Resurgence, exact WKB and quantum geometry

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Resurgence in Gauge and String Theories, Lisboa, 2016

based on:
1501.05671 with G.Dunne, 16xx.xxxx with G.Dunne, M. Ünsal

related: Monte-Carlo dynamics, Lefschetz thimbles and the sign problem
1510.03258, 1512.08764, 1604.00956, 1605.08040, 1606.02742
with A. Alexandru, P. Bedaque, G. Ridgway, N. Warrington

A Primer on Resurgent Transseries and Their Asymptotics

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coming soon...

Many expansions in physics are asymptotic:

$$f(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n \quad , \quad c_n \sim n!$$

some examples: *(beware! highly incomplete list)*

- quartic/cubic oscillator, Mathieu, Zeeman, Stark, ...
- Euler-Heisenberg, QFT in dS/AdS background, large N, ...
- genus expansion in string theory ($c_g \sim (2g)!$) [Shenker]
- hydrodynamics [Heller,Spalinski; GB, Dunne; Aniceto, Spalinski]

Resurgence

$$f(\hbar) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} \underbrace{c_{n,k,l}\,\hbar^n}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{\hbar}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{\hbar}\right]\right)^l}_{\text{quasi-zero-modes}}$$

a characteristic feature of resurgence is there are stringently relations between $c_{n,k,l}$s where large order growth of perturbative series is related to low order coefficients of fluctuations around instantons and so on...

Punchline of this talk:

"Beyond resurgence"

[Dunne Ünsal; GB, Dunne]

For certain Schrödinger equations (relevant for SUSY QFTs) in addition to the large order - low order relations between perturbative and non-perturbative expansions, there is a surprising low order - low order relation between them.
It can be understood in terms of the geometry of the spectral curve.

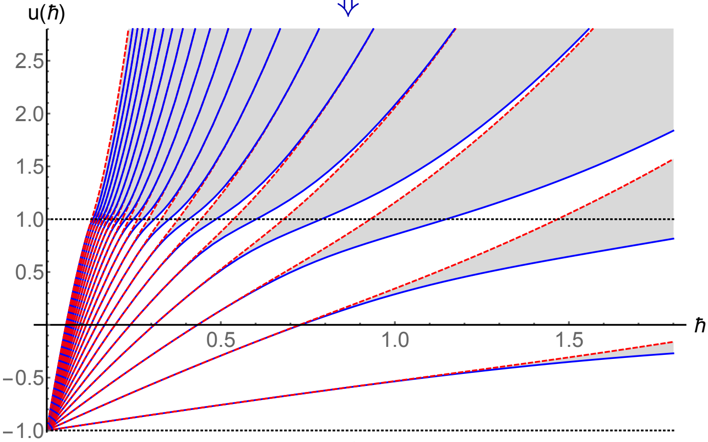
Mathieu equation [GB, Dunne; 1501.05671]

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dz^2} + cos(z)\,\psi = u(N,\hbar)\,\psi$$

- NS limit of the $\mathcal{N}=2$, $SU(2)$ theory, $u \Leftrightarrow \mathrm{tr}\langle\Phi^2\rangle$, moduli space coord. [see talks by Hatsuda, Kashani-Poor, Russo]
- Wilson loops in $\mathcal{N}=4$ (via AdS/CFT and Pohlmeyer Reduction) [Kruczenski et. al]
- ...
- more generally, ODE $\Leftrightarrow$ 2D integrable models [Dorey, Tateo; Voros; Bazhanov, Fateev, Lukyanov, Zamolodchikov; ...]

Strong coupling expansion: $N\hbar := \lambda \gg 1$

$\Downarrow$



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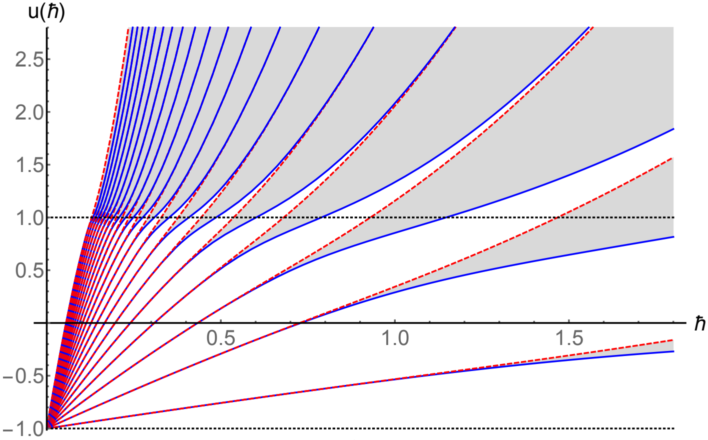
Weak coupling expansion: $\lambda \ll 1$



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$\textit{Weak coupling expansion: } \lambda \ll 1$

Trans-series

near $u \sim -1$, tightly bound states, tunneling exponentially suppressed

$$
\begin{aligned}
u(N,\hbar) \quad &\sim \quad -1 + \hbar\left[N+\frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right] \\
&\quad -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3 + \frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots \\
&\quad + \underbrace{e^{-\frac{S_{inst}}{\hbar}} \sum_n \hbar^n f_n(N)\, \cos\theta}_{1-instanton} + \underbrace{e^{-\frac{2S_{inst}}{\hbar}} \sum_n \hbar^n g_n(N,\theta)}_{2-instanton} \\
&\quad + \dots
\end{aligned}
$$

trans-monomials:

$\hbar^n$ (perturbative fluctuations), $e^{\frac{-k\,S_{inst}}{\hbar}}$ (multi instantons), $\log(-1/\hbar)^l$ (quasi zero modes)

Resurgence relations

large order growth of perturbative series:

$$c_n(N=0) \sim \frac{n!}{2S_{\mathcal{I}}^n}\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

instanton anti-instanton fluctuations: (leading ambiguity)

$$\mathcal{I}m\, u(0,\hbar) \quad \sim \quad \pi\, e^{-2S_{inst}/\hbar}\left(1 - \frac{5}{2}\cdot\left(\frac{\hbar}{16}\right)^2 - \frac{13}{8}\cdot\left(\frac{\hbar}{16}\right)^4 - \dots\right)$$

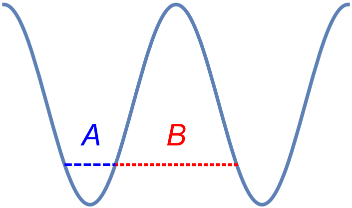
Beyond resurgence

- In addition to the large order - low order relations between perturbative and non-perturbative expansions, there is a surprising low order - low order relation between them!
- allows one to *fully construct* the non-perturbative fluctuations from perturbative data.
- valid everywhere in the spectrum

WKB expansion

$$\psi \sim e^{\frac{i}{\hbar}Q(z,u;\hbar)} \;\Rightarrow\; Q'^2 + i\hbar Q'' - 2(u - V(z)) = 0 \quad \text{(Ricatti eqn.)}$$

$$Q(z) \sim \sum_{n=0}^{\infty} \hbar^n Q_n(z,u) = \int \sqrt{2(u-V)}dz + \sum_{n=1}^{\infty} \hbar^n Q_n(z,u)$$



WKB actions: [Dunham]

$$a(u;\hbar) = \frac{1}{2\pi}\int_A Q' dz \sim \sum_{n=0}^{\infty} a_n(u)\hbar^{2n}$$

$$a^D(u;\hbar) = \frac{1}{2\pi}\int_B Q' dz \sim \sum_{n=0}^{\infty} a_n^D(u)\hbar^{2n}$$

perturbative : $a(u;\hbar) = \frac{\hbar}{2}(N + 1/2) \Rightarrow u_{pt.}(N)$

non-perturbative (tunneling): $\Delta u = \frac{2}{\pi}\frac{\partial u_{pt.}}{\partial N} e^{-\frac{2\pi}{\hbar}\mathcal{I}m[a^D]}$

$a(u)$ and $a^D(u)$ are related order by order in $\hbar$!

WKB expansion

$$\psi \sim e^{\frac{i}{\hbar}Q(z,u;\hbar)} \;\Rightarrow\; Q'^{\,2} + i\hbar Q'' - 2(u - V(z)) = 0 \quad \text{(Ricatti eqn.)}$$

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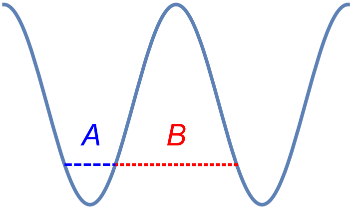

WKB actions: [Dunham]

$$a(u;\hbar) \;=\; \frac{1}{2\pi}\int_A Q'dz \sim \sum_{n=0}^{\infty} a_n(u)\hbar^{2n}$$

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perturbative : $a(u;\hbar) = \frac{\hbar}{2}(N+1/2) \Rightarrow u_{pt.}(N)$

non-perturbative (tunneling): $\Delta u = \frac{2}{\pi}\frac{\partial u_{pt.}}{\partial N} e^{-\frac{2\pi}{\hbar}\mathcal{I}m[a^D]}$

$a(u)$ and $a^D(u)$ are related order by order in $\hbar$! $\Rightarrow P = NP$

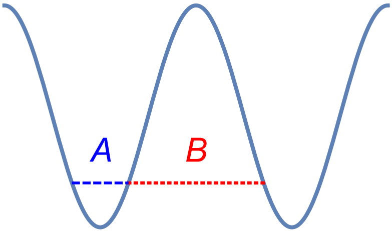
Geometry and WKB

- Set $\hbar = 0$ for now.
- Classically the (complex) phase space can be identified with the moduli space of complex tori.
- $u \Leftrightarrow$ moduli space parameter

$$u = \frac{p^2}{2} + \cos z \quad \Rightarrow \quad x \equiv \cos z, \quad y = \frac{\dot{x}}{\sqrt{2}}$$

$$y^2 = (x^2 - 1)(x - u) \quad \text{genus-1 elliptic curve}$$

Geometry and WKB



WKB actions: integrals of abelian differentials over the two independent cycles of torus

$$a_0(u) = \frac{\sqrt{2}}{2\pi} \int_A \sqrt{u - V(z)}\, dz = \frac{\sqrt{2}}{\pi} \int_A \frac{u - x}{y} dx$$

$$a_0^D(u) = \frac{\sqrt{2}}{2\pi} \int_B \sqrt{u - V(z)}\, dz = \frac{\sqrt{2}}{\pi} \int_B \frac{u - x}{y} dx$$

$\equiv$ Seiberg-Witten differentials for $SU(2)$ [Gorsky, Krichever, Marshakhov, Mironov, Morozov]

Geometry and WKB

a_0 and a_0^D are related via *Riemann bilinear identity*

$$a_0 \frac{da_0^D}{du} - a_0^D \frac{da_0}{du} = \frac{i}{2} \frac{S_{inst}}{T}$$

$T = 2\pi$ =period of the harm. oscll. at the bottom of the well

- a_0, a_0^D: satisfy a Picard-Fuchs equation

$$a_0''(u) - \frac{1}{4(1-u^2)} a_0(u) = 0$$

- Bilinear identity $\Leftrightarrow$ Wronskian
- alternatively: $a_0^D(u) = \tau_0(u)\, a_0(u) - i \frac{S_{inst}}{\omega_0(u)}$

  where $\omega_0 = a_0'$, modular parameter: $\tau_0 = \omega_0^D / \omega_0$

Geometry and WKB: Quantum corrections

$$a(u;\hbar) \sim \sum_{n=0}^{\infty} a_n(u)\hbar^{2n} \quad , \quad a^D(u;\hbar) \sim \sum_{n=0}^{\infty} a_n^D(u)\hbar^{2n}$$

All higher order actions are encoded in the lowest order (classical) action

$$a_n(u) = p_n(u)a_0(u) + q_n(u)a_0'(u)$$

$$a_n^D(u) = p_n(u)a_0^D(u) + q_n(u)a_0^{D\prime}(u)$$

- p_n, q_n: rational functions that can be derived from Schrödinger eqn.

Geometry and WKB: Quantum corrections

"quantum corrections" to the bilinear identity

[GB, Dunne]

$$\left(a - \hbar \frac{\partial a}{\partial \hbar}\right) \frac{\partial a^D}{\partial u} - \left(a^D - \hbar \frac{\partial a^D}{\partial \hbar}\right) \frac{\partial a}{\partial u} = \frac{2i}{\pi}$$

- connects the perturbative expansion to non-perturbative fluctuations order by order
- valid *everywhere* in the spectrum
- SUSY inspired proof via Matone's relation [Gorsky, Milekhin]

quantum corrections to the Picard Fuchs equation:

[GB, Dunne, Ünsal, in prep]

$$a''(u) + F(u)a'(u) + G(u)a(u) = 0$$

$$F(u) := \sum_{n=0}^{\infty} \hbar^n f_n(u) \quad , \quad G(u) := \sum_{n=0}^{\infty} \hbar^n g_n(u)$$

quantum corrections: higher order poles

$$f_0(u) = 0 \quad , \quad g_0 = \frac{1}{8(-1+u)} - \frac{1}{8(1+u)}$$

$$f_1(u) = -\frac{1}{96(u+1)^2} - \frac{1}{96(u-1)^2} \quad , \quad g_1 = \frac{1}{96(u+1)^3} + \frac{1}{384(u+1)^2} + \dots$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

no new singularities!

P = NP

perturbative expansion:

$$u^{pt.}(N,\hbar) \;\sim\; -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N + \frac{1}{2}\right)^2 + \frac{1}{4}\right] + \ldots$$

$$\Updownarrow$$

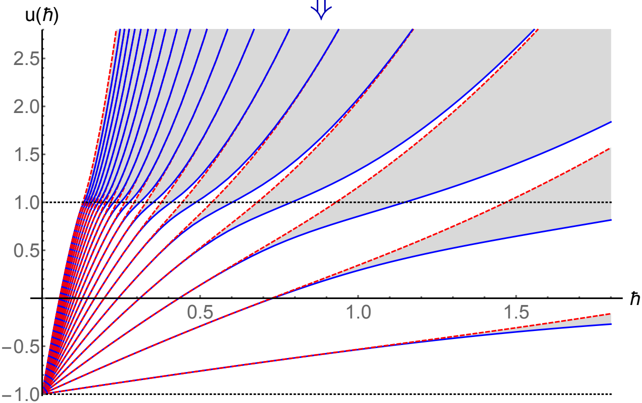
band width (non-perturbative, 1-instanton+fluctuations) :

$$\Delta u_{1\,inst.}(N,\hbar) = \frac{\partial u^{pt.}}{\partial N}\; e^{S_{inst}\int_0^{\hbar}\frac{d\hbar'}{\hbar'^{\,3}}\left(\frac{\partial u^{pt.}(N,\hbar')}{\partial N} - \hbar' + \frac{\hbar'^{\,2}(N+\frac{1}{2})^2}{S_{inst}}\right)}$$

checked up to 3 loops via explicit calculation [Escobar-Ruiz, Shuryak, Turbiner]

Strong coupling expansion: $N\hbar \gg 1$

$\Downarrow$



$$u^{pert.}(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \dots\right)$$

Strong coupling $(\lambda \gg 1)$

gauge theory detour [Alday, Gaiotto, Tachikawa; Marshakov et. al.; ...]

$$Z^{inst.}(a;\epsilon_1,\epsilon_2) = \sum_{n=0}^{\infty} \left(\frac{\Lambda^2}{\epsilon_1\epsilon_2}\right)^{2n} Q_\Delta^{-1}([1^n],[1^n]), \qquad Q_\Delta(Y,Y') = \langle\Delta|L_Y\, L_{-Y'}|\Delta\rangle$$

- from AGT: $\Delta = \frac{1}{\epsilon_1\epsilon_2}\left(a^2 - \frac{(\epsilon_1+\epsilon_2)^2}{4}\right) \quad , \quad c = 1 - \frac{6(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$
- $\epsilon_2 \to 0$ limit, *twisted superpotential:* [Nekrasov, Shatashvili]

$$\mathcal{W}_{NS}^{inst.}(a;\epsilon_1) \equiv -\frac{\epsilon_1}{4\pi i} \lim_{\epsilon_2\to 0} \epsilon_2 \log\left(Z^{inst.}(a,\epsilon_1,\epsilon_2)\right)$$

- identify $\epsilon_1 = \hbar$, $a = N\hbar/2$

$$u = \frac{i\pi}{2}\Lambda\frac{\partial\mathcal{W}_{NS}^{inst.}}{\partial\Lambda} = \frac{\hbar^2}{8}\left(\frac{8\Lambda^4}{(N^2-1)\,\hbar^4} + \frac{8\Lambda^8\left(5N^2+7\right)}{(N^2-4)\,(N^2-1)^3\,\hbar^8} + \dots\right)$$

Strong coupling ($\lambda \gg 1$)

back to QM: level splitting

- $u^{pert}(N, \hbar)$ is not the whole story
- In the limit $N, \lambda \gg 1$ there are exponentially small *gaps* in the spectrum
- $u^{pert}(N, \hbar)$ determines the center of the gap

gap width:

$$\Delta u_N^{\rm gap} \approx \frac{\hbar^2}{4} \frac{1}{\left(2^{N-1}(N-1)!\right)^2} \left(\frac{2}{\hbar}\right)^{2N} \left[1 + \mathcal{O}\left(\hbar^{-4}\right)\right]$$

$$\approx \frac{N\,\hbar^2}{2\pi} \left(\frac{e}{N\,\hbar}\right)^{2N}$$

Strong coupling: complex instantons

Is there a semi-classical interpretation of the exponentially small gaps at strong coupling, similar to the instantons for the case of exponentially small bands at weak coupling?

YES! complex instantons

- For $u > 1$, the turning points are complex.
- a^D goes around these complex turning points.
- when $\hbar \ll 1$ and $N \gg \hbar^{-1}$ $(u \gg 1)$ semi-classically:

$$\Delta u_N^{\text{gap}} \sim \frac{2}{\pi} \frac{\partial u^{pert.}}{\partial N} e^{-\frac{2\pi}{\hbar} \mathcal{I}m\, a^D} \sim \frac{N\,\hbar^2}{2\pi} \left(\frac{e}{N\,\hbar} \right)^{2N}$$

Strong coupling ($\lambda \gg 1$)

A physical analogy:

Schwinger effect in monochromatic electric field $\mathcal{E}\cos(\omega t)$

- Pair production rate behaves differently for different ωs
- Keldysh adiabaticity parameter: $\gamma \equiv \frac{m\,\omega}{\mathcal{E}}$
- $\gamma \ll 1 \leftrightarrow$ constant field, $\gamma \gg 1 \leftrightarrow$ multi-photon limit
- In our analogy: $\hbar \equiv \frac{4\omega^2}{\mathcal{E}} \quad , \quad N \equiv \frac{m}{\omega} \quad , \quad \lambda = 2\gamma$

$$P_{\text{QED}} = e^{-\frac{m^2\pi}{\mathcal{E}}\, g(\gamma)} \sim \begin{cases} e^{-\pi\,\frac{m^2}{\mathcal{E}}} \quad , \quad \gamma \ll 1 \\ \\ e^{-\frac{m^2\pi}{\mathcal{E}}\,\frac{4}{\pi\gamma}\log(4\gamma)} = \left(\frac{\mathcal{E}}{4m\,\omega}\right)^{4m/\omega} \quad , \quad \gamma \gg 1 \end{cases}$$

- in the worldline formalism:
  $\gamma \ll 1 \leftrightarrow$ real instantons, $\gamma \gg 1 \leftrightarrow$ complex instantons

Fluctuations around complex instantons

- "quantum bilinear identity" relates $u^{pert.}(N,\hbar)$ to Δu_N^{gap}

$$
\begin{aligned}
u(N,\hbar) \quad \sim \quad & \frac{\hbar^2}{8}\sum_{n=1}^{N-1}\frac{P_n(N)}{\prod_{k=1}^{n}(N^2-k^2)^{2\lfloor\frac{n}{k}\rfloor-1}}\left(\frac{4}{\hbar^2}\right)^{2n} \\
& \pm\frac{1}{(2^{N-1}(N-1)!)^2}\left(\frac{2}{\hbar}\right)^{2N-1}\sum_{n=1}^{N-1}\frac{R_n(N)}{\prod_{k=1}^{n}(N^2-k^2)^{2\lfloor\frac{n}{k}\rfloor}}\left(\frac{4}{\hbar^2}\right)^{2n} \\
& +\ldots
\end{aligned}
$$

- The level splitting term (*gap width*) has the same structure with the leading perturbative expansion.
- $P_n(N)$, $R_n(N)$ are related! [GB, Dunne, Ünsal, in prep]
- New results for Mathieu equation!!

How general is the P- NP connection?

Mathieu (classical, $\hbar = 0$)

modular parameter: $$\tau_0(u) = \frac{\omega_0^D(u)}{\omega_0(u)} = i\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;\frac{1-u}{2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;\frac{1+u}{2}\right)}$$

τ_0 satisfies a Schwarzian equation: $\{\tau_0, u\} + Q_0(u) = 0$

where:

$$\{\tau_0, u\} := \frac{\tau_0'''}{\tau_0'} - \frac{3}{2}\left(\frac{\tau_0''}{\tau_0'}\right)^2 \quad , \quad Q_0(u) = \frac{1}{4(u-1)^2} + \frac{1}{8(u+1)} + \frac{1}{4(u+1)^2} - \frac{1}{8(u-1)}$$

spectrum can be obtained by inversion ($q_0 := e^{i\pi\tau_0}$):

$$u(q_0) = -1 + \lambda(q_0) = -1 + 32q_0 - 256q_0^2 + 1408q_0^3 + \ldots$$

How general is the P- NP connection?

Mathieu (quantum, $\hbar \neq 0$)

quantum correction to the Schwarzian equation:

$$\{\tau, u\} + Q(u) = 0$$

where: $Q(u) = \sum_{n=0}^{\infty} \hbar^{2n} \underbrace{Q_n(u)}_{\sum \text{poles at } u=\pm 1}$

inversion $\rightarrow$ spectrum:

$$u(q) = -1 + \lambda(q) + \sum_{n=1}^{\infty} \hbar^{2n} f_n(q)$$

there are 3 more cases that have the same P = NP structure as the Mathieu equation upon quantization:

- quantum bilinear identity:

$$\left(a - \hbar \frac{\partial a}{\partial \hbar}\right) \frac{\partial a^D}{\partial u} - \left(a^D - \hbar \frac{\partial a^D}{\partial \hbar}\right) \frac{\partial a}{\partial u} = \frac{i S_{inst}}{2\pi}$$

- Schwarzian & Picard-Fuchs equations:

$$\{\tau, u\} + Q^{(M)}(u) = 0$$
$$a''(u) + F^{(M)}(u) a'(u) + G^{(M)}(u) a(u) = 0$$

$Q^{(M)}(u) := \sum_{n=0}^{\infty} \hbar^n Q_n^{(M)}(u)$, etc...

$\hbar \neq 0 \rightarrow F_n^{(M)}(u), G_n^{(M)}(u), Q_n^{(M)}(u)$: sum over higher order poles at the *same locations* as the classical curve

they have a remarkable connection to number theory, quasi-modular forms and Hecke groups and possibly superconformal $\mathcal{N} = 2$ SUSY theories [GB, Dunne, Ünsal, in progress]

and more examples with more complicated P = NP relations:

generic genus-1: 2^{nd} order Picard-Fuchs eqn. for $a_0'(u)$

- Lamé equation $V(z) = \mathcal{P}(z; \mathfrak{t})$ related to $\mathcal{N} = 2^* SU(2)$
  [GB, Dunne; Kashani-Poor, Troost, ...]
- $V(z) = \cos(z) + \frac{2m_1 m_2}{\cos(z)+1} + \frac{(m_1 - m_2)^2}{\sin^2(z)}$
  related to $\mathcal{N} = 2, SU(2), N_f = 2$
- Double sine gordon $V(z) = \sin^2(z) + \mu \sin(z)$
- Asymmetric double well $V(z) = (z^2 - 1)^2 + \mu z$
- ...

Conclusions

- In an infinite class of QM systems in addition to the standard resurgence relations there is a low order -low order relation between perturbative and non-perturbative sectors
- Classically it is related to the topology of the spectral curve
- It is valid everywhere in the spectrum even though the series are drastically different (asymptotic vs. convergent) in different regions
- Quantization preserves this $P = NP$ relation
- 4 examples such that $P = NP$ is particularly simple