The Hidden Geometry

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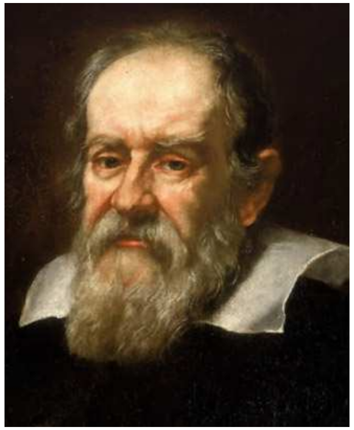
Galileu Galilei (1564–1642)

"Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it."

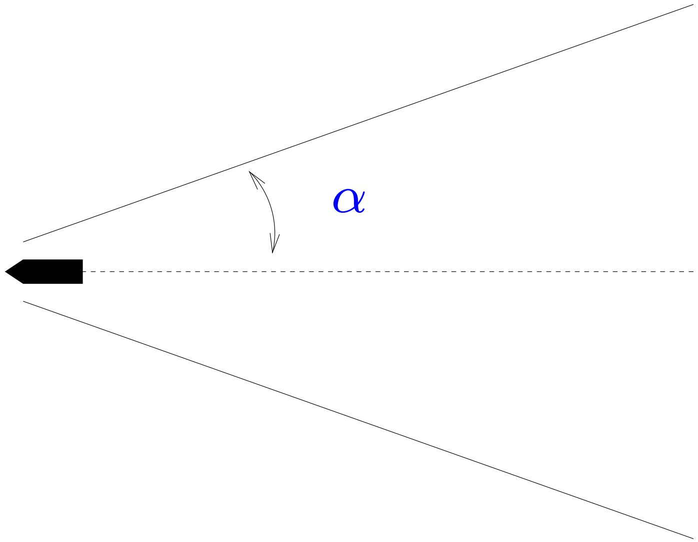


Manneken Pis (Brussels).

$$\begin{cases} x = v_0 t \\ y = -\dfrac{g}{2}t^2 \end{cases} \Rightarrow$$

$$\boxed{y = -\frac{g}{2}\left(\frac{x}{v_0}\right)^2}$$

Kelvin's Theorem (1824–1907)



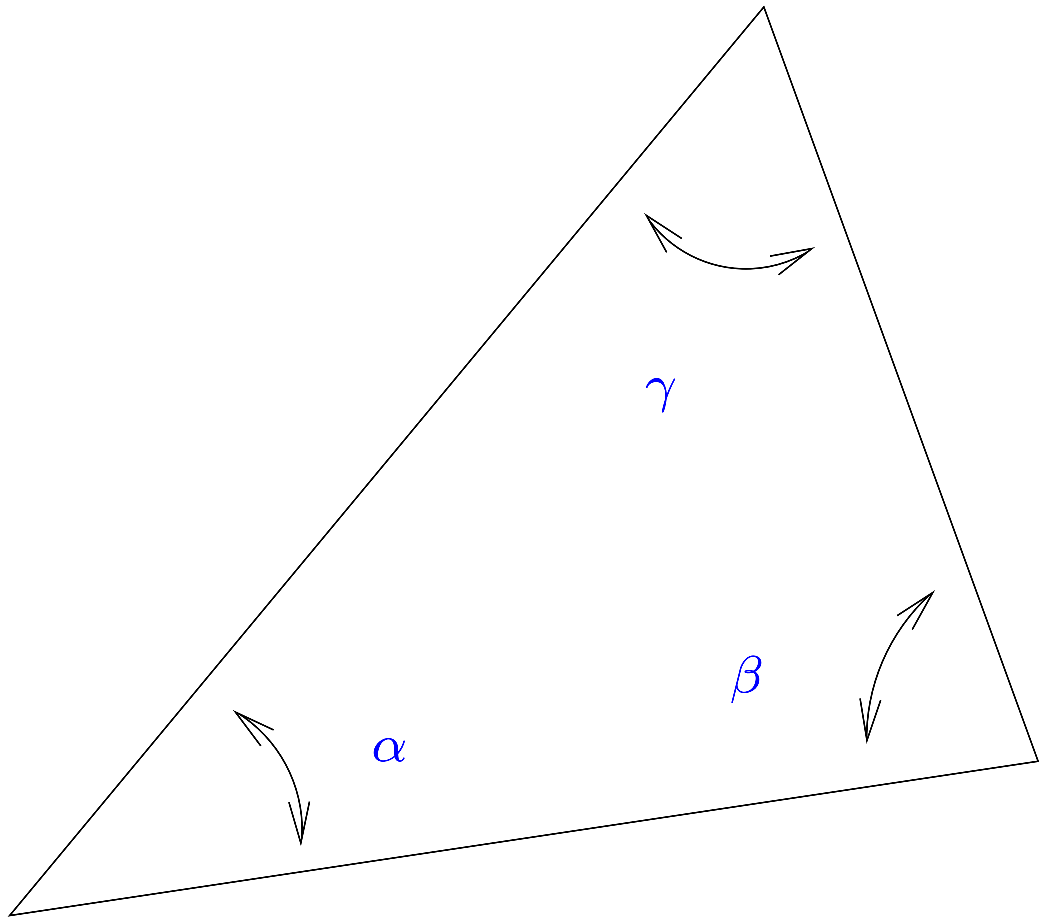
$\sin\alpha = \frac{1}{3}$ (so $\alpha \simeq 19.5°$).

$$\zeta(x,y) = -\frac{\epsilon F^2 \operatorname{sgn}(x)}{\pi^2}\int_0^{\frac{\pi}{2}} \cos\theta \int_0^{+\infty} \frac{k e^{-k|x|}\cos(ky\sin\theta) g(k,\theta)}{F^4k^2+\cos^2\theta} dk d\theta$$
$$+ \frac{\epsilon H(x)}{\pi}\int_{-\infty}^{+\infty} \xi e^{-F^2\xi^2}\cos(x\xi)\cos(y\xi\lambda) d\lambda$$

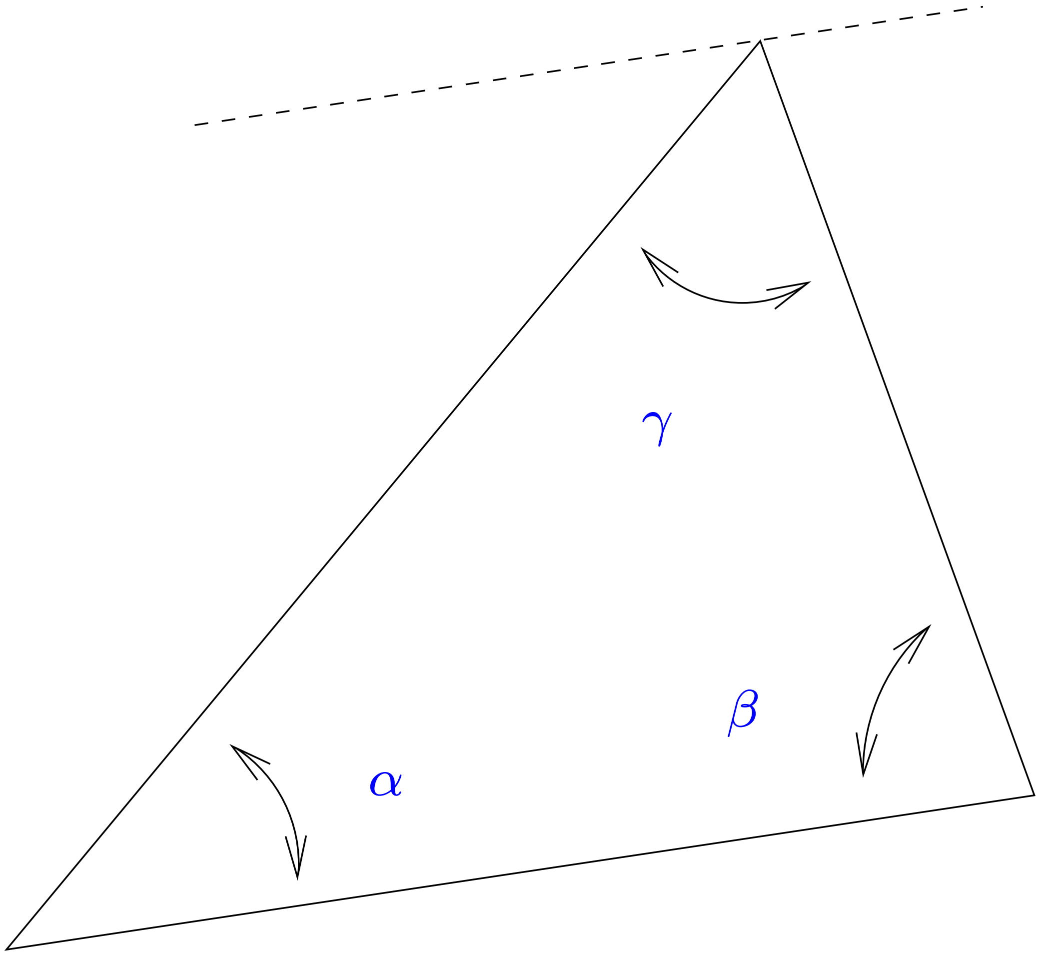
$$g(k,\theta) = F^2 k\sin(k\cos\theta) + \cos\theta\cos(k\cos\theta)$$
$$\xi(\lambda) = \frac{1}{F^2}\sqrt{\lambda^2+1}$$

Euclidean geometry (4^{th} – 3^{rd} century b.C.)

- Two distinct lines intersect at most once.
- There are lines which do not intersect (parallel).
- The internal angles of a triangle add up to 180°.



$$\alpha + \beta + \gamma = 180^\circ$$

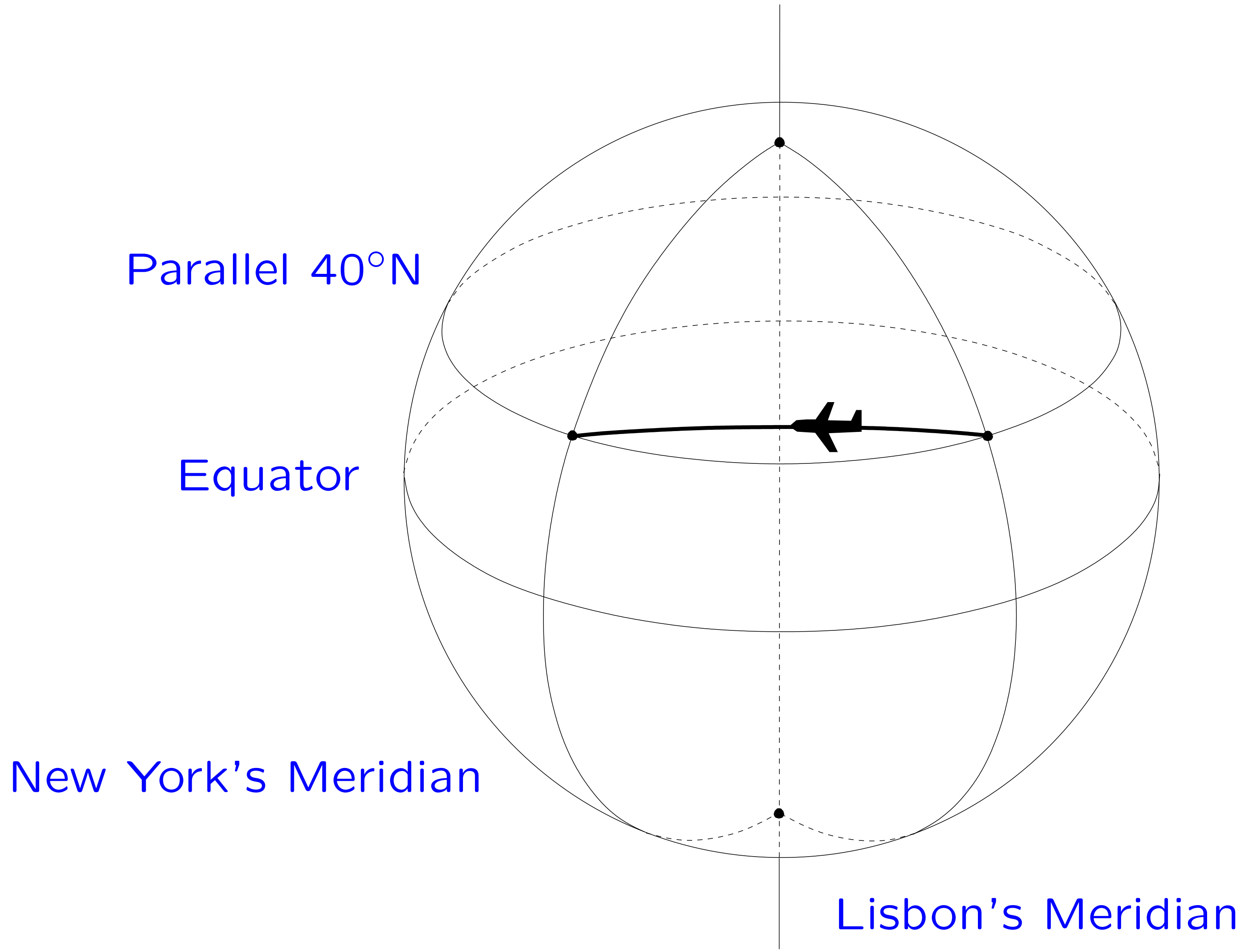


$$\alpha + \beta + \gamma = 180^\circ$$

Riemannian geometry (1826–1866)

• It is the geometry of curved surfaces (spaces). Instead of lines we have geodesics (curves of minimum length).

• In the sphere, for instance, the geodesics are great circles, as the equator or the meridians. That is why to go from Lisbon to New York the plane does not fly directly westwards.


Parallel 40°N
Equator
New York's Meridian
Lisbon's Meridian

Hudson Bay
Labrador Sea
JFK
LIS
United Kingdom
Ireland
Denmark
Germany
France
Italy
Spain
Portugal
North Atlantic Ocean
Morocco
Algeria
Tunisia
Western Sahara
Mauritania
Mali
Niger
Gulf of Mexico
Cuba
Puerto Rico
Guatemala
Caribbean Sea
Map data ©2019 Google, INEGI
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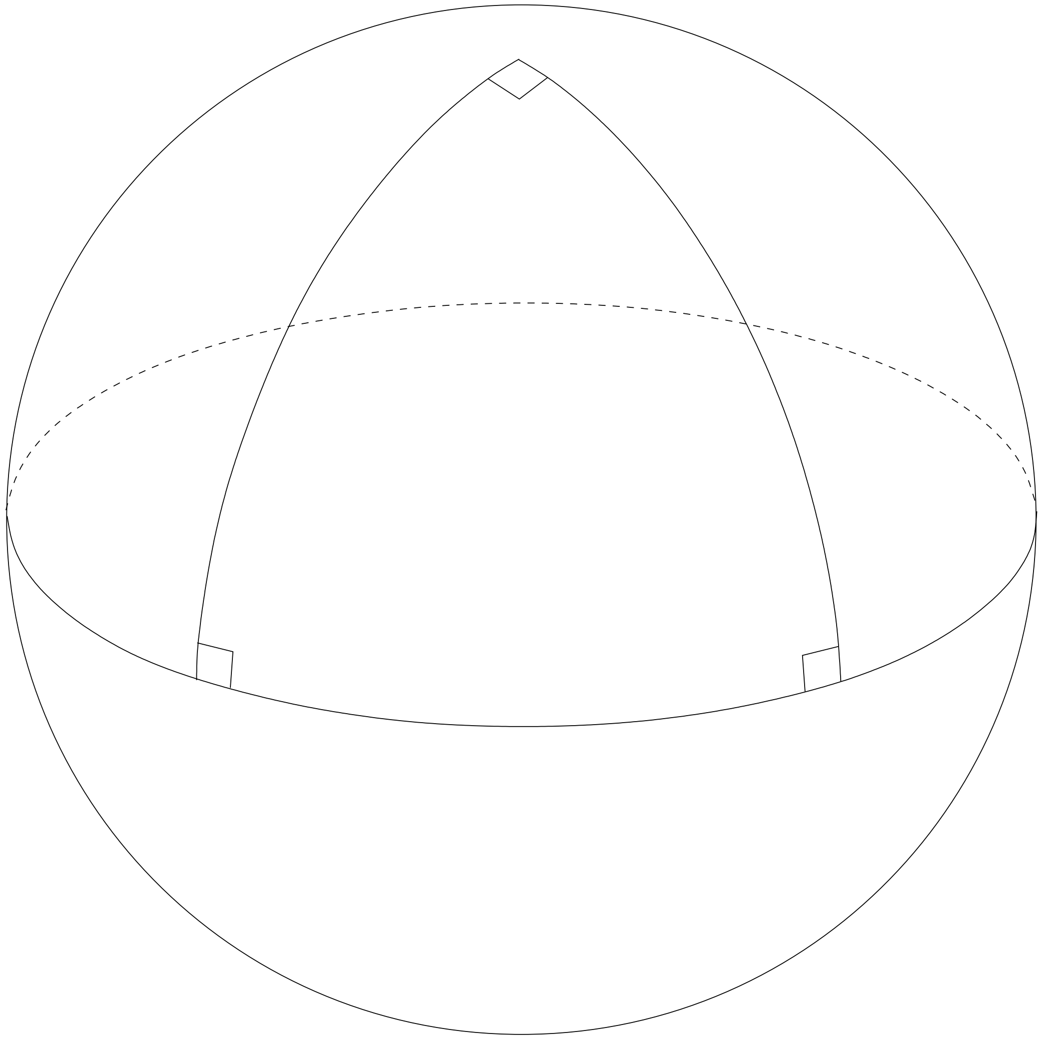
Sphere's geometry

- Two distinct geodesics intersect at exactly two points (there are no parallel geodesics).

- The internal angles of a triangle add up to more than 180°.

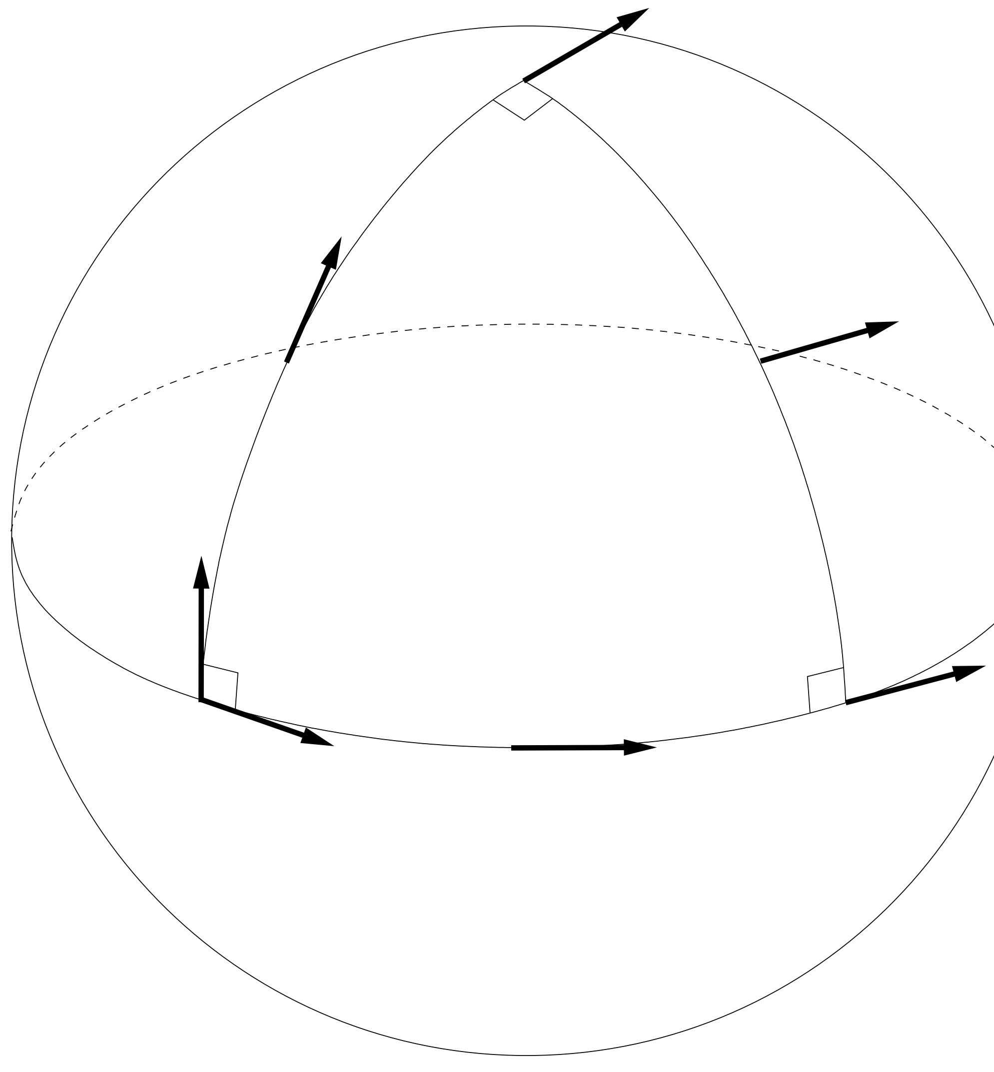


• Average curvature $= \dfrac{\text{Excess angle}}{\text{Triangle's area}} = \dfrac{\frac{\pi}{2}}{\frac{4\pi R^2}{8}} = \dfrac{1}{R^2}$.

• In the sphere all triangles have the same average curvature (constant curvature surface). In general, the curvature of a surface at a point is the limit of the average curvature of increasingly smaller triangles.

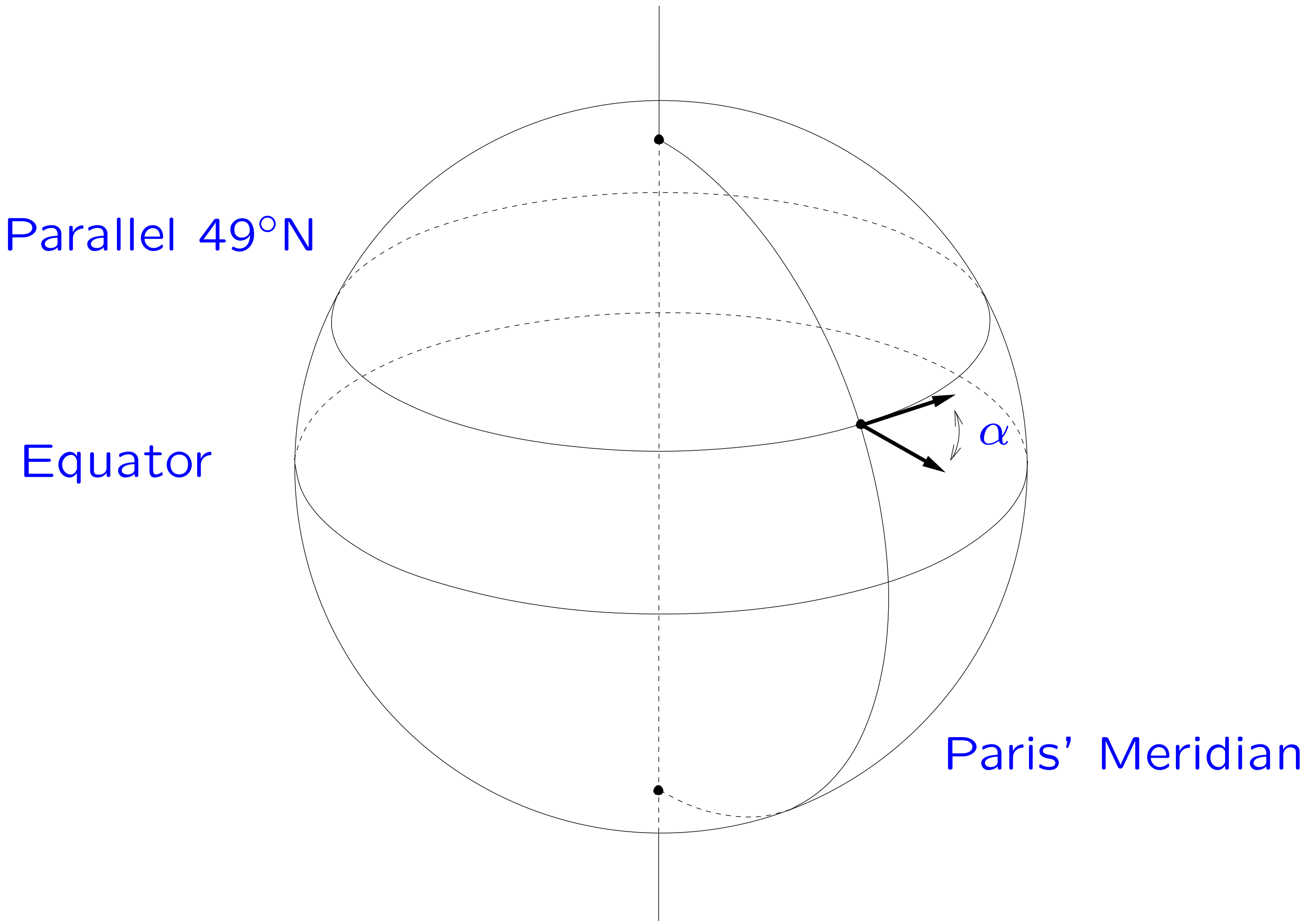
Parallel transport

• A tangent vector which is parallel-transported along a closed curve returns to the initial point rotated by an angle equal to the excess.



- That is what happens with the Foucault pendulum:

Foucault pendulum at the Panthéon in Paris.


Parallel 49°N
Equator
α
Paris' Meridian

- At latitude λ, Foucault's pendulum rotates

$$\alpha = 2\pi - \overbrace{\underbrace{2\pi R(R - R\,\mathsf{sen}\,\lambda)}_{\text{area}}\quad\underbrace{\frac{1}{R^2}}_{\text{curvature}}}^{\text{excess}} = 2\pi\,\mathsf{sen}\,\lambda$$

radians per day.

Mathematical formulation

- Parameterization: $\mathbf{r} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$

- Metric: $g_{ij} = \dfrac{\partial \mathbf{r}}{\partial x^i} \cdot \dfrac{\partial \mathbf{r}}{\partial x^j}, \quad (g^{ij}) = (g_{ij})^{-1}$

- Christoffel symbols: $\Gamma^i_{jk} = \dfrac{1}{2} \sum_l g^{il} \left(\dfrac{\partial g_{lk}}{\partial x^j} + \dfrac{\partial g_{jl}}{\partial x^k} - \dfrac{\partial g_{jk}}{\partial x^l} \right)$

- Vector field: $\mathbf{V} = \sum_i V^i \dfrac{\partial \mathbf{r}}{\partial x^i}$

- Parallel transport: $\dfrac{dV^i}{dt} + \sum_{j,k} \Gamma^i_{jk} \dfrac{dx^j}{dt} V^k = 0$

- Geodesics: $\dfrac{d^2x^i}{dt} + \sum_{j,k} \Gamma^i_{jk} \dfrac{dx^j}{dt}\dfrac{dx^k}{dt} = 0$

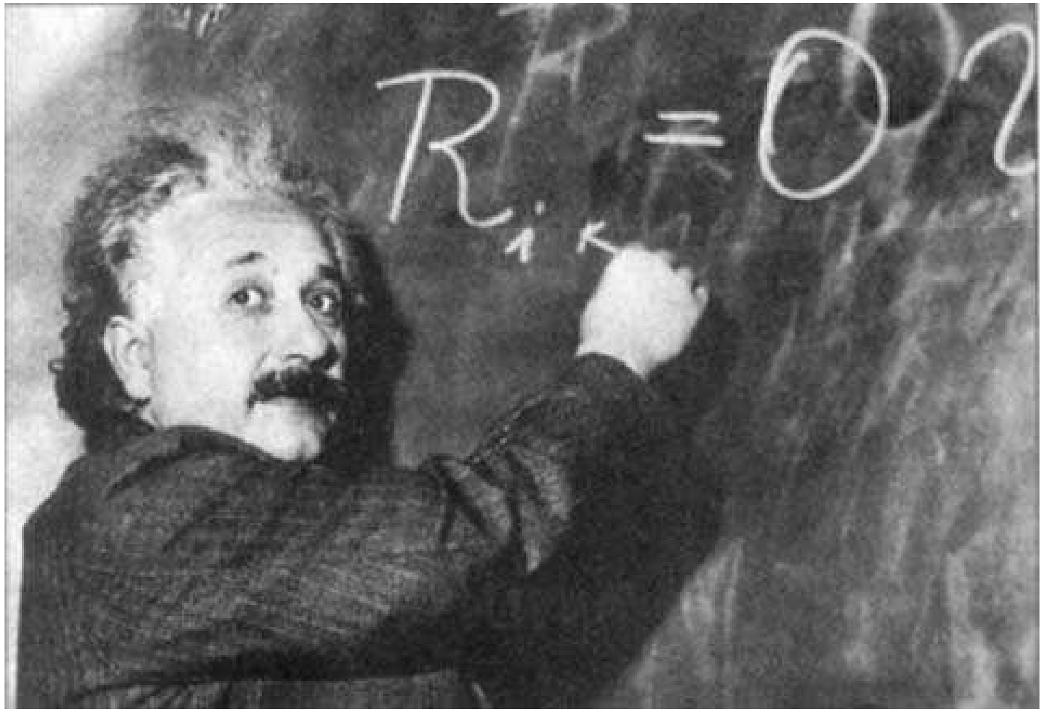
- Riemann tensor: $R_{ij}{}^k{}_l = \dfrac{\partial \Gamma^k_{jl}}{\partial x^i} - \dfrac{\partial \Gamma^k_{il}}{\partial x^j} + \sum_m \left(\Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il} \right)$

- Ricci tensor: $R_{ij} = \sum_k R_{ki}{}^k{}_j$

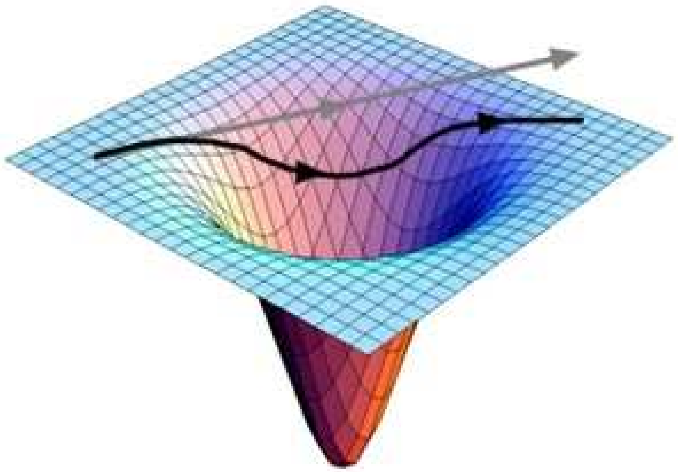
- Scalar curvature: $R = \sum_{ij} g^{ij} R_{ij}$

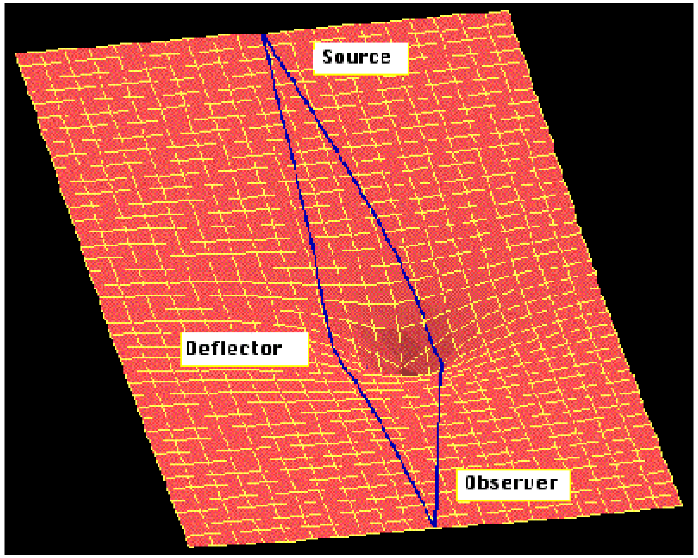
Einstein (1879–1955)

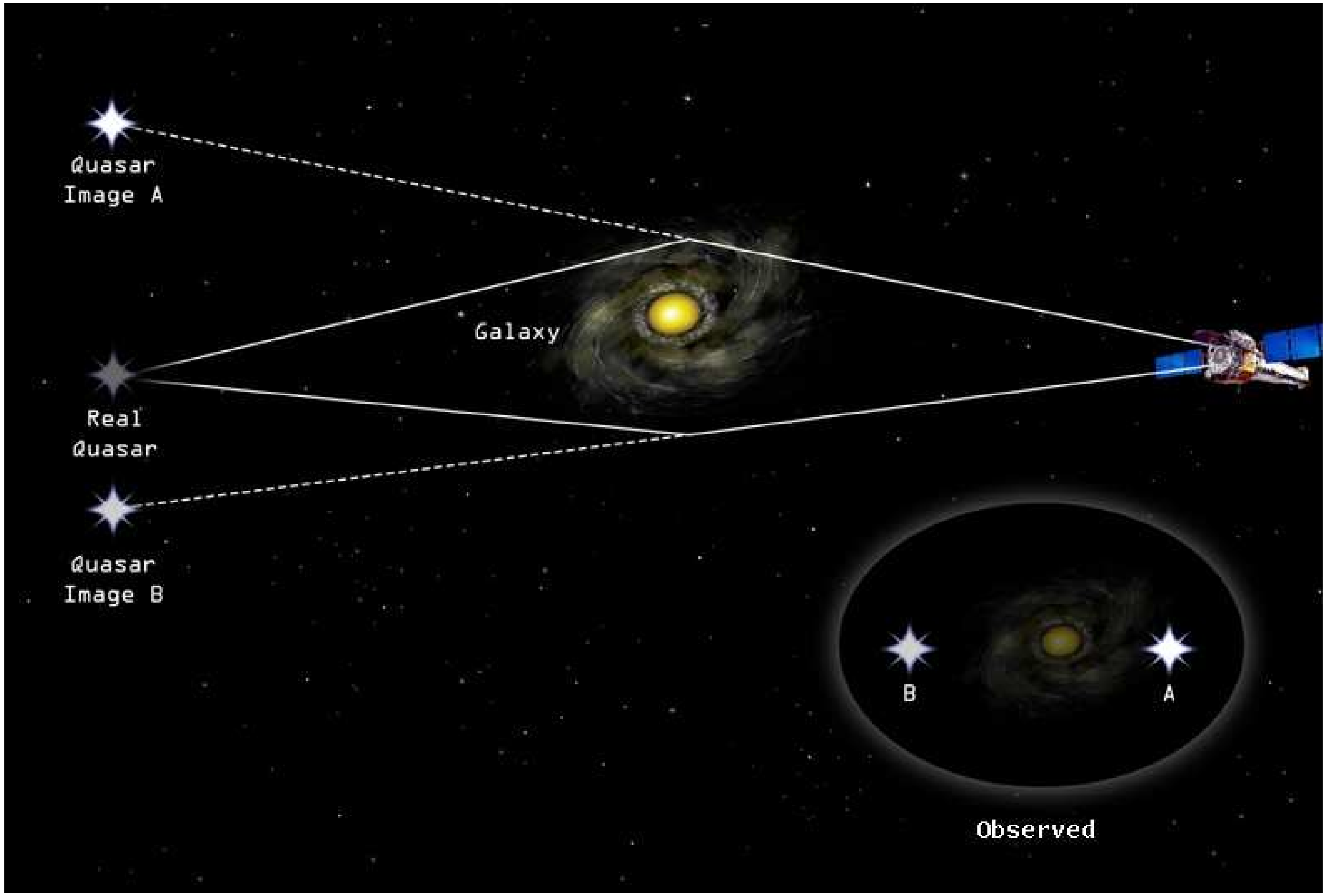
Matter curves space(-time), and light rays follow geodesics.

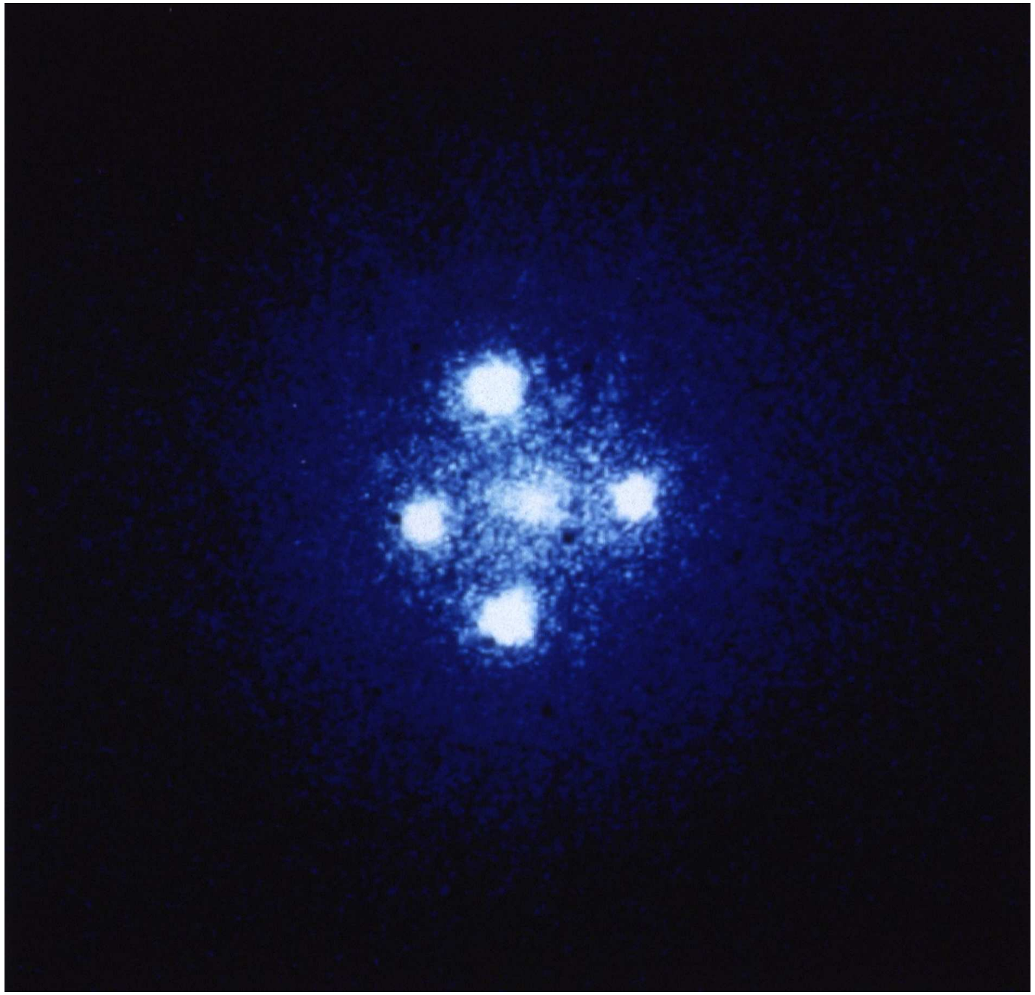


One consequence is the gravitational lens effect, which originates multiple images of astronomical objects.





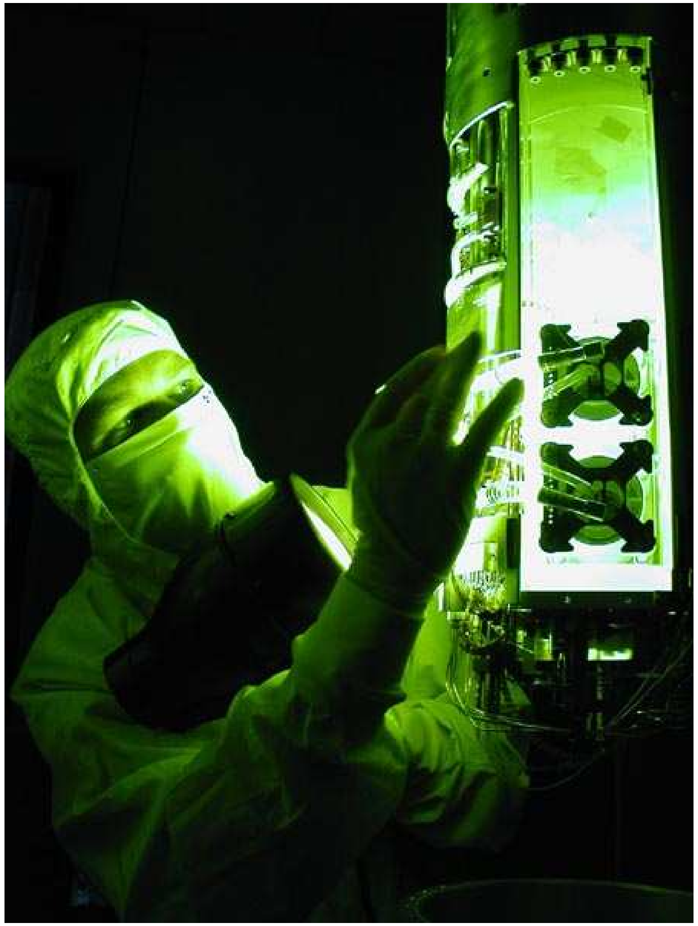

Quasar
Image A
Galaxy
Real
Quasar
Quasar
Image B
B
A
Observed



Einstein Cross.

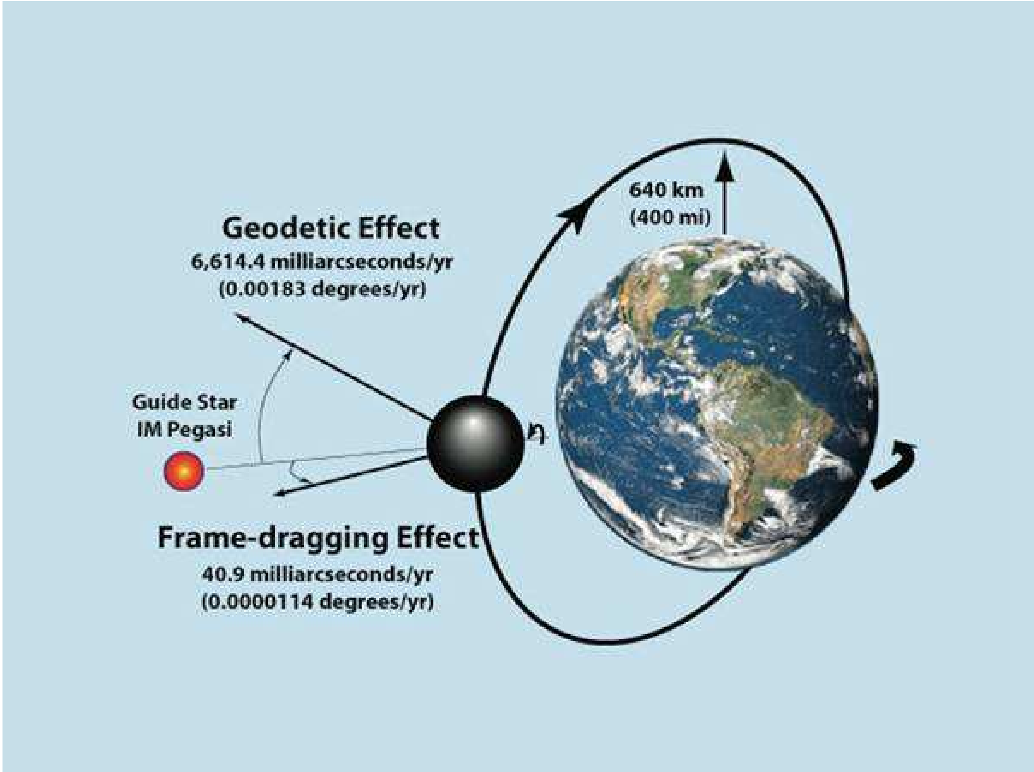
Gravity Probe B (launched in 2004)

The most spherical spheres in the world...

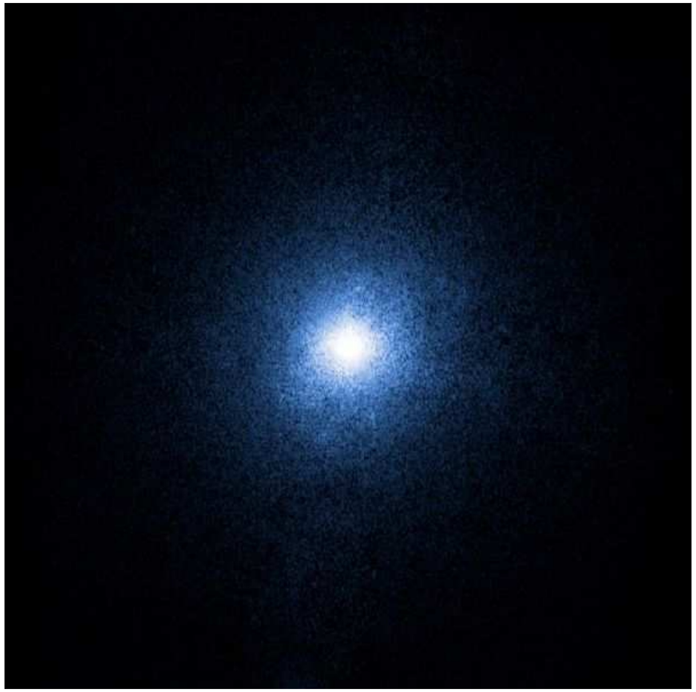


...cooled below -271°C...

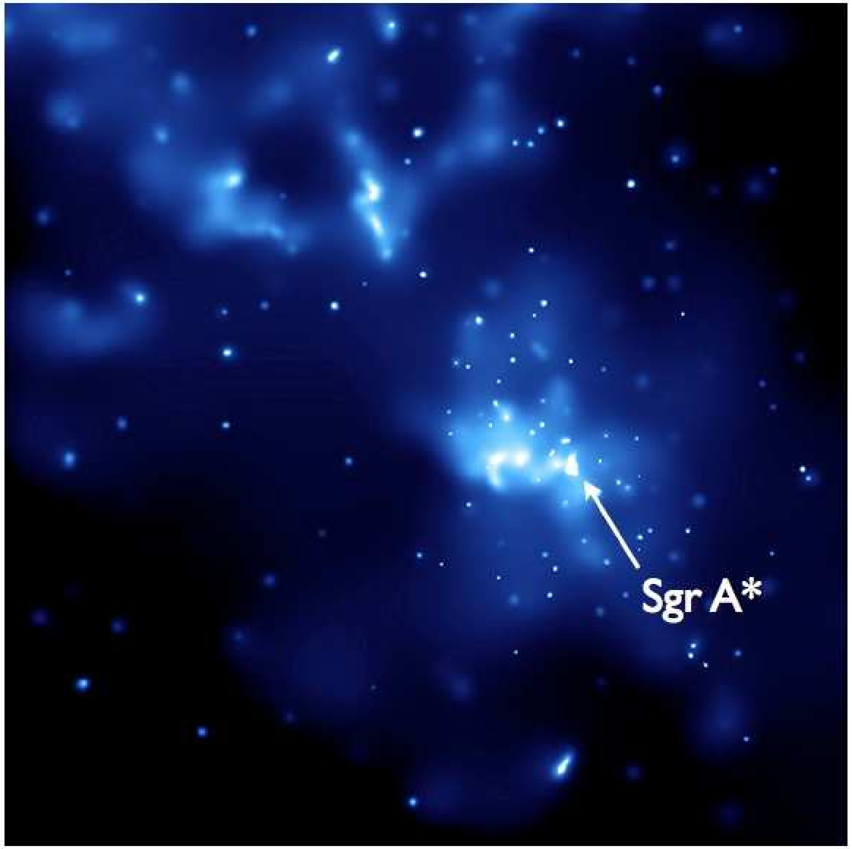
...were spun up and put in orbit.


Geodetic Effect
6,614.4 milliarcseconds/yr
(0.00183 degrees/yr)
640 km
(400 mi)
Guide Star
IM Pegasi
Frame-dragging Effect
40.9 milliarcseconds/yr
(0.0000114 degrees/yr)

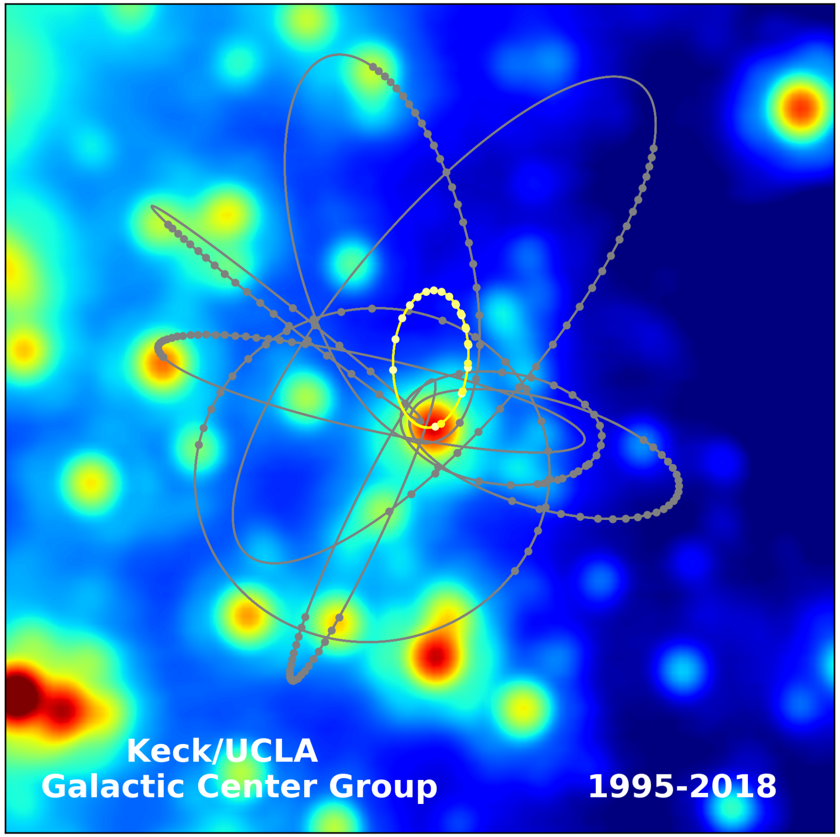
Black holes



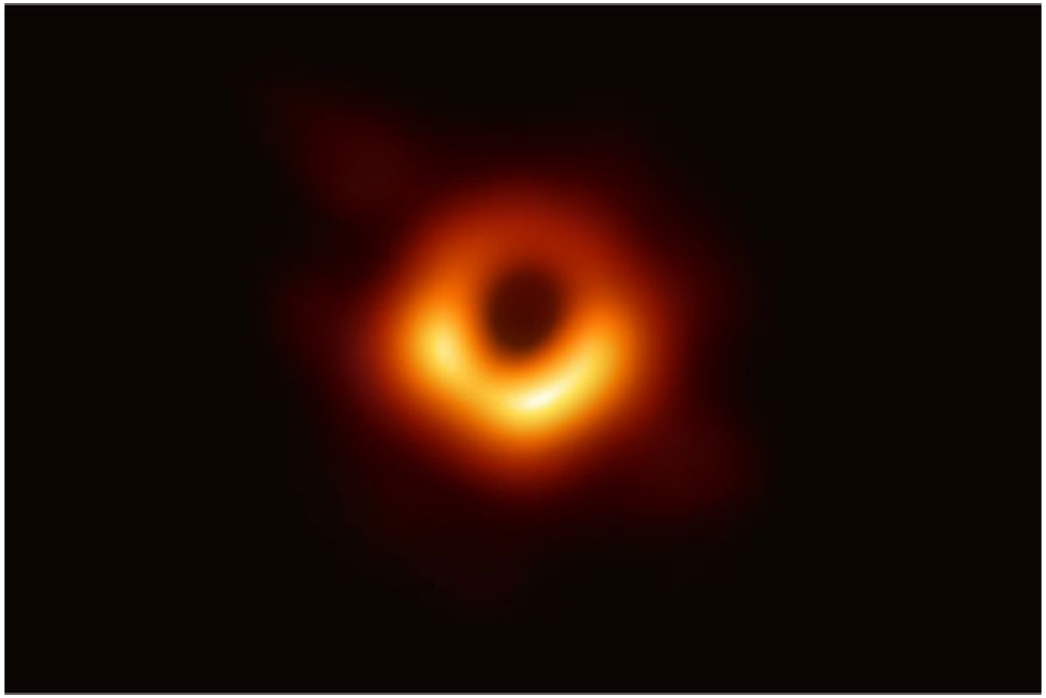
Cygnus X-1 (X-ray image by Chandra).



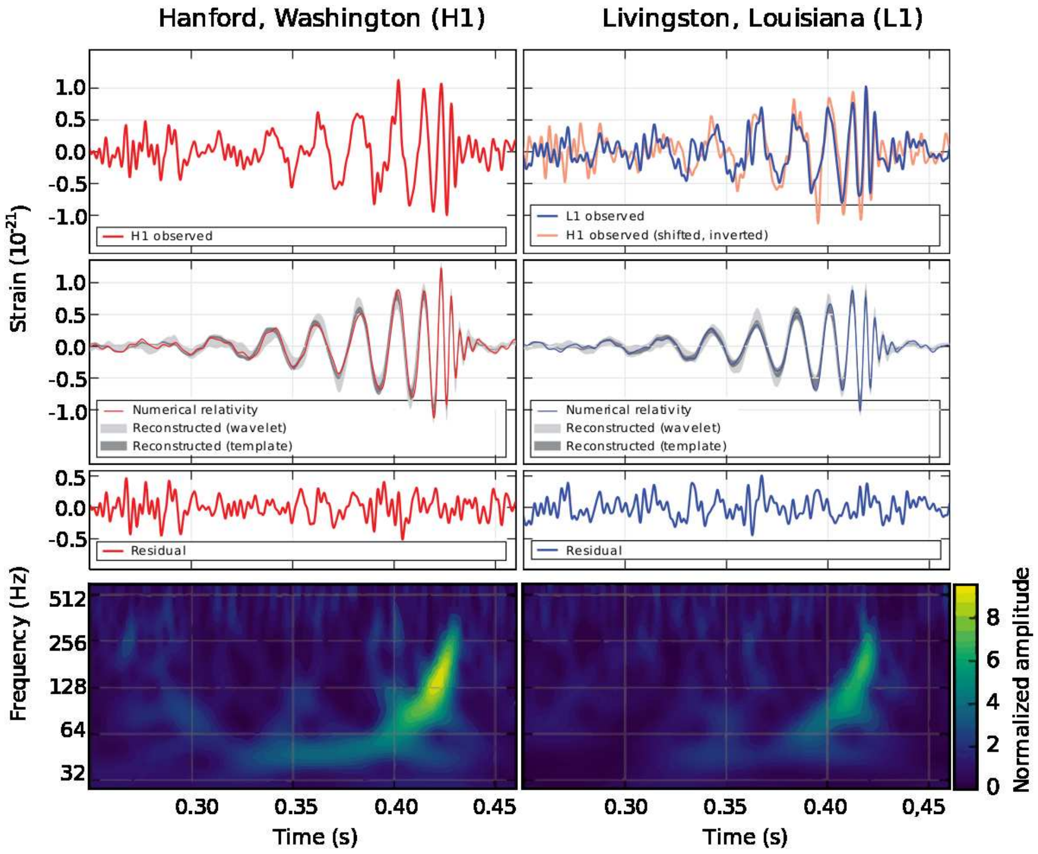
Sagittarius A* (X-ray image by Chandra).



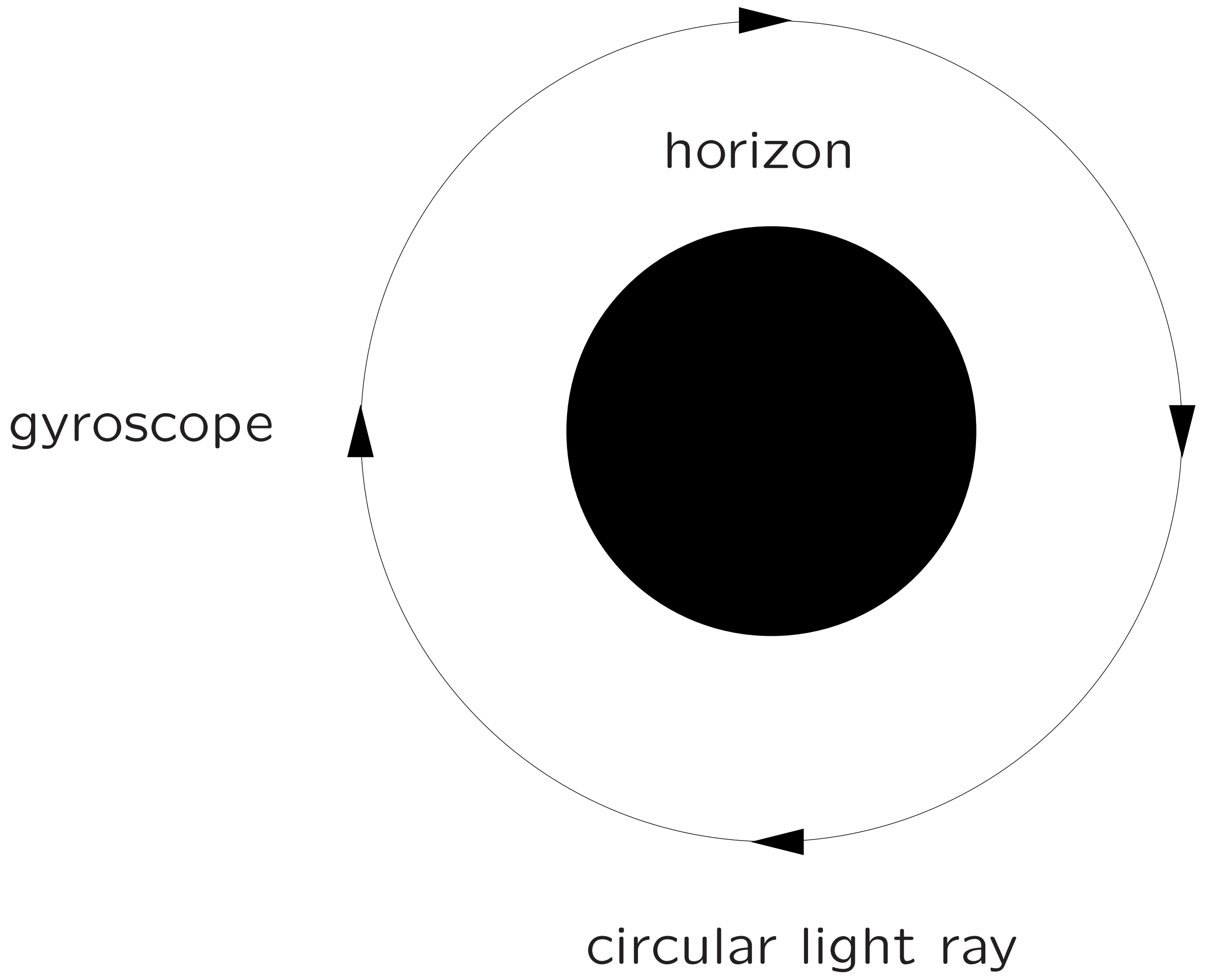
Orbits of stars around Sagittarius A*
(infrared images by the Keck Observatory).

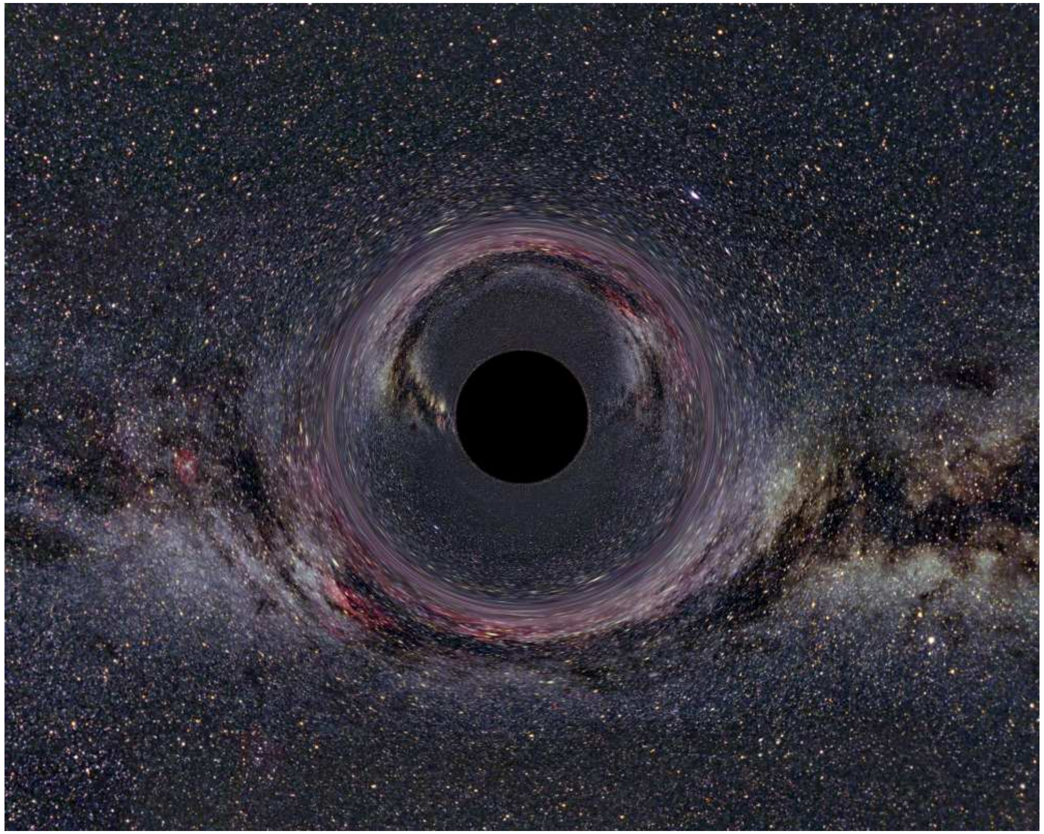


Black hole at the center of the M87 galaxy
(radio interferometry image by the Event Horizon Telescope).

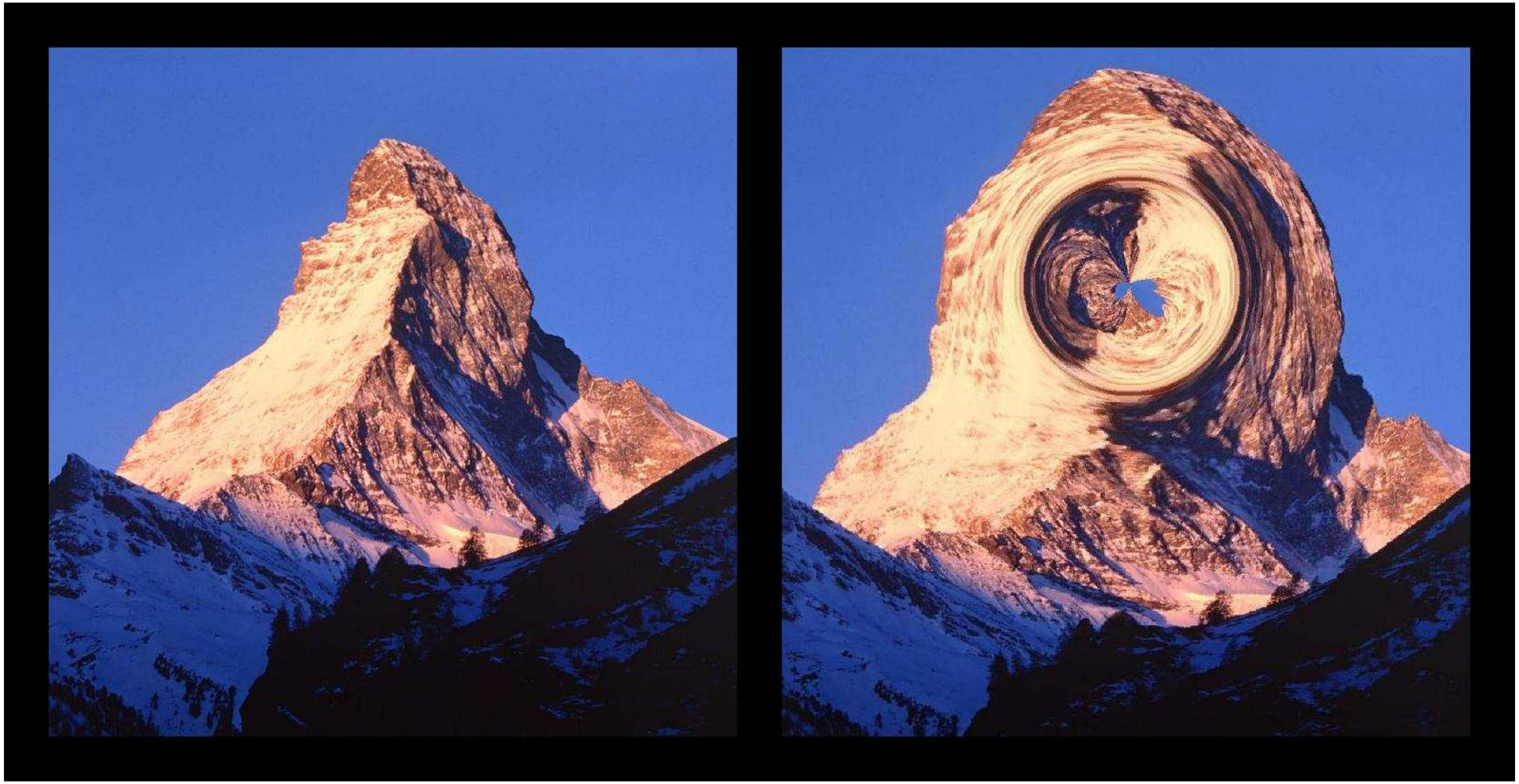


Gravitational wave signal from the merger of two black holes (measured by the Hanford/Livingston LIGO interferometers).


horizon
gyroscope
circular light ray



Black hole 600 kilometers away...



...and 600 meters away.

Gauge theories

• At each point of space(-time) there exists an internal space, i.e. a complex vector space with a Hermitian inner product.

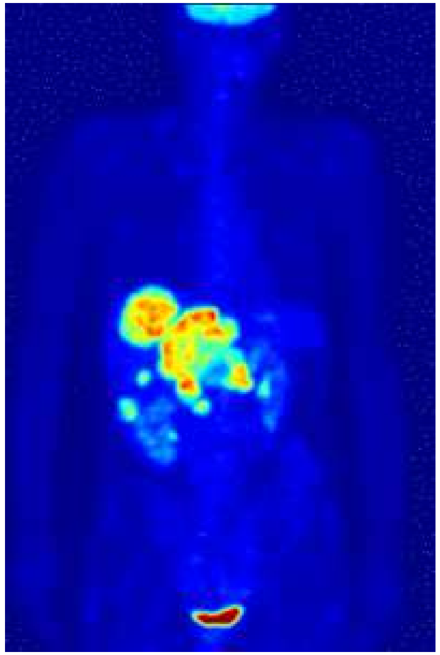
• The fundamental forces arise from the curvature of parallel transport of internal vectors.

• Parallel transport: $\dfrac{d\psi^\alpha}{dt} + \sum_{i,\beta} \Gamma^\alpha_{i\beta} \dfrac{dx^i}{dt} \psi^\beta = 0$

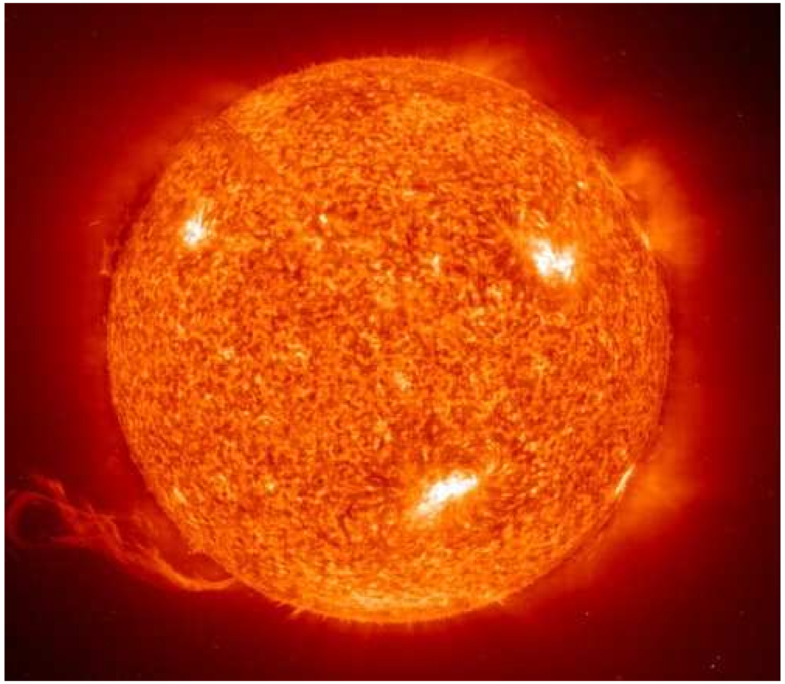
• Curvature: $F_{ij}{}^\beta{}_\alpha = \dfrac{\partial \Gamma^\alpha_{j\beta}}{\partial x^i} - \dfrac{\partial \Gamma^\alpha_{i\beta}}{\partial x^j} + \sum_\gamma \left(\Gamma^\alpha_{i\gamma} \Gamma^\gamma_{j\beta} - \Gamma^\alpha_{j\gamma} \Gamma^\gamma_{i\beta} \right)$

• One dimension: $F_{ij} = \dfrac{\partial \Gamma_j}{\partial x^i} - \dfrac{\partial \Gamma_i}{\partial x^j}$ has 6 independent components

Electromagnetic force – 1-dimensional internal space.



Weak nuclear force – 2-dimensional internal space.



Strong nuclear force – 3-dimensional internal space.

Galileu Galilei (1564–1642)

"Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it."