Defining a Conformal Field Theory as a scaling limit of discrete models

The Ising model example

Henrique Santos

Intern, Banco de Portugal
MSc Student at IST

Winter School of Mathematics, February 2021

Mathematics Department
Instituto Superior Técnico

Motivation

The Ising model is a probabilistic model of ferromagnetism.

Motivation

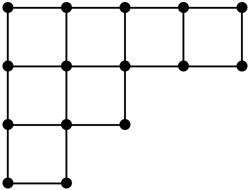
The Ising model is a probabilistic model of ferromagnetism.

Consider a graph.

Motivation

The Ising model is a probabilistic model of ferromagnetism.

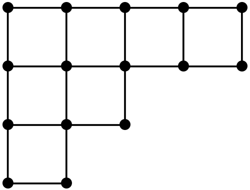
Consider a graph.



Motivation

The Ising model is a probabilistic model of ferromagnetism.

Consider a graph.

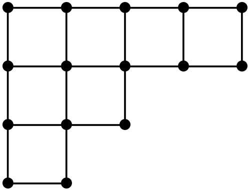


The vertices represent magnetic dipoles that can be aligned in one of two ways, positive or negative.

Motivation

The Ising model is a probabilistic model of ferromagnetism.

Consider a graph.



The vertices represent magnetic dipoles that can be aligned in one of two ways, positive or negative.

The edges link dipoles that interact with each other. Interacting dipoles tend to align.

Definition

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

Definition

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

To each vertex $v \in \mathcal{V}$ one associates a variable $\sigma_v \in \{\pm 1\}$, the *spin* of v.

Definition

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

To each vertex $v \in \mathcal{V}$ one associates a variable $\sigma_v \in \{\pm 1\}$, the *spin* of v.

For a configuration of spins $\sigma = (\sigma_v)_{v \in \mathcal{V}}$, its *Hamiltonian* is given by

$$H(\sigma) = -\sum_{(uv) \in \mathcal{E}} \sigma_u \sigma_v$$

Definition

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

To each vertex $v \in \mathcal{V}$ one associates a variable $\sigma_v \in \{\pm 1\}$, the *spin* of v.

For a configuration of spins $\sigma = (\sigma_v)_{v \in \mathcal{V}}$, its *Hamiltonian* is given by

$$H(\sigma) = - \sum_{(uv) \in \mathcal{E}} \sigma_u \sigma_v$$

The probability of a configuration σ is given by the Gibbs measure

$$\mathbb{P}(\sigma) \propto e^{-\beta H(\sigma)}$$

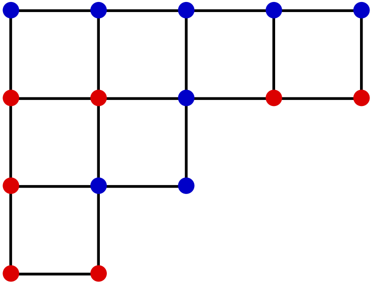
where $\beta > 0$ is constant (inverse temperature).

Probability of a configuration

$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$

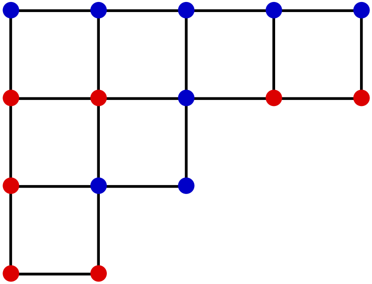
Probability of a configuration

$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$



Probability of a configuration

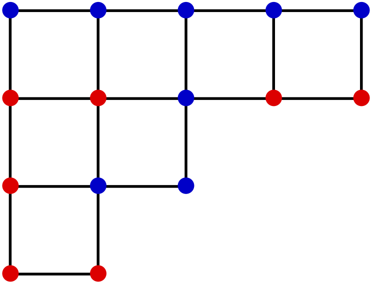
$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$



$$\sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v =$$

Probability of a configuration

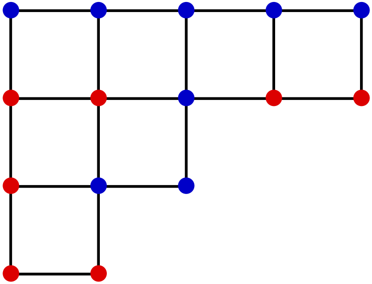
$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$



$$\sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v = \left[12 \cdot (+1) + 9 \cdot (-1)\right]$$

Probability of a configuration

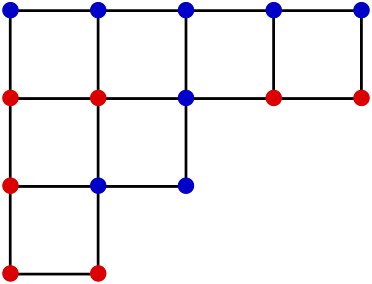
$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$



$$\sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v = \left[12 \cdot (+1) + 9 \cdot (-1)\right] = 3$$

Probability of a configuration

$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$

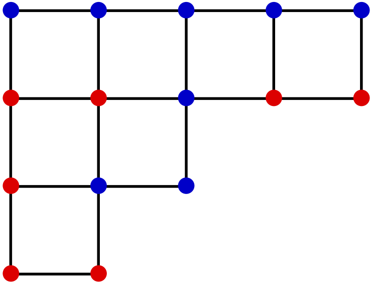


$$\sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v = \left[12 \cdot (+1) + 9 \cdot (-1)\right] = 3$$

$$H(\sigma) = -3$$

Probability of a configuration

$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)} = \frac{1}{Z_\beta} e^{\beta \sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v}$$



$$\sum_{(uv)\in\mathcal{E}} \sigma_u \sigma_v = \left[12 \cdot (+1) + 9 \cdot (-1)\right] = 3$$

$$H(\sigma) = -3$$

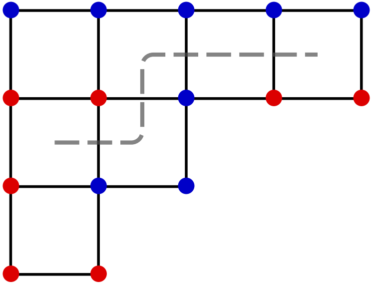
$$\mathbb{P}(\sigma) \propto e^{3\beta}$$

Disorder lines

We can add disorder lines to the model, along which the spins behave as if the neighbour had the opposite spin.

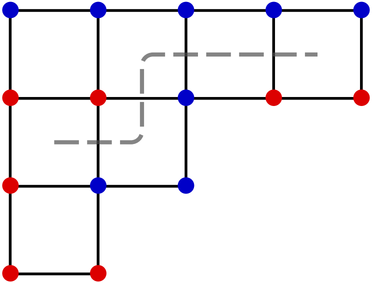
Disorder lines

We can add disorder lines to the model, along which the spins behave as if the neighbour had the opposite spin.



Disorder lines

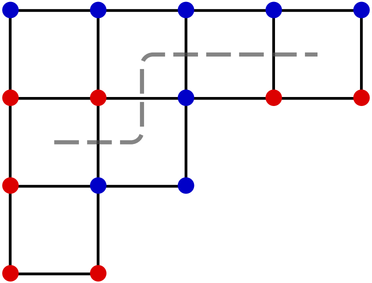
We can add disorder lines to the model, along which the spins behave as if the neighbour had the opposite spin.



Without disorder line: $H(\sigma) = -\big[12 \cdot (+1) + 9 \cdot (-1)\big] = -3$

Disorder lines

We can add disorder lines to the model, along which the spins behave as if the neighbour had the opposite spin.



Without disorder line: $H(\sigma) = -\left[12 \cdot (+1) + 9 \cdot (-1)\right] = -3$

With disorder line: $H(\sigma) = -\left[14 \cdot (+1) + 7 \cdot (-1)\right] = -5$

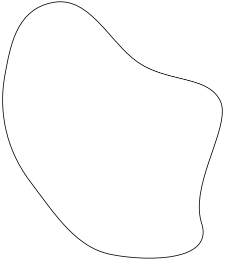
Ising model on domains

Ising model on domains

For a smooth bounded domain $\Omega \subset \mathbb{R}^d$, one can define an Ising model.

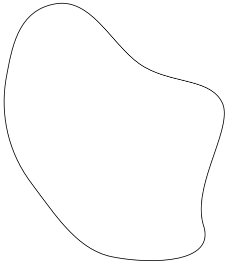
Ising model on domains

For a smooth bounded domain $\Omega \subset \mathbb{R}^d$, one can define an Ising model.



Ising model on domains

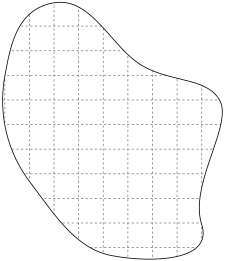
For a smooth bounded domain $\Omega \subset \mathbb{R}^d$, one can define an Ising model.



Fix a small $\delta > 0$, take the mesh $\delta\mathbb{Z}^d$ and consider a discretization of Ω in the lattice.

Ising model on domains

For a smooth bounded domain $\Omega \subset \mathbb{R}^d$, one can define an Ising model.



Fix a small $\delta > 0$, take the mesh $\delta\mathbb{Z}^d$ and consider a discretization of Ω in the lattice.

From a Physics perspective, δ should be very small.

From a Physics perspective, δ should be very small.

It would be nice if there were a "continuous version", with a spin variable for every point of Ω, which describes the discrete version when δ is close to 0.

From a Physics perspective, δ should be very small.

It would be nice if there were a “continuous version”, with a spin variable for every point of Ω, which describes the discrete version when δ is close to 0.

We are looking for a Statistical Field Theory.

Statistical Field Theory (SFT)

A SFT is a description of some random system with infinite degrees of freedom.

Statistical Field Theory (SFT)

A SFT is a description of some random system with infinite degrees of freedom.

- These systems show very different behaviours under different external parameters, and the *phase transitions* a system shows are at the core of a SFT.

Statistical Field Theory (SFT)

A SFT is a description of some random system with infinite degrees of freedom.

- These systems show very different behaviours under different external parameters, and the *phase transitions* a system shows are at the core of a SFT.
- It is intimately linked to Quantum Field Theory and Mean Field Theory.

The sweet dream...

Informally, this is what we want to do:

The sweet dream...

Informally, this is what we want to do:

For a domain $\Omega \subset \mathbb{R}^d$, the SFT is some random process ϕ, a random *field* which is a function of Ω.

The sweet dream...

Informally, this is what we want to do:

For a domain $\Omega \subset \mathbb{R}^d$, the SFT is some random process ϕ, a random *field* which is a function of Ω.

This random field comes with a "probability measure" $\mathbb{P}(\phi) \propto e^{-\mathcal{S}[\phi]}$, where $\mathcal{S}$ is a functional of ϕ called *action*.

The sweet dream...

Informally, this is what we want to do:

For a domain $\Omega \subset \mathbb{R}^d$, the SFT is some random process ϕ, a random *field* which is a function of Ω.

This random field comes with a "probability measure" $\mathbb{P}(\phi) \propto e^{-\mathcal{S}[\phi]}$, where $\mathcal{S}$ is a functional of ϕ called *action*.

The SFT becomes a Conformal Field Theory when it is invariant under conformal maps of Ω.

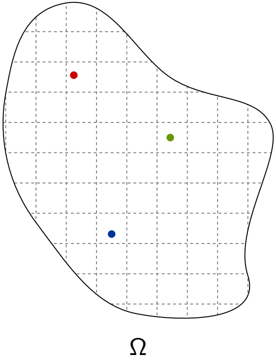
A conformal map is a function Φ that locally preserves angles.

A conformal map is a function Φ that locally preserves angles.

- For maps of $\mathbb{C}$, Φ is conformal if and only if Φ is holomorphic and has a non-zero derivative.

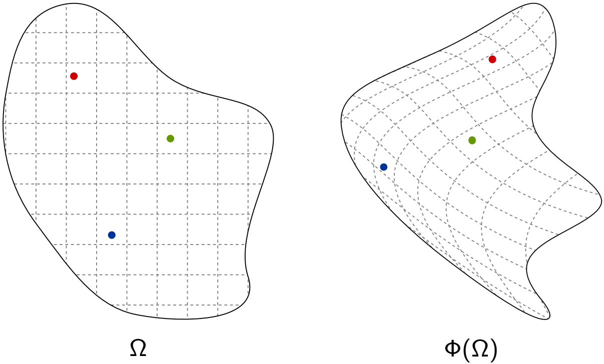
A conformal map is a function Φ that locally preserves angles.

- For maps of $\mathbb{C}$, Φ is conformal if and only if Φ is holomorphic and has a non-zero derivative.



A conformal map is a function Φ that locally preserves angles.

- For maps of $\mathbb{C}$, Φ is conformal if and only if Φ is holomorphic and has a non-zero derivative.



...and the harsh reality

The previous idea is difficult to formalize.

...and the harsh reality

The previous idea is difficult to formalize.

One solution is to define the continuous model as the scaling limit of discrete models.

What does "conformal theory" really mean?

Using scaling limits, this property becomes much more concrete.

What does “conformal theory” really mean?

Using scaling limits, this property becomes much more concrete.

An example:

$$\mathbb{E}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}]$$

What does "conformal theory" really mean?

Using scaling limits, this property becomes much more concrete.

An example:

$$\mathbb{E}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}] \xrightarrow{\delta\to 0} f_\Omega(a_1, a_2, \ldots, a_n)$$

What does “conformal theory” really mean?

Using scaling limits, this property becomes much more concrete.

An example:

$$\mathbb{E}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}] \xrightarrow{\delta\to 0} f_\Omega(a_1, a_2, \ldots, a_n)$$

is *conformally covariant*.

What does “conformal theory” really mean?

Using scaling limits, this property becomes much more concrete.

An example:

$$\mathbb{E}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}] \xrightarrow{\delta\to 0} f_\Omega(a_1, a_2, \ldots, a_n)$$

is *conformally covariant*. That is, if $\Phi : \Omega \to \widetilde{\Omega}$ is a conformal map, then

What does "conformal theory" really mean?

Using scaling limits, this property becomes much more concrete.

An example:

$$\mathbb{E}_{\Omega_\delta}[\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_n}] \xrightarrow{\delta\to 0} f_\Omega(a_1, a_2, \ldots, a_n)$$

is *conformally covariant*. That is, if $\Phi : \Omega \to \widetilde{\Omega}$ is a conformal map, then

$$f_\Omega(a_1, \ldots, a_n) = f_{\widetilde{\Omega}}\big(\Phi(a_1), \ldots, \Phi(a_n)\big) \cdot \prod_{k=1}^{n} |\Phi'(a_k)|^\alpha$$

Ising model as a SFT

The 2D Ising model is the quintessential model for SFT. It is one of the simplest models to feature a phase transition.

Ising model as a SFT

The 2D Ising model is the quintessential model for SFT. It is one of the simplest models to feature a phase transition.

The Ising model belongs to a family of 2D lattice models which is conjectured to be a conformally invariant at criticality.

Conformal invariance results

This setup allowed to rigorously define and prove conformal invariance for some models: percolation and Ising model.

Conformal invariance results

This setup allowed to rigorously define and prove conformal invariance for some models: percolation and Ising model.

For the Ising model case, it was a conjecture in Physics that certain objects – *fermions* – were conformally invariant.

Conformal invariance results

This setup allowed to rigorously define and prove conformal invariance for some models: percolation and Ising model.

For the Ising model case, it was a conjecture in Physics that certain objects – *fermions* – were conformally invariant.

[2011] Theorem (Chelkak, Smirnov)

The discrete fermions converge uniformly to the square root of a conformal map.

Conformal invariance results

This setup allowed to rigorously define and prove conformal invariance for some models: percolation and Ising model.

For the Ising model case, it was a conjecture in Physics that certain objects – *fermions* – were conformally invariant.

[2011] Theorem (Chelkak, Smirnov)

The discrete fermions converge uniformly to the square root of a conformal map.

[2015] Theorem (Chelkak, Hongler, Izyurov)

$\mathbb{E}^+[\sigma_{a_1} \cdots \sigma_{a_n}]$ is conformally covariant.

Conformal invariance results

This setup allowed to rigorously define and prove conformal invariance for some models: percolation and Ising model.

For the Ising model case, it was a conjecture in Physics that certain objects – *fermions* – were conformally invariant.

[2011] Theorem (Chelkak, Smirnov)

The discrete fermions converge uniformly to the square root of a conformal map.

[2015] Theorem (Chelkak, Hongler, Izyurov)

$\mathbb{E}^+[\sigma_{a_1} \cdots \sigma_{a_n}]$ is conformally covariant.

Theorem (Thesis)

$\mathbb{E}^\Gamma[\sigma_{a_1} \cdots \sigma_{a_n}]$ and $\mathbb{E}^{\Gamma,+}[\sigma_{a_1} \cdots \sigma_{a_n}]$ are conformally covariant.

The future

The future

This framework allows to rigorously describe the properties which are conjectured to hold in many SFT from Physics.

The future

This framework allows to rigorously describe the properties which are conjectured to hold in many SFT from Physics.

These results give hope of accurate proofs for other models, especially 2D.

References

Chelkak, Dmitry ; Cimasoni, David ; Kassel, Adrien:
Revisiting the combinatorics of the 2D Ising model (2017).

Chelkak, Dmitry ; Hongler, Clément ; Izyurov, Konstantin:
Conformal invariance of spin correlations in the planar Ising model (2015).

Chelkak, Dmitry ; Smirnov, Stanislav:
Discrete complex analysis on isoradial graphs (2011).

Hongler, Clément:
Conformal invariance of Ising model correlations (2014).

Kadanoff, Leo P. ; Ceva, Horacio:
Determination of an operator algebra for the two-dimensional Ising model (1971).

Smirnov, Stanislav:
Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model (2010).

Smirnov, Stanislav:
Towards conformal invariance of 2D lattice models (2006).