Predator-prey reaction-diffusion systems with application to population dynamics

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Table of Contents

Table of Contents

References

H. Berestycki and A. Zilio.
Predator-prey models with competition: The emergence of territoriality.
The American Naturalist, 193(3):436–446,2019.

H. Berestycki and A. Zilio.
Predators-prey models with competition, Part I: existence, bifurcation and qualitative properties.
Communications in Contemporary Mathematics, 20(07):1850010, 2018.

The predator-prey model

- Equation for the prey:

$$\underbrace{\frac{\partial u}{\partial t}}_{\substack{\text{rate of}\\ \text{population}\\ \text{growth}}} - \underbrace{D\Delta u}_{\substack{\text{random}\\ \text{motion}}} = \underbrace{\lambda u\left(1-\frac{u}{\mathcal{K}}\right)}_{\substack{\text{population}\\ \text{growth rate}}} - \underbrace{\sum_{i=1}^{N} p_i w_i u}_{\text{predation}}$$

The predator-prey model

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- Equation for each predator:

$$\underbrace{\frac{\partial w_i}{\partial t}}_{\substack{\text{rate of}\\\text{population}\\\text{growth}}} - \underbrace{d_i\Delta w_i}_{\substack{\text{random}\\\text{motion}}} = \underbrace{p_i u w_i}_{\text{predation}} - \underbrace{l_i w_i}_{\text{mortality}} - \underbrace{a_{ii} w_i^2}_{\substack{\text{intraspecific}\\\text{competition}}} - \underbrace{\beta \sum_{j\neq i} a_{ij} w_j w_i}_{\substack{\text{interspecific}\\\text{competition}}}$$

The predator-prey model

The complete model reads

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = \left(\lambda - \dfrac{\lambda}{\mathcal{K}} u - \displaystyle\sum_{i=1}^{N} p_i w_i\right) u & \text{in } \Omega \times (0, +\infty)\,, \\ \dfrac{\partial w_i}{\partial t} - d_i \Delta w_i = \left(p_i u - l_i - a_{ii} w_i - \beta \displaystyle\sum_{j \neq i} a_{ij} w_j\right) w_i & \text{in } \Omega \times (0, +\infty)\,, \\ \partial_\nu u = \partial_\nu w_i = 0 & \text{on } \partial\Omega \times (0, +\infty)\,, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ w_i(x,0) = w_{i,0}(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where ν is the outward normal vector at the boundary.

Existence of solution

Lemma [2, Lemma 2.1]:

Given a sufficiently regular initial condition $(u_0, w_{1,0}, \dots, w_{N,0}) \in C^{0,\alpha}(\overline{\Omega})$ there exists a unique global solution $(u, w_1, \dots, w_N) \in C_x^{2,\alpha} C_t^{1,\alpha/2}(\Omega \times (0, +\infty))$ of problem (1).

Moreover, the solution is bounded and, for any $\varepsilon > 0$

$$\sup_{(x,t)\in\Omega\times[T_\varepsilon,+\infty)} u(x,t) \le \mathcal{K} + \varepsilon$$

and

$$\sup_{(x,t)\in\Omega\times[T_\varepsilon,+\infty)} w_i(x,t) \le \frac{\mathcal{K} p_i - l_i}{a_{ii}} + \varepsilon.$$

Extinction of predators

Consequence: if there exists an index $i \in \{1, \dots, N\}$ such that $\mathcal{K} p_i \leq l_i$, then

$$\sup_{x\in\Omega} w_i(x,t) \to 0,$$

as $t \to +\infty$.

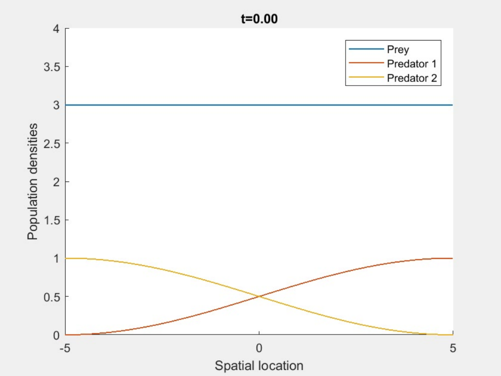


Figure: Simulation that gave rise to extinction of both predators ($\mathcal{K} = 2$, $p_i = 2$ and $l_i = 4$ for $i = 1, 2$).

Table of Contents

The Influence of the competition

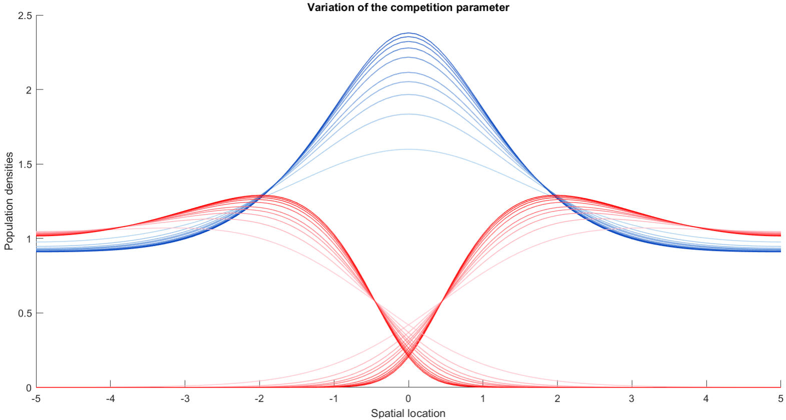
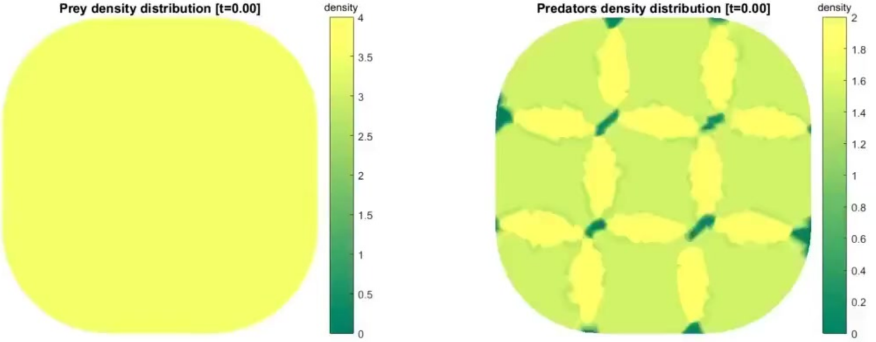


Figure: Impact of the competition parameter β on the predator-prey model (1). Lighter colours correspond to small values of β, from 2, while darker colours correspond to higher values, up to 35. This figure is consistent and replicates the results of [1, Figure 1].

Simulations in 2D: possible shapes of territories


Prey density distribution [t=0.00]

Predators density distribution [t=0.00]

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Figure: Simulation with 9 indistinguishable groups of predators.

Simulations in 2D: possible shapes of territories

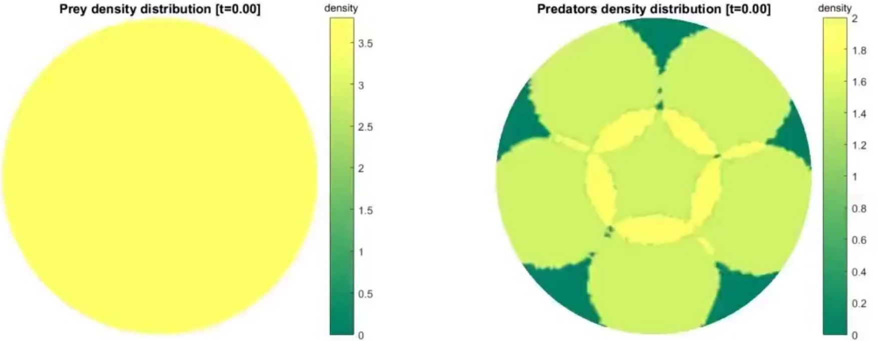

Figure: Simulation considering Dirichlet boundary condition for the prey and Neumann boundary conditions for the 6 groups of predators.

The influence of the consumption rate p_i

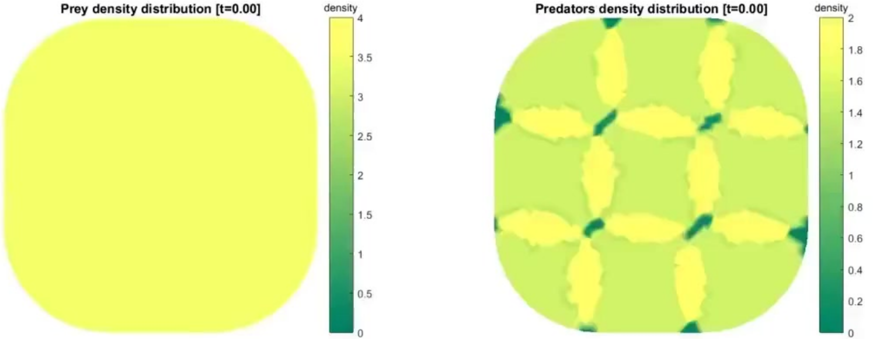

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Figure: Simulation with $p_1 = p_9 = 1.4$ and $p_i = 1$ for $i = 2, \cdots, 8$. On the left we show the evolution of the density of prey and on the right the cumulative density of predators also along time.

Type II functional response model

Handling time (T_i): the average time predator i spends on a captured prey.

The model with type II functional response:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = \left(\lambda - \dfrac{\lambda}{\mathcal{K}} u\right) u - u \displaystyle\sum_{i=1}^{N} \dfrac{p_i}{1 + p_i T_i u} w_i & \text{in } \Omega, \\ \dfrac{\partial w_i}{\partial t} - d_i \Delta w_i = \left(-l_i - a_{ii} w_i\right) w_i + \dfrac{p_i}{1 + p_i T_i u} u w_i - \beta w_i \displaystyle\sum_{j \neq i} a_{ij} w_j & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu w_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Original model vs Type II

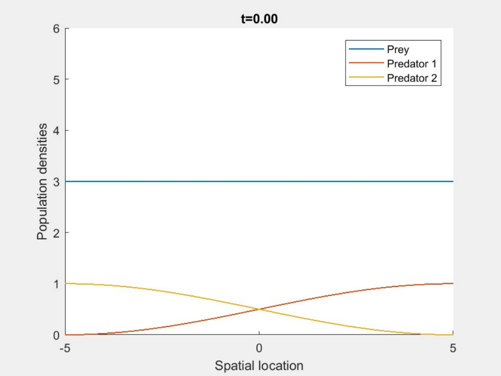

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Figure: Evolution of the solution of the original model.

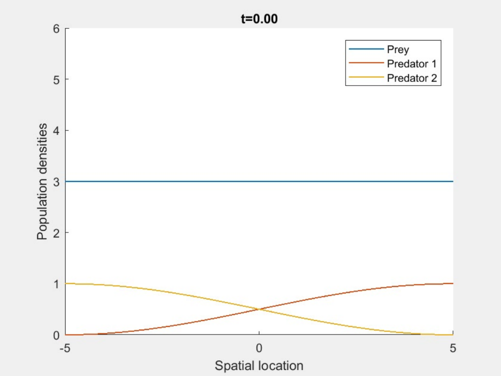

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Figure: Evolution of the solution of the model with type II functional response (here $T_1 = T_2 = 0.25$).

What happens when $T_1 > T_2$?

Predator-prey interaction
The model
Existence of solution
Results
The influence of the competition
Simulations in 2D: possible shapes of territories
Influence of the consumption rate
Type II functional response
Conclusion

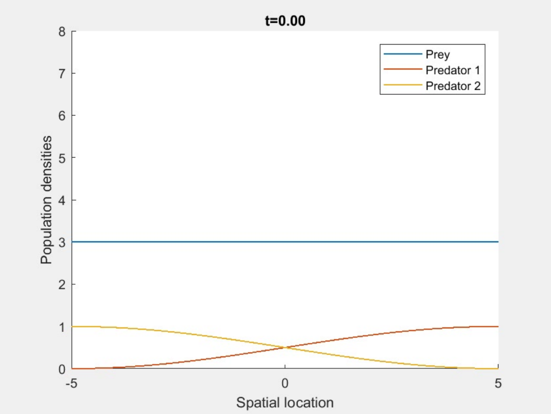

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Figure: Simulation with $T_1 = 0.5$ and $T_2 = 0.25$ leading to extinction of predator 1 (red curve). We consider here strong competition ($\beta = 100$).

Table of Contents

Conclusion

- Territoriality is an emergent property of the model giving rise to a buffer zone benefiting both the populations involved.

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- Consumption rate of a given predator increases its territory size.
- Handling time increased the prey population inside the buffer zone.
- The territory size decreases with an increase in the handling time until it reaches a rupture point and the predator becomes extinct.

thank you!