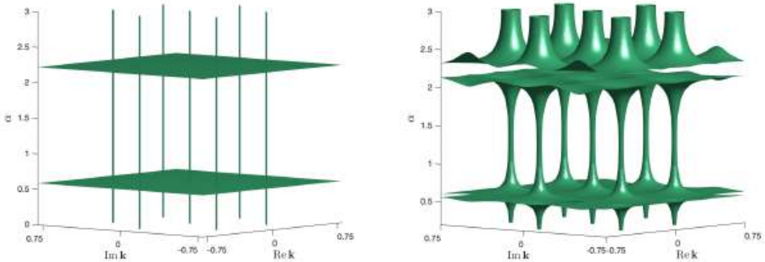

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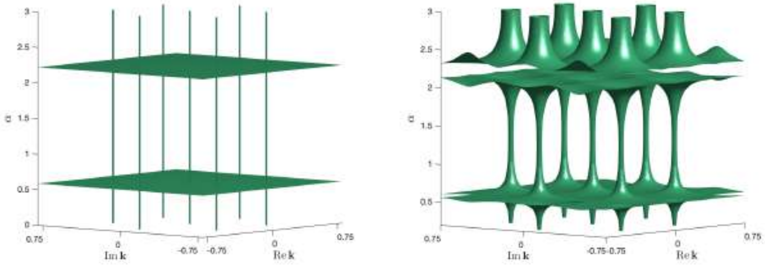

On the right: level surface of $\mathbf{k} \mapsto \|(D(\alpha) - \mathbf{k})^{-1}\| = 10^2$ as α varies: we see that the norm of the resolvent $(D(\alpha) - \mathbf{k})^{-1}$ grows as we approach the first two magic α's (near 0.586 and 2.221), at which it blows up for all k.

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PHYSICAL REVIEW LETTERS **122,** 106405 (2019)

Editors' Suggestion

Origin of Magic Angles in Twisted Bilayer Graphene

Grigory Tarnopolsky, Alex Jura Kruchkov,* and Ashvin Vishwanath

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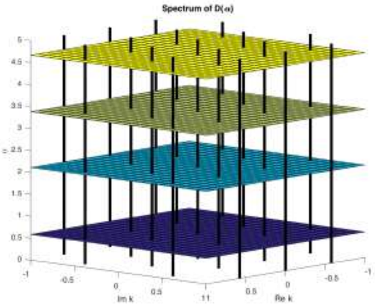

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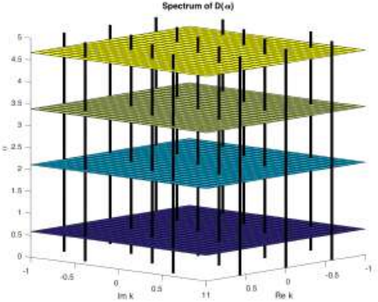

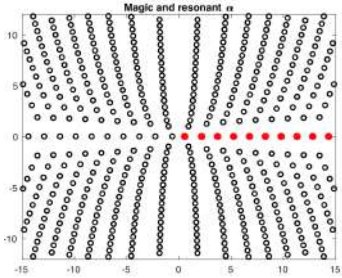

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Movie: level sets of $\lambda \mapsto \|(D(\alpha) - \lambda)^{-1}\|$

Protected state from Heisenberg group over $\mathbb{Z}_3$

Symmetries of $D = \begin{pmatrix} 2D_{\bar{z}} & \alpha U(z) \\ \alpha U(-z) & 2D_{\bar{z}} \end{pmatrix}$ and $H = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$.

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Decompose into irreducible represention of this Heisenberg group:

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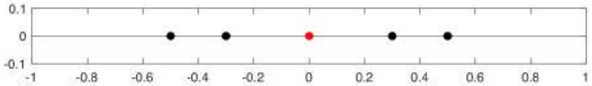
That is because the spectrum of $H(\alpha)|_{L^2_{\rho_{1,0}}}$ is even...



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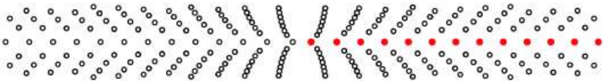

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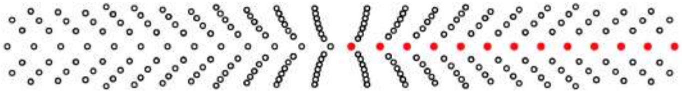

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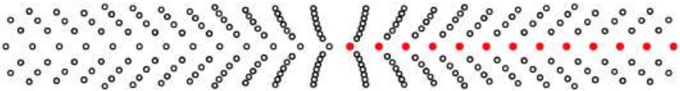

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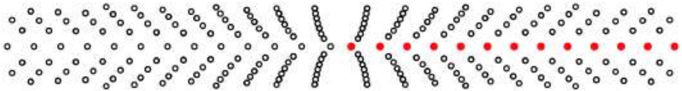

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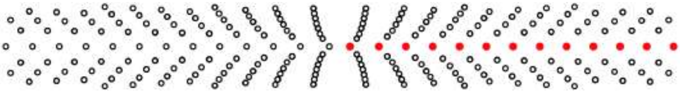

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Also, the spectral characterization allows more efficient calculation of α's

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Works for general potentials with $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_3$ symmetries

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Flat bands

The bands are eigenvalues of $H_{\mathbf{k}} = \begin{pmatrix} 0 & D^* - \bar{\mathbf{k}} \\ D - \mathbf{k} & 0 \end{pmatrix}$ as function of $\mathbf{k} \in \mathbb{C}/\Gamma^* \simeq \mathbb{R}^2/\mathbb{Z}^2$:

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Tarnopolsky et al '19: consider $\mathbf{u} \in L^2_{\rho_{1,0}}(\mathbb{C}/\Gamma; \mathbb{C}^2)$, $D(\alpha)\mathbf{u} = 0$

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Problem: $f_{\mathbf{k}}$ with these properties will have poles

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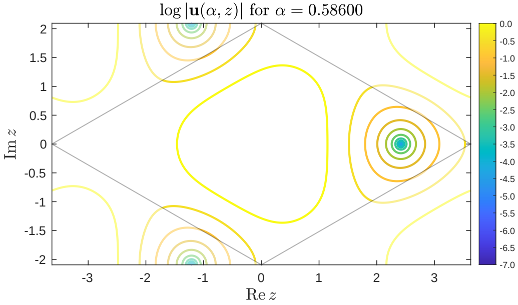

$$\mathbf{u}(\alpha, z_S) = 0, \quad \alpha \in \mathcal{A}, \quad z_S = \frac{4\sqrt{3}}{9}\pi, \quad z_S \equiv \omega z_S \mod \Gamma/3$$

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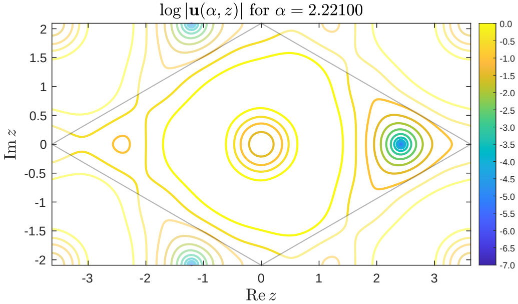

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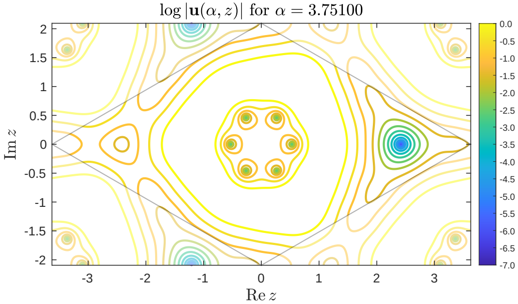

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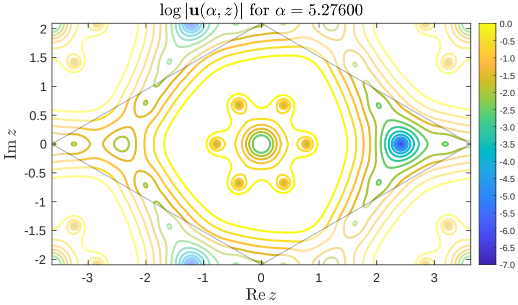

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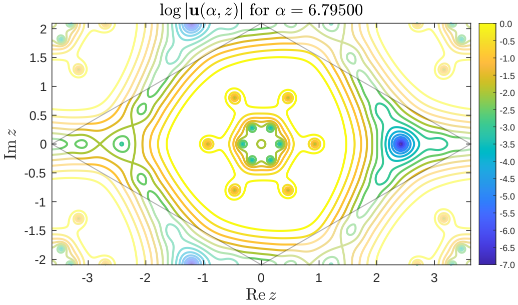

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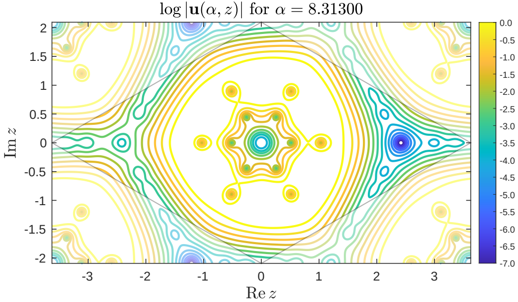

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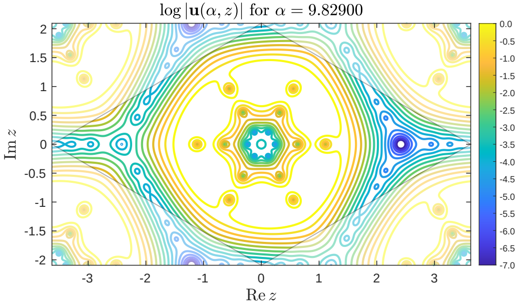

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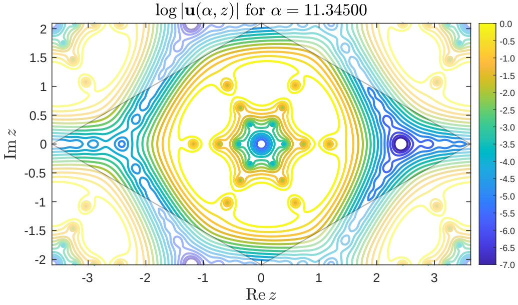

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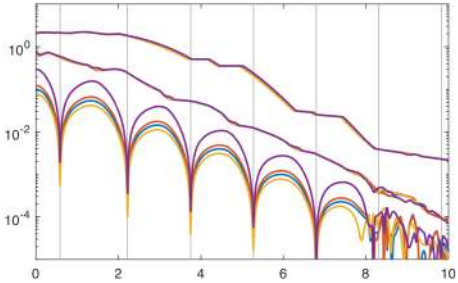

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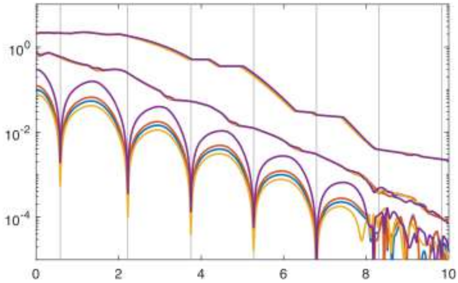

Every small angle $\theta \sim 1/\alpha$ wants to be magical...

Exponential squeezing of bands via solvability of PDE

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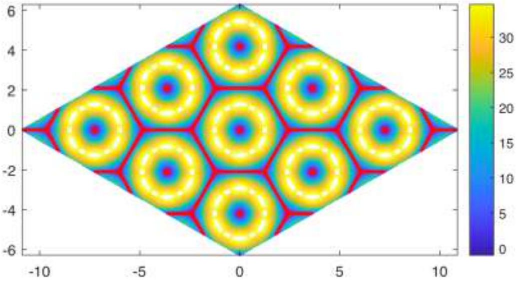

A contour plot of $|\{q,\bar{q}\}|$, $q(x,\xi) = 2\bar{\zeta} - U(z)U(-z)$, $z = x_1 + ix_2$, $2\zeta = \xi_1 - i\xi_2$.

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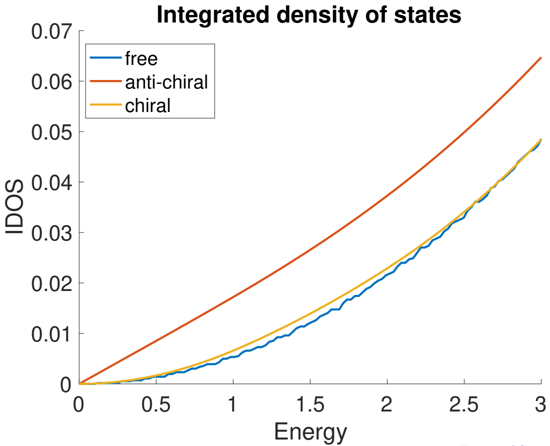

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