Facets and Parallel Universes of Modal Logic

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Roadmap

A possible Octopus of Modal Logic

- Semantic Facets and their Relationship ✓
- Fancier Semantics and Model Theory
- Kripke Semantics for Intuitionistic Logic ✓
- Labelled Deduction
- Combination of Logics
- Parallel Universes ✓
- Open Questions ✓
- Challenges ✓

Formulas

Let P be a countable set (proposicional symbols). The set of modal formulas L_P is inductively defined as follows:

- $P \subseteq L_P$;
- $\neg\, \varphi, \Box \varphi$ whenever $\varphi \in L_P$;
- $\varphi_1 \supset \varphi_2$ whenever $\varphi_1, \varphi_2 \in L_P$.

$\Diamond \varphi =_{\text{abb}} \neg\, \Box\, \neg\, \varphi$.

(Set-Theoretic) Semantic Facets

- Relational
- Algebraic
- Topological

Relational via Kripke structures

A Kripke frame is a pair (W, R) where W is a non-empty set and $R \subseteq W^2$.

A Kripke structure is a triple (W, R, V) where (W, R) is a Kripke frame and $V : P \to \wp W$ is a map. A Kripke structure (W, R, V) and $w \in W$ (locally) satisfy φ, written

$$(W, R, V), w \Vdash \varphi$$

whenever

- $(W, R, V), w \Vdash p$ whenever $w \in V(p)$
- $(W, R, V), w \Vdash \neg\psi$ whenever $(W, R, V), w \not\Vdash \psi$
- $(W, R, V), w \Vdash \Box\psi$ whenever

  $(W, R, V), w' \Vdash \psi$ for all $w' \in W$ such that wRw'
- $(W, R, V), w \Vdash \psi_1 \supset \psi_2$ whenever either $(W, R, V), w \not\Vdash \psi_1$ or $(W, R, V), w \Vdash \psi_2$.

Relational via Kripke structures

A Kripke structure (W, R, V) satisfies φ, written

$$(W, R, V) \Vdash \varphi$$

whenever $(W, R, V), w \Vdash \varphi$ for every $w \in W$. Similarly, a Kripke frame (W, R) satisfies φ, written

$$(W, R) \Vdash \varphi$$

whenever (W, R, V) satisfies φ for ever map $V : P \to \wp W$.

Validity over a class $\mathfrak{F}$ of frames:

$\vDash_{\mathfrak{F}} \varphi$ whenever $(W, R) \Vdash \varphi$ for every $(W, R) \in \mathfrak{F}$.

Different properties of R lead to different logics:

For instance, **S**4 is the modal logic where R is reflexive and transitive. Observe that R can have properties that are not elementary.

Hilbert Calculus for Modal Logic

Modal **K** (no properties on R)

Axioms

The usual ones for propositional logic plus

K $(\Box(\varphi_1 \wedge \varphi_2)) \equiv ((\Box\varphi_1) \wedge (\Box\varphi_2))$;

Rules

MP $\dfrac{\varphi_1 \quad \varphi_1 \supset \varphi_2}{\varphi_2}$ and **Nec** $\dfrac{\varphi}{\Box\varphi}$

Modal **S**4 (R reflexive and transitive) is

Modal **K** + Axioms **T** $(\Box\varphi) \supset \varphi$ and **4** $(\Box\varphi) \supset (\Box\Box\varphi)$.

General Form of Completeness Results

Proposition

$$\vdash_{\mathbf{K}} \varphi \quad \text{if and only if} \quad \vDash_{\mathfrak{F}} \varphi$$

where $\mathfrak{F}$ is the class of all frames.

$$\vdash_{\mathbf{S4}} \varphi \quad \text{if and only if} \quad \vDash_{\mathfrak{F}_{\mathrm{rt}}} \varphi$$

where $\mathfrak{F}_{\mathrm{rt}}$ is the class of all reflexive and transitive frames.

Is it always possible to obtain such nice results? The end of the story is not here...

General Kripke Frames: Motivation

The so called McKinsey formula

$$(\Box\Diamond\varphi) \supset (\Diamond\Box\varphi)$$

is not satisfied in the frame $(\mathbb{N}, <)$.

But it is a valid formula whenever a valuation assigns a finite or a co-finite set of natural numbers to each $p \in P$.

That is, we need to work with general Kripke frames.

General Kripke Frames

A general Kripke frame is a triple $(W, R, \mathcal{A})$ where (W, R) is a frame and $\mathcal{A} \subseteq \wp W$ fulfils the following closure properties:

- if $A \in \mathcal{A}$ then $W \setminus A \in \mathcal{A}$;
- if $A_1, A_2 \in \mathcal{A}$ then $A_1 \cup A_2 \in \mathcal{A}$;
- if $A \in \mathcal{A}$ then $\{w \in W : \forall w' \in W,\ w' \in A \text{ whenever } wRw'\} \in \mathcal{A}$;

A general Kripke structure is a tuple $(W, R, \mathcal{A}, V)$ where $(W, R, \mathcal{A})$ is a general Kripke frame and $V : P \to \mathcal{A}$ is a map.

Then

$$(\mathbb{N}, <, \mathcal{A}, V) \Vdash (\Box\Diamond\varphi) \supset (\Diamond\Box\varphi)$$

where $\mathcal{A}$ is the class of all finite and cofinite subsets of $\mathbb{N}$.

Algebraic Semantics via Modal Algebras

A modal algebra is a tuple $\mathfrak{A} = (A, \sqcap, \sqcup, \sqsupset, -, \top, \underline{\square})$ where $(A, \sqcap, \sqcup, \sqsupset, -, \top)$ is a Boolean algebra, $\underline{\square} : A \to A$ satisfies the following identities:

$$\underline{\square}(a_1 \sqcap a_2) = (\underline{\square} a_1 \sqcap \underline{\square} a_2) \quad \text{and} \quad \underline{\square}\top = \top.$$

A modal algebraic structure is a pair $(\mathfrak{A}, V)$ where $V : P \to A$ is a map.

The denotation of φ in

$$[\![\varphi]\!]^{\mathfrak{A},V} \in A$$

is inductively defined as follows:

- $[\![p]\!]^{(\mathfrak{A},V)} = V(p)$ for $p \in P$;
- $[\![\neg\,\psi]\!]^{(\mathfrak{A},V)} = -[\![\psi]\!]^{(\mathfrak{A},V)}$;
- $[\![\psi_1 \supset \psi_2]\!]^{(\mathfrak{A},V)} = [\![\psi_1]\!]^{(\mathfrak{A},V)} \sqsupset [\![\psi_2]\!]^{(\mathfrak{A},V)}$;
- $[\![\square\psi]\!]^{(\mathfrak{A},V)} = \underline{\square}[\![\psi]\!]^{(\mathfrak{A},V)}$.

Observation

The denotation of φ in

- Kripke semantics is such that

$$[\![\varphi]\!]^{(W,R,V)} \in \wp W$$

- general Kripke semantics is such that

$$[\![\varphi]\!]^{(W,R,\mathcal{A},V)} \in \mathcal{A}$$

- modal algebra semantics

$$[\![\varphi]\!]^{(\mathfrak{A},V)} \in A$$

Stone Representation (The Simpler View)

Modal Algebraic Structures $\longrightarrow^{g}$ D General Kripke Structures

D General Kripke Structures $\longrightarrow^{h}$ Modal Algebraic Structures

Stone Representation

Let g be such that

$$g(\mathfrak{A}, V) = (W, R, \underline{V})$$

where

- W is $\{U \subseteq A : U \text{ is an ultrafilter of } \mathfrak{A}\}$
- URU' whenever for every $a \in A$ if $\underline{\Box}a \in U$ then $a \in U'$
- $\underline{V}(p) = \{U \in W : V(p) \in U\}$.

Proposition

For every $U \in W$,

$$[\![\varphi]\!]^{\mathfrak{A},V} \in U \quad \text{if and only if} \quad (W, R, \underline{V}), U \Vdash \varphi.$$

Furthermore,

$$(\mathfrak{A}, V) \Vdash \varphi \quad \text{if and only if} \quad (W, R, \underline{V}) \Vdash \varphi.$$

Stone Representation

A filter of a modal algebra $\mathfrak{A}$, is a set $F \subseteq A$ such that:

- $\top \in F$
- if $a, b \in F$ then $a \sqcap b \in F$
- if $a \in F$ and $a \leq b$ then $b \in F$.

A filter F is a ultrafilter whenever:

- $\bot \notin F$
- for every $a \in A$ either $a \in F$ or $-a \in F$.

Stone Representation

Let h be such that

$$h(W, R, \mathcal{A}, V) = (\mathcal{A}, \cap, \cup, -.\underline{\Box}, W, V)$$

where

- $-Z = W \setminus Z$;
- $\underline{\Box}Z = \{w \in W : w' \in Z \text{ whenever } wRw'\}$.

Proposition

For every $w \in W$,

$$(W, R, \mathcal{A}, V), w \Vdash \varphi \quad \text{if and only if} \quad w \in [\![\varphi]\!]^{((\mathcal{A}, \cap, \cup, -.\underline{\Box}, W), V)}.$$

Furthermore,

$$(W, R, \mathcal{A}, V) \Vdash \varphi \quad \text{if and only if} \quad ((\mathcal{A}, \cap, \cup, -.\underline{\Box}, W), V) \Vdash \varphi.$$

Topological Semantics for **S4**

A topological structure is a triple (X, τ, V) where (X, τ) is a topological space and $V : P \to \wp X$ is a map. A topological structure (X, τ, V) and $x \in X$ (locally) satisfy φ, written

$$(X, \tau, V), x \Vdash \varphi$$

whenever

- $(X, \tau, V), x \Vdash p$ whenever $x \in V(p)$;
- $(X, \tau, V), x \Vdash \neg \psi$ whenever $(X, \tau, V), x \nVdash \psi$;
- $(X, \tau, V), x \Vdash \Box\psi$ whenever there is $U \in \tau$ such that $x \in U$ and $(X, \tau, V), y \Vdash \psi$ for all $y \in U$;
- $(X, \tau, V), x \Vdash \psi_1 \supset \psi_2$ whenever either $(X, \tau, V), x \nVdash \psi_1$ or $(X, \tau, V), x \Vdash \psi_2$;

Topological Semantics for **S4**

The denotation of a formula

$$[\![\varphi]\!]^{(X,\tau,V)} \subseteq X$$

Proposition

$$[\![\Box\psi]\!]^{(X,\tau,V)} = ([\![\psi]\!]^{(X,\tau,V)})^{\circ}.$$

Topological Semantics for **S4**

- $[\![\Box(\varphi \wedge \psi)]\!]^{(X,\tau,V)} = [\![\Box\varphi]\!]^{(X,\tau,V)} \cap [\![\Box\psi]\!]^{(X,\tau,V)}$ **K**
- $([\![\varphi \wedge \psi)]\!]^{(X,\tau,V)})^\circ = ([\![\varphi]\!]^{(X,\tau,V)})^\circ \cap ([\![\psi]\!]^{(X,\tau,V)})^\circ$
- $[\![\varphi \wedge \psi]\!]^{(X,\tau,V)} = [\![\varphi]\!]^{(X,\tau,V)} \cap [\![\psi]\!]^{(X,\tau,V)}$;
- $([\![\varphi]\!]^{(X,\tau,V)} \cap [\![\psi]\!]^{(X,\tau,V)})^\circ = ([\![\varphi]\!]^{(X,\tau,V)})^\circ \cap ([\![\psi]\!]^{(X,\tau,V)})^\circ$.

$$X^\circ = ([\![\top]\!]^{(X,\tau,V)})^\circ = [\![\Box\top]\!]^{(X,\tau,V)} = [\![\top]\!]^{(X,\tau,V)} = X$$

$$([\![\varphi]\!]^{(X,\tau,V)})^\circ \subseteq [\![\varphi]\!]^{(X,\tau,V)}$$

$$([\![\varphi]\!]^{(X,\tau,V)})^\circ \subseteq (([\![\Box\varphi]\!]^{(X,\tau,V)})^\circ)^\circ$$

Topological Semantics for **S**4

Let g and h be such that

$$g(W,R) = (W, \{A \subseteq W : A \text{ is } R \text{ upclosed of } (W,R)\})$$

and

$$h(X,\tau) = (X, \{(x,y) : y \in \mathrm{Cl}(x)\})$$

Proposition

S4 is sound and complete with respect to topological semantics.

Kripke Semantics for Intuitionistic Logic

The propositional connectives are not abbreviations of each other. In the sequel $\sim$ is the symbol adopted for negation.

The Gödel map provided a translation from

$$\tau : L_{\mathbf{Int}} \to L_{\mathbf{S4}}$$

as follows

- $\tau(p) = \Box p$
- $\tau(\varphi_1 *' \varphi_2) = \tau(\varphi_1) * \tau(\varphi_2)$ where $* \in \{\wedge, \vee\}$
- $\tau(\varphi_1 \supset' \varphi_2) = \Box(\tau(\varphi_1) \supset \tau(\varphi_2))$
- $\tau(\sim \varphi) = \Box \neg \tau(\varphi)$.

Intuitionistic Structures

An intuitionistic structure is a triple (D, S, V) where (D, S) is a preorder and $V : P \to \wp D$ fulfilling the hereditary property:

$$\text{if } d \in V(p) \text{ then } d' \in V(p), \text{ for every } d' \in D \text{ such that } dSd'.$$

Surprise:

$$p \vee \sim p \quad \text{is no longer a valid formula.}$$

The triple (D, S, V) where $D = \{d_1, d_2\}$, $S = \{(d_1, d_1), (d_2, d_2), (d_1, d_2)\}$ and $V(p) = \{d_2\}$ is an intuitionistic structure. However,

$$(D, S, V), d_1 \not\Vdash p \quad \text{and} \quad (D, S, V), d_1 \not\Vdash \sim p$$

Intuitionistic Logic and S4 Modal Logic

Intutionistic Language $\longrightarrow^{\tau}$ Modal language

Intuitionistic Structures $\longrightarrow^{\text{id}}$ **S4** Kripke Structures

S4 Kripke Structures $\longrightarrow^{f}$ Intuitionistic Structures

From Intuitionistic Logic to **S4**

Proposition

$$(D, S, V), d \Vdash_{\mathbf{Int}} \varphi \quad \text{if and only if} \quad (D, S, V), d \Vdash_{\mathbf{S4}} \tau(\varphi).$$

In particular,

$$(D, S, V), d \Vdash_{\mathbf{Int}} p \quad \text{if and only if} \quad (D, S, V), d \Vdash_{\mathbf{S4}} \Box p.$$

From **S4** to Intuitionistic Logic

Let f be such that

$$f(W, R, V) = (W, R, V')$$

where $V'(p) = \{w \in W : (W, R, V), w \Vdash \Box p\}$.

Proposition

$$(W, R, V'), w \Vdash_{\mathbf{Int}} \varphi \quad \text{if and only if} \quad (W, R, V), w \Vdash_{\mathbf{S4}} \tau(\varphi).$$

In particular,

$$(W, R, V'), w \Vdash_{\mathbf{Int}} p \quad \text{if and only if} \quad (W, R, V), d \Vdash_{\mathbf{S4}} \Box p.$$

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Parallel Worlds: Linear Temporal Logic

Propositional Language + $\mathbf{X}\varphi,\ \varphi_1\mathbf{U}\varphi_2 \in L_P$:

Semantics is a family $V = \{V_k\}_{k\in\mathbb{N}}$ such that $V_k \subseteq P$

- $V, k \Vdash p$ whenever $p \in V_k$
- $V, k \Vdash \mathbf{X}\varphi$ whenever $V, k+1 \Vdash \varphi$
- $V, k \Vdash \varphi_1\mathbf{U}\varphi_2$ whenever there is $n \in \mathbb{N}$ such that $V, k+i \Vdash \varphi_1$ for every $i < n$ and $V, k+n \Vdash \varphi_2$
- $\vDash \varphi$ whenever $V, 0 \Vdash \varphi$ for every V.

Important abbreviations

$$\mathbf{F}\varphi =_{\text{abb}} \top\mathbf{U}\varphi \quad \text{and} \quad \mathbf{G}\varphi =_{\text{abb}} \neg\,\mathbf{F}\,\neg\,\varphi$$

Parallel Worlds: Linear Temporal Logic

- LTL is not compact: indeed,

$$\{\mathbf{X}^k\varphi : k \in \omega\} \vDash \mathbf{G}\varphi$$

  but there is no finite subset Ψ of $\{\mathbf{X}^k\varphi : k \in \omega\}$ such that $\Psi \vDash \varphi$

- LTL is not (strongly) complete

Parallel Worlds: Linear Temporal Logic

LTL is a fusion (with interaction) of modal logics

- Axiom $\neg \mathbf{X}\varphi \equiv \mathbf{X} \neg \varphi$
- Axiom $(\varphi_1 \mathbf{U} \varphi_2) \supset \mathbf{F}\varphi_2$
- Axiom $(\varphi_1 \mathbf{U} \varphi_2) \equiv (\varphi_2 \vee (\varphi_1 \wedge \mathbf{X}(\varphi_1 \mathbf{U} \varphi_2))$

Parallel Worlds: Deontic Logic

SDL axiomatized as modal logic **K** plus

D $(\Box\varphi) \supset (\Diamond\varphi)$

SDL^{+} axiomatized as modal logic **SDL** plus

D^{+} $\Box((\Box\varphi) \supset \varphi)$

How to avoid axiom **K**?

Parallel Worlds: Neighbourhood Semantics

A neighbourhood structure is a pair (W, R, V) where W is a non empty set, $R \subseteq W^3$ and $V : P \to \wp W$. Then

$$(W, R, V), w \Vdash \Box\varphi$$

whenever, for every $w', w'' \in W$,

if $wRw'w''$ then

either $(W, R, V), w' \Vdash \varphi$ or $(W, R, V), w'' \Vdash \varphi$.

Observe that

$$\not\models ((\Box\varphi_1) \wedge (\Box\varphi_2)) \supset (\Box(\varphi_1 \wedge \varphi_2)).$$

Parallel Worlds: Logic of Programs

Let Π be a set of atomic programs. The set L_Π of programs is inductively defined as follows:

- $\Pi \subseteq L_\Pi$;
- $\alpha; \beta, \alpha \cup \beta \in L_\Pi$ whenever $\alpha, \beta \in L_\Pi$;
- $\alpha^* \in L_\Pi$ whenever $\alpha \in L_\Pi$;
- $\varphi? \in L_\Pi$ whenever $\varphi \in L_P$;

Moreover, L_P is a propositional language with mixed formulas of the form

$$[\alpha]\varphi$$

where $\alpha \in L_\Pi$ and $\varphi \in L_P$. Each $[\alpha]$ is a modal operator.

Parallel Worlds: Logic of Programs

- Semantics given by multimodal Kripke structures

$$M = (W, \{R_\alpha\}_{\alpha \in L_\Pi}, V)$$

- $[\![[\alpha;\beta]]\!]^M = [\![\alpha]\!]^M \circ [\![\beta]\!]^M$
- $[\![[\alpha^*]]\!]^M = ([\![\alpha]\!]^M)^*$
- $[\![[\alpha \cup \beta]]\!]^M = [\![\alpha]\!]^M \cup [\![\beta]\!]^M$
- $[\![\varphi?]\!]^M = \{(w,w) \in W^2 : w \in [\![\varphi]\!]^M\}$;
- $[\![[\alpha]\varphi]\!]^M = \{w \in W : \text{if } wR_\alpha w' \text{ then } w' \in [\![\varphi]\!]^M, w' \in W\}$.

Some Axioms

- $[\alpha;\beta]\varphi \equiv [\alpha][\beta]\varphi$
- $[\varphi?]\psi \supset (\varphi \supset \psi)$
- $\varphi \supset (([\alpha^*](\varphi \supset [\alpha]\varphi)) \supset [\alpha^*]\varphi)$

Parallel Worlds: Hoare Logic

An assertions

$$\{\varphi\}\,\alpha\,\{\psi\}$$

is an abbreviation of $\varphi \supset ([\alpha]\psi)$.

While Rule

$$\frac{\{\varphi \wedge \psi\}\,\alpha\,\{\psi\}}{\{\psi\}\,\textbf{while}\,\varphi\,\textbf{do}\,\alpha\{(\neg\,\varphi) \wedge \psi\}}$$

where **while** φ **do** α is an abbreviation of

$$(\varphi?;\alpha)^*;(\neg\,\varphi)?$$

Parallel Worlds: Epistemic Logic

Multimodal Logic with a finite family of operators

$$\{\mathcal{K}_a\}_a \in A$$

where A is a finite set.

We can write formulas like this one

$$(\mathcal{K}_a\mathcal{K}_b p) \supset \mathcal{K}_b\mathcal{K}_a\mathcal{K}_b p.$$

A possible axiom is

$$(\neg\mathcal{K}_a p) \supset \mathcal{K}_a \neg\mathcal{K}_a p$$

which is true in Euclidean frames.

Parallel Worlds: Evidence Logic

Assume that

- ϕ is either $t \cdot p$ or $-(t \cdot p)$ meaning that p holds at t and p does not hold at t
- $a : \phi$ and $-a : \phi$ meaning that agent a claims ϕ and agent a does not claim ϕ, respectively
- $a : \boxdot\phi$ states that there are no agents at least as trustworthy with respect to ϕ as a that state $-\phi$. That is, each agent at least as trustworthy with respect to p as a does not claim $-\phi$
- $a \trianglelefteq_{var_P(\phi)} b$ means that agent b is more trustworthy than agent a with respect to the variable in ϕ

Parallel Worlds: Evidence

Axiomatization of $\boxdot$

$$\left(\bigwedge_{b\in A}(a \trianglelefteq_{\mathsf{var}_P(\phi)} b \supset \neg\, b : -\phi)\right) \supset a : \boxdot\phi$$

and

$$(a \trianglelefteq_{var_P(\phi)} a' \wedge a' : -\phi) \supset \neg\, a : \boxdot\phi$$

Parallel Worlds: Provability Logic GL (Gödel-Löb)

After Gödel Second Incompleteness Theorem

Modal Logic **K** + **GL** $(\Box((\Box p) \supset p)) \supset (\Box p)$

The binary relation $R \subseteq W^2$ is converse wellfounded if for every non-empty set $X \subseteq W$ there is $w \in X$ such that for every $x \in X$ it is not the case that wRx.

Proposition

$$\vdash_{\mathbf{GL}} \varphi \quad \text{if and only if} \quad \vDash_{\mathfrak{F}_{\text{cwf}}} \varphi$$

where $\mathfrak{F}_{\text{cwf}}$ is the class of all converse wellfounded.

Parallel Worlds: Spatial

- **S4.2** is **S4**+ the axiom

$$(\Diamond\Box p) \supset (\Box\Diamond p)$$

- corresponding to the first-order property

$$\forall x_1, x_2((xRx_1 \wedge xRx_2) \supset (\exists y\,(x_1Ry \wedge x_2Ry))$$

- It is related to Relativity:

  if there are two different causal futures of some world then there is a common future history.

Open Questions in Heaven

Generalized modalities for axiomatizing qualitative properties in measure theory and in probability concepts like almost everywhere and very unlikely

$$\Box_{\mathbf{AE}}\,\varphi \quad \text{and} \quad \Box_{\mathbf{VL}}\,\varphi$$

Open Questions in Heaven

Complete axiomatization of geometric notions like betweeness

Enrichment of propositional with the operator **B** leading to formulas of the kind $\mathbf{B}(\varphi, \psi)$. Let $R \subseteq W^3$ be a relation. Then

$$(W, R, V), w \Vdash \mathbf{B}(\varphi, \psi)$$

whenever there are $w_1, w_2 \in W$ such that

$$\begin{cases} R(w_1, w, w_2) \\ (W, R, V), w_1 \Vdash \varphi \text{ and } (W, R, V), w_2 \Vdash \psi. \end{cases}$$

Then

$$(W, R, V), w \Vdash \mathbf{B}(\varphi, \mathbf{B}(\psi, \delta)) \supset \mathbf{B}(\mathbf{B}(\varphi, \psi), \delta)$$

holds of and only if (Pash axiom)

$$\forall t{,}x{,}y{,}z{,}u((R(x, t, u) \wedge R(y, u, z)) \supset (\exists v(R(x, v, y) \wedge R(v, t, z))$$

Open Questions in Heaven

Algebraic semantics for non-normal modal logics in particular and neighbourhood semantics in general

Dynamic quantum logic incorporating compatible observables

Many Questions in Parallel Universes

Many Questions in Combination of Logics

Challenges on Earth and in Hell

Is modal logic relevant for

- AI **H** *yes* $\supset$ **G** *yes*
- Security **P** *yes* $\supset$ **G** *yes*
- ML That's a good question! **FG** *yes*
- Climate change I believe...
- Medical applications I believe...
- Real jobs
  Yes e.g Specification and Verification, Model Checking and Information Security
- a CEO Quantum superposition of Yes and No