The SSEP with open boundaries - Hydrodynamic Limit and Matrix Product Ansatz

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Outline

Main Goal

Mathematically ”sound” approach to justify the description of the evolution of thermodynamic characteristics of a fluid.

- System with *large* number of components.
- A precise description of the microscopic state of the system is very intractable.
- Reduce to study the collective behavior of these ”particles”.
- Characterize the equilibrium states of the system by some *small* number of macroscopic quantities (temperature, pressure, density . . .)
- Drive the system out of equilibrium while maintaining a local (and ”mesoscopic”) equilibrium.

Figure 1: Ludwig Boltzmann

Modern Approach

- Simplification by considering **stochastic** underlying microscopic dynamics.
- Pioneering work [1]
- Deep "phenomanological" interest, explaining macrospocic behavior via stochastic particle systems:
    - Interacting random walkers on some lattice under some local interaction.
- Ecompasses mass-transport, polarization, coalescence . . . via systems evolving under a Markovian law.

Figure 2: Frank Spitzer

Some more examples

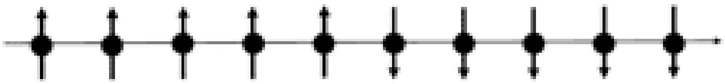


Figure 3: Spin Chain

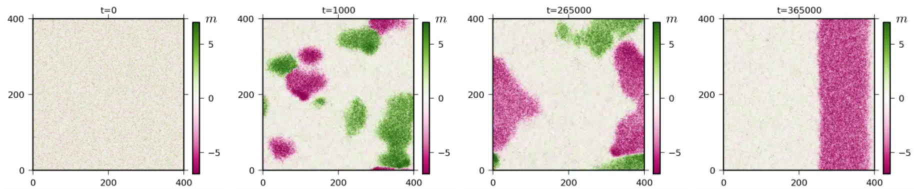
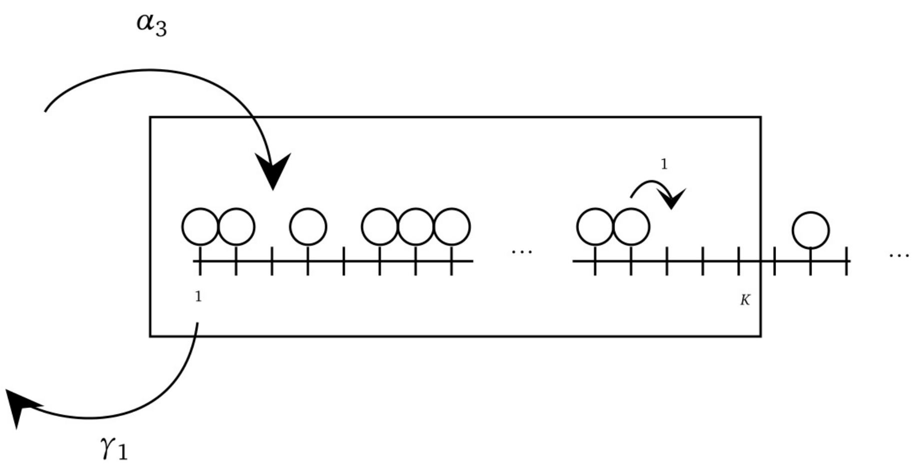


Figure 4: From Flocking with discrete symmetry, the 2d active Ising model, Solon & Tailleur 2015 [13, 13]

Dynamics



- Lattice: $\Lambda_N = \{1, \ldots, N-1\}$;
  - Site: $x \in \Lambda_N$;
  - Bond: $\{x, y\}$ with $x, y \in \Lambda_N$;
- Process: $\eta = (\eta(1), \ldots, \eta(N-1))$;
- State Space: $\Omega_N = \{0, 1\}^{\Lambda_N}$;
- "Boundary": $I_- = \{1, \ldots, K\}$ and $I_+ = \{N-1-K, \ldots, N-1\}$ for $K \geq 1$.

Main result

Theorem (Hydrodynamic Limit)

- ***Assumption:*** *The finite sequences $\alpha, \gamma, \beta, \delta$ are non-increasing.*
- *The **macroscopic density** of particles, $\rho_t(u)$, is solution **in a weak sense** of the **Heat Equation**, $\partial_t \rho_t = \Delta \rho_t$, with boundary conditions*

$\theta = 1$: "Nonlinear" Robin

$$\begin{cases} \partial_u \rho_t(0) + D_{\alpha,\gamma}\rho_t(0) = 0, \\ \partial_u \rho_t(1) - D_{\beta,\delta}\rho_t(1) = 0, \end{cases}$$

$\theta > 1$: Neumann

$$\begin{cases} \partial_u \rho_t(0) = 0, \\ \partial_u \rho_t(1) = 0, \end{cases}$$

where for $\lambda = (\lambda_1, \ldots, \lambda_K), \sigma = (\sigma_1, \ldots, \sigma_K)$ and $f : [0,1] \to \mathbb{R}$,

$$D_{\lambda,\sigma} f := \sum_{x=1}^{K} \{\lambda_x (1-f) f^{x-1} - \sigma_x f (1-f)^{x-1}\}.$$

On the Msc. Thesis

Motivation

Dynamics introduced in *Truncated correlations in the stirring process with births and deaths* [4]

- Following works: [5] (LLN for empirical density and current), [6] (LLN w.r.t. the stationary measure)

Goals

- Study [4, 5] ✓
- Show the LLN without resorting to the machinery on [4]
  - Later included Fick's Law and Hydrostatic Limit [8] (joint work with Clément Erignoux, "INRIA Lille Nord")
- Study the applicability of the Matrix Product Ansatz (and possible extension)
  - Ongoing project (joint work with Gunter Schutz, "Institute of Biological Information Processing" and "Institute for Advanced Simulation")

Markov Chains

Stochastic Process

Sequence of r.v.'s, $\{X_i\}_{i \in J}$, defined on a common probability space $\{\Omega, \mathcal{F}, P\}$.

- For each $\omega \in \Omega$ let $X_i(\omega) \in S$ where (S, Σ) is some measurable space.

Remark

- $X_i(\cdot) : \Omega \to S$ and $X_\cdot(\omega) : J \to S$ are random variables.
- We can define the law of the ***stochastic process*** as $\mu := P \circ X^{-1}$.

(Time Homogenous) Markov Chain

Satisfies the Markov property:

$$P(X_{n+1} = x \mid X_n = x_n, \dots X_0 = x_0) = P(X_{n+1} = x \mid X_n = x_n) := p^{(1)}_{x_n, x}$$

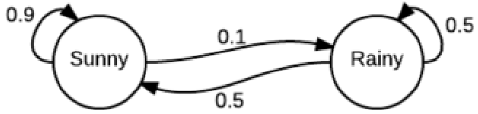


Figure 5: Markov Chain (Straight outta Wikipedia)

From transitions to rates

From discrete to continuous time

- We can go from discrete to continuous time by considering a DTMC as the ***skeleton*** of a CTMC and appropriatedly interpolating on (random) times.
- This leads to an important "linearization" for the transition probabilities:

$$P(X_{t+\epsilon} = j \mid X_t = i) = \delta_{ij} + q_{ij}\epsilon + o(\epsilon), \qquad \epsilon \searrow 0$$

and we find that

$$\frac{d}{dt}p_t(\eta) = \sum_{\eta' \in S} (c(\eta', \eta)p_t(\eta') - p_t(\eta)c(\eta, \eta')) \rightsquigarrow \frac{d}{dt}P(t) = P(t)Q, \quad [P(t), Q] = 0.$$

- Another level of abstraction shows that the above is part of something larger

$$\frac{d}{dt}S_t f = \mathcal{L}S_t f = S_t \mathcal{L} f$$

$$S_t f(\eta_0) = \mathbb{E}(f(\eta_t) \mid \eta_0), \qquad \mathcal{L}f(\eta) = \sum_{\xi \in S} c(\eta, \xi)(f(\eta') - f(\eta)).$$

Generator description

Definition (Generator)

$$\mathcal{L}_N = \mathcal{L}_{N,0} + \frac{1}{N^\theta}\mathcal{L}_{N,b} \rightsquigarrow \mathcal{L} = N^2\mathcal{L}_N,$$

where

$$(\mathcal{L}_{N,0}f)(\eta) = \sum_{x=1}^{N-2}[\eta(x)(1-\eta(x+1)) + (1-\eta(x))\eta(x+1)]\{f(\eta^{x,x+1}) - f(\eta)\},$$

$$(\mathcal{L}_{N,b}f)(\eta) = (\mathcal{L}_{N,-}f)(\eta) + (\mathcal{L}_{N,+}f)(\eta),$$

$$(\mathcal{L}_{N,\pm}f)(\eta) = \sum_{I_\pm} c_x^\pm(\eta)\{f(\eta^x) - f(\eta)\}$$

Rates

$$c_x^-(\eta) = \boldsymbol{\alpha}_x(\eta)(1-\eta(x)) + \boldsymbol{\gamma}_x(1-\eta)\eta(x)$$

$$c_x^+(\eta) = (1-\eta(x))\boldsymbol{\beta}_x(\eta) + \eta(x)\boldsymbol{\delta}_x(1-\eta).$$

Empirical Measure and Time-scalling

Definition (Empirical measure)

$$\pi^N(\eta, du) = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta(x) \delta_{\frac{x}{N}}(du)$$

where we have a natural mapping

$$(D([0,T], \{0,1\}^{\Lambda_N}), \mathbb{P}_\mu^N) \to (D([0,T], \mathcal{M}), \mathbb{Q}^N := \mathbb{P}_{\mu^N} \circ \pi^{-1})$$
$$\{\eta_t\}_{t \in [0,T]} \mapsto \{\pi_t^N\}_{t \in [0,T]}$$

Dynkin's Martingale

Let $\{X_t\}_{t \geq 0}$ be a Markov process with generator $\mathcal{L}$ and countable state space S, and $f : \mathbb{R}^+ \times S \longrightarrow \mathbb{R}$ bounded *with some regularity assumptions*. For all $t \geq 0$ let

$$M_t(f) := f(t, X_t) - f(0, X_0) - \int_0^t (\partial_s + \mathcal{L}) f(s, X_s) ds.$$

Then $\{M_t(f)\}_{t \geq 0}$ is a martingale w.r.t. the natural filtration of $\{X_t\}_{t \geq 0}$.

Identifying the Hydrodynamic Equation

Computing this martingale for $f(t, X_t) \equiv \langle \pi_t^N, G_t \rangle$ we see that

$$M_t^N = \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, (\partial_s + \Delta_N) G_s \rangle ds$$
$$- \int_0^t \left[\nabla_N^+ G_s(0) \eta_{N^2 s}(1) - \nabla_N^- G_s(1) \eta_{N^2 s}(N-1) \right] ds$$
$$- \frac{N^2}{N^\theta} \int_0^t \left[\langle \pi^N (D_{\alpha,\gamma}^{N,-} \eta_{N^2 s}, \cdot), G_s \rangle + \langle \pi^N (D_{\beta,\delta}^{N,+} \eta_{N^2 s}, \cdot), G_s \rangle \right] ds$$

Weak Formulation

$$0 = \langle \rho_t, G_t \rangle - \langle f_0, G_0 \rangle - \int_0^t \langle \rho_s, \left(\partial_u^2 + \partial_s \right) G_s \rangle ds$$
$$+ \int_0^t \left\{ \rho_s(1) \partial_u G_s(1) - \rho_s(0) \partial_u G_s(0) \right\} ds$$
$$- 1_{\theta=1} \left(\int_0^t G_s(1)(D_{\beta,\delta} \rho_s)(1) ds + \int_0^t G_s(0)(D_{\alpha,\gamma} \rho_s)(0) ds \right) =: F_\theta(\rho, G, t).$$

Results

Definition (Associated profile)

A sequence of probability measures $\{\mu_N\}_{N\geq 1}$ on Ω_N is associated with a profile $\rho_0 : [0,1] \to [0,1]$ if for any continuous function $G : [0,1] \to \mathbb{R}$ and every $\delta > 0$,

$$\lim_{N\to\infty} \mu_N \left(\eta \in \Omega_N : \left| \langle \pi^N, G \rangle - \langle \rho_0, G \rangle \right| > \delta \right) = 0$$

Theorem (Hydrodynamic Limit)

Let $f_0 : [0,1] \to [0,1]$ be a measurable function and $\{\mu_N\}_{N\geq 1}$ a sequence of probability measures in Ω_N associated with f_0 in the sense above. Then, for any $t \in [0,T]$ and every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N} \left(\left| \langle \pi_t^N, G \rangle - \langle \rho_t, G \rangle \right| > \delta \right) = 0,$$

where ρ is the unique weak solution for the heat equation with boundary conditions and formulation as previously.

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Entropy Method

Strategy

- Convergence in subsequences:
  - Prokhorov's Theorem + Aldous' criterion for tightness;
- Characterization of the Limit points:
  - Absolute continuity: $\pi_t(du) = \rho_t(u)du$;
  - Existence of solutions via microscopic system;
  - Replacement Lemmas (mean field estimates to control correlation terms) [2, ?];
- Uniqueness of the Limit (PDE's problem):
  - Choice of test function (backwards heat equation) [3]