Supersymmetric Gauge Theories And Resurgence

Jorge Russo
ICREA – University of Barcelona

There is now a large class of observables in different gauge theories in various dimensions that have been computed exactly, allowing us to explore fascinating aspects of gauge theories, such as dualities, large N physics, resurgence, non-perturbative phenomena, integrable systems, Wilson loops.

Today we discuss:

- Weak coupling expansion in four-dimensional $\mathcal{N}=2$ gauge theories
- Seiberg-Witten theory for SQCD with two massive flavors.
- $\mathcal{N}=2$ gauge theories on ellipsoids. Connection with WKB expansion in quantum mechanics and Nekrasov-Shatashvili limit.

SUPERSYMMETRIC LOCALIZATION

Consider SU(N) $\mathcal{N}$ = 2 supersymmetric YM theories on $\mathbf{S^4}$, radius R

Vector multiplet $(A_\mu, \psi^1_\alpha, \psi^1_\alpha, \Phi + i\Phi')$

Matter hypermultiplet mass M $(\phi, \chi_\alpha, \tilde{\chi}_\alpha, \tilde{\phi})$ adjoint or fundamental

Exact partition function for $\mathcal{N}$ = 2 supersymmetric YM theories on $\mathbf{S^4}$, with arbitrary matter content . [Pestun, 0712.2824]

Partition function localizes to a finite dimensional integral over Coulomb moduli

$$\langle \Phi \rangle = diag(a_1, \dots, a_N)$$

VEV of scalar of vector multiplet

Round S^4

$$Z(g) = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-S_{cl}(a)} \, z_{1-loop}(a) \quad \left| z_{inst}(a;\tau) \right|^2$$

$Z = Z(g)$

Exact g dependence

$$S_{cl} = \frac{1}{4g^2} \int_{S^4} d^4x \sqrt{G} \, \mathrm{R} \, \mathrm{tr}\, \Phi^2 = R^2 \frac{8\pi^2}{g^2} \sum_i a_i^2$$

$z_{1\text{-}loop}$ is expressed in terms of a single function $H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$

$$Z_{inst} = \sum_{k=0}^{\infty} q^k \, z_k(M, a, \varepsilon_1, \varepsilon_2) \quad , \qquad q = e^{2\pi i \tau}, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

$$\varepsilon_1 = \varepsilon_2 = \frac{1}{R}$$

Z = Z(g) is given in terms of a very complicated integral which must still be computed to be able to understand how the partition function depends on the coupling.

The one-loop factor

$$H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$$

Related to Barnes G-function:

$$H(x) \equiv e^{-(1+\gamma)x^2} G(1+ix)G(1-ix)$$

The different multiplets contribute as follows:

Vector multiplet

$$\prod_{i<j} H^2(a_i - a_j)$$

Adjoint hypermultiplet

$$\prod_{i<j} \frac{1}{H(a_i - a_j - M)H(a_i - a_j + M)}$$

Fundamental hypermultiplet

$$\prod_{i} \frac{1}{H(a_i + M)}$$

How can we find $Z(g)$?

• Only for the $\mathcal{N} = 4$ theory the integral defining $Z(g)$ can be carried out exactly.

In $\mathcal{N} = 2$ theories, we may consider limits :

I) **Weak coupling.** Perturbation series in g^2

II) **Large N, R arbitrary** ($\lambda = g^2 N$ fixed)
This implies two big simplifications that will allow us to determine Z exactly.

a) At $N \to$ Infinity the integral is exactly determined by a saddle-point.
b) Instantons do not contribute. $z_{inst} \to 1$, since

$$|q| = e^{-\frac{8\pi^2}{g^2}} = e^{-\frac{8\pi^2 N}{\lambda}} \xrightarrow[\lambda\ fixed]{N\to\infty} 0$$

III) **Finite *N*** (e.g. SU(2)) **but *R* → Infinity** [J.R. arxiv 1411.2602]

a) The integral is also exactly determined by a saddle-point, as long as a saddle-point exists
b) Instanton contribution can be incorporated exactly using Seiberg-Witten curve.

Weak coupling expansion in four-dimensional $\mathcal{N}$ =2 gauge theories

Some supersymmetric observables have been computed exactly in $\mathcal{N}$ =2 four-dimensional gauge theories on S^4. This includes

- Free energy
- ½ BPS Circular Wilson loop
- ‘t Hooft loops
- n-point correlation functions of chiral primaries

The structure of perturbation series for all $\mathcal{N}$ =2 four-dimensional gauge theories on S^4 turns out to be qualitatively similar:

In all cases, perturbation series is asymptotic and Borel summable.

Discussions can be found in:

- [Aniceto, Schiappa, J.R., 1410.5834]; [J.R., 1203.5061]
- [Dunne, Shifman, Unsal, 1502.06680]
- [Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski and Pufu, 1602.05971]
- [M. Honda, 1604.08653]

Weak coupling expansion in SU(2) $\mathcal{N}$ =2 SCF theory

By localization, we can compute the VEV of the circular Wilson loop or the free energy.
The partition function is given by

$$Z(g)=\int_{-\infty}^{\infty} da\, a^2\, \mathrm{e}^{-\frac{16\pi^2}{g^2}a^2} \prod_{n=1}^{\infty} \frac{\left(1+\frac{4a^2}{n^2}\right)^{2n}}{\left(1+\frac{a^2}{n^2}\right)^{8n}} \left|z_{inst}^{SU(2)}(a;g^2)\right|^2$$

The Nekrasov instanton factor is $\quad Z_{inst}^{SU(2)}=1+\sum_{k=1}^{\infty} e^{2\pi i k\tau} Z_k(a) \quad , \quad \tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}$

Each distinct topological sector can be addressed separately, as the resurgent structure of different topological sectors does not mix.

Consider first the zero-instanton sector, $k =$ 0.

The perturbative series is then obtained by expanding the one-loop factor in powers of a and performing the integration.

A direct approach is to consider the original integral and change integration variable
$s = 2A\, a^2 = (4\,\pi)^2\, a^2$

$$Z^{(0)}(g) = \int_0^\infty ds\ \mathrm{e}^{-\frac{s}{g^2}}\ \sqrt{s} \prod_{n=1}^{\infty} \frac{\left(1+\frac{4s}{2An^2}\right)^{2n}}{\left(1+\frac{s}{2An^2}\right)^{8n}}$$

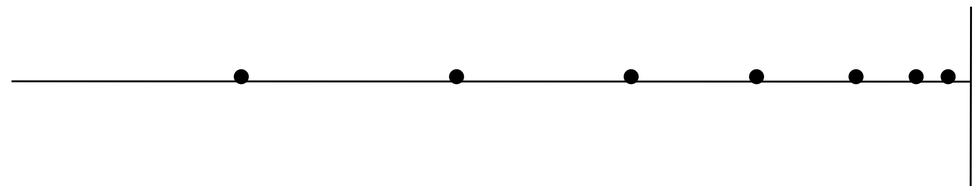
From this formula we can directly read off the Borel transform of Z:

$$\mathcal{B}[Z^{(0)}](s) = \sqrt{s} \prod_{n=1}^{\infty} \frac{\left(1+\frac{4s}{2An^2}\right)^{2n}}{\left(1+\frac{s}{2An^2}\right)^{8n}}$$

Thus the Borel transform has poles at

$$s_n = -2An^2 = -(4\pi)^2\, n^2 \quad , \qquad n = 1, 2, \cdots$$

There is a single Stokes line at $\theta = \pi$



The discontinuity across the Stokes line can be found from the behavior of the Borel transform around each pole [Aniceto, Schiappa, J.R]

$$\mathcal{B}[Z^{(0)}](s) = \frac{1}{(s-s_n)^{4n}} \sum_{\ell=0}^{\infty} \frac{f_\ell^{(n)}}{\ell!} (s-s_n)^\ell$$

One finds

$$\mathrm{Disc}_\pi\, Z_0(\lambda) = -\sum_{n=1}^{\infty} e^{-\frac{s_n}{\lambda}} \sum_{\ell=0}^{4n-1} \frac{2\pi i\, f_\ell^{(n)}\, \lambda^{\ell-4n-\frac{1}{2}}}{(-1)^{4n-\ell-1}\Gamma(4n-\ell)\ell!} \quad , \qquad \lambda \equiv g_{YM}^2$$

The discontinuity encodes the complete information of the asymptotic series $Z_0(\lambda)$
In particular, one can determine the large-order behavior of the coefficients of the series from the Cauchy dispersion relation

$$Z_0(\lambda) = \frac{1}{2\pi i} \int_0^{-\infty} dw \frac{\mathrm{Disc}_\pi\, Z_0(\lambda)}{w-\lambda}$$

This leads to a precise determination of the large-order behavior of the perturbation series, which exactly matches the analytically computed coefficients (in terms of Riemann $\zeta(2k+1)$).

$$Z = \sum_{n=0}^{\infty} g^{2n} Z_n^{(0)} \qquad\qquad A = 8\pi^2$$

$$Z_n^{(0)} = \frac{\Gamma(n+\frac{9}{2})}{(-2A)^{n+\frac{9}{2}}} \left(c_0 + \frac{Ac_1}{n+\frac{7}{2}} + \frac{A^2 c_2}{(n+\frac{5}{2})(n+\frac{7}{2})} + \ldots \right) + \frac{\Gamma(n+\frac{17}{2})}{(-8A)^{n+\frac{17}{2}}} \left(b_0 + \frac{Ab_1}{n+\frac{15}{2}} + \frac{A^2 b_2}{(n+\frac{13}{2})(n+\frac{15}{2})} + \ldots \right) + \cdots$$

Instanton sectors

The instanton partition function is given in terms of factors which are rational functions of a^2. In particular

$$Z_{k=1} = \tfrac{1}{2}(a^2 - 3) \quad , \qquad Z_{k=2} = \frac{8a^8 + a^6 - 91a^4 - 60a^{32} + 132}{4(4a^2+9)^2}$$

They have poles at the same location as zeros of $Z^{(0)}$

They modify the order of the zeros, but they do not add extra poles.

This property applies to all $\mathcal{N} = 2$ theories.

In general, the instanton partition function $Z_{inst}(a;\varepsilon_1,\varepsilon_2)$ has poles at [Pestun, 0712.2824]

$$a_i - a_j = i\left(n_1\varepsilon_1 + n_2\varepsilon_2\right), \qquad n_1,\ n_2 = 1,2,3,\cdots$$

For SU(2) gauge theories on the four-sphere, this implies poles at

$$2a = \pm i\, n\,, \qquad n = n_1 + n_2$$

There are exactly n of such poles for n_1 =1,2,…,n-1, leading to a pole of order n in Z_{inst} and a pole of order 2n in $|Z_{inst}|^2$, which cancels against a zero of order 2n in the one-loop factor.

As a result, the resurgence properties in instanton sectors are qualitatively the same as in the zero instanton sector: resurgence acts within each instanton sector alone, with no mixing between different instanton sectors.

Thus non-perturbative ambiguities cancel out at fixed topological charge.

In some $\mathcal{N}$ =2 theories, the perturbation series does not exhibit large order *n!* behavior.

For example, consider the partition function for SU(2) SQCD with N_f = 2 massless flavors on **S^4**
The perturbative part is

$$Z_{SQCD}^{SU(2)}\Big|_{k=0} = \int_0^\infty dt\, e^{-\frac{t}{\alpha}}\, \mathcal{B}(t) \quad , \qquad \alpha = \frac{g_{YM}^2}{16\pi^2} ,$$

$$\mathcal{B}(t) = \sqrt{t} \prod_{n=1}^{\infty} \frac{(1+4t/n^2)^{2n}}{(1+t/n^2)^{4n}} = \sqrt{t} \prod_{n=1}^{\infty} \left(1+\frac{4t}{(2n-1)^2}\right)^{2n}$$

The Borel transform is regular. Thus, perturbation series is convergent.
Here there are gauge-theory instantons but they do not induce n! behavior.

This does not imply that the theory contain "less" Feynman diagrams at each loop order.
There are massive cancellations due to the high amount of supersymmetries.

Another example: Pure SU(2) SYM

$$Z_{SYM}^{SU(2)}\Big|_{k=0} = \int_0^\infty dt\, e^{-\frac{t}{\alpha}}\, \mathcal{B}(t) \quad , \qquad \mathcal{B}(t) = \sqrt{t} \prod_{n=1}^{\infty} \left(1+\frac{4t}{n^2}\right)^{2n}$$

Perturbation theory has a finite radius of convergence, $|g_{YM}| < 2.8$

The theory contains instantons, but they do not induce n! behavior.

Higher rank gauge groups

So far we discussed $\mathcal{N}$=2 theories with SU(2) gauge group.

Higher rank groups have been recently discussed by [Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski and Pufu, 1602.05971] and more generally by [M. Honda, 1604.08653].

Starting with the matrix integral, and changing integration variables to "spherical" coordinates $a_i = \sqrt{t}\ \hat{x}_i$ the partition function takes the form

$$Z(g) = \int_0^\infty dt \quad e^{-\frac{t}{g}}\ f(t)$$

where $f(t)$ is given by the integral over the "angular" coordinates $\hat{x}_i$ of the integrand of Z.

Thus $f(t)$ can be viewed as the Borel transform of the original perturbation series.

The structure of Borel singularities is not as explicit as in the SU(2) case.
However, one can argue that the small *t* expansion converges uniformly, which allows to exchange the order of integrals over $\hat{x}_i$ and perturbative sum. There are no singularities along the positive real *t* axes and as a result perturbation series for any $\mathcal{N}$= 2 gauge theory is Borel summable.

Correlation functions of chiral primary operators

Chiral primary operators (CPO): Annihilated by all supercharges of one chirality.

Consider 2 pt functions: $$\left\langle O_n(x)\overline{O}_{\overline{m}}(0)\right\rangle = \frac{G_{n\overline{m}}}{|x|^{2\Delta_n}}\,\delta_{\Delta_n\Delta_{\overline{m}}}$$

These correlation functions can be computed exactly in any $\mathcal{N}=2$ superconformal gauge theory by a construction based on localization recently developed by
[Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski and Pufu, 1602.05971]

The method can be used to determine correlators of the form

$$\left\langle O_{I_1}(x_1)\cdots O_{I_n}(x_n)\overline{O}_{\overline{J}}(y)\right\rangle$$

The deformed matrix models is

$$Z(g_n) = \int d^N a \prod_{i<j}(a_i - a_j)^2\, \mathrm{e}^{-S_{cl}(a)-\delta S(a)}\, z_{1-loop}(a)\quad \left|z_{inst}(a;g^2)\right|^2 \quad , \qquad S_{cl} + \delta S = 2\sum_{n=2}^{N}\pi^{n/2}\,\mathrm{Im}\, g_n \sum_i a_i^n$$

The perturbation series is of the form

$$\left\langle \mathrm{Tr}[\phi^m(0)]\mathrm{Tr}[\overline{\phi}^m(\infty)]\right\rangle = \sum_{n=0}^{\infty} a_{m,n}\left(\frac{g^2}{4\pi}\right)^n$$

This was carried out for SU(2), SU(3) and SU(4) gauge groups. They are asymptotic with the leading behavior

$$a_{m,n} \approx (-1)^n n!$$

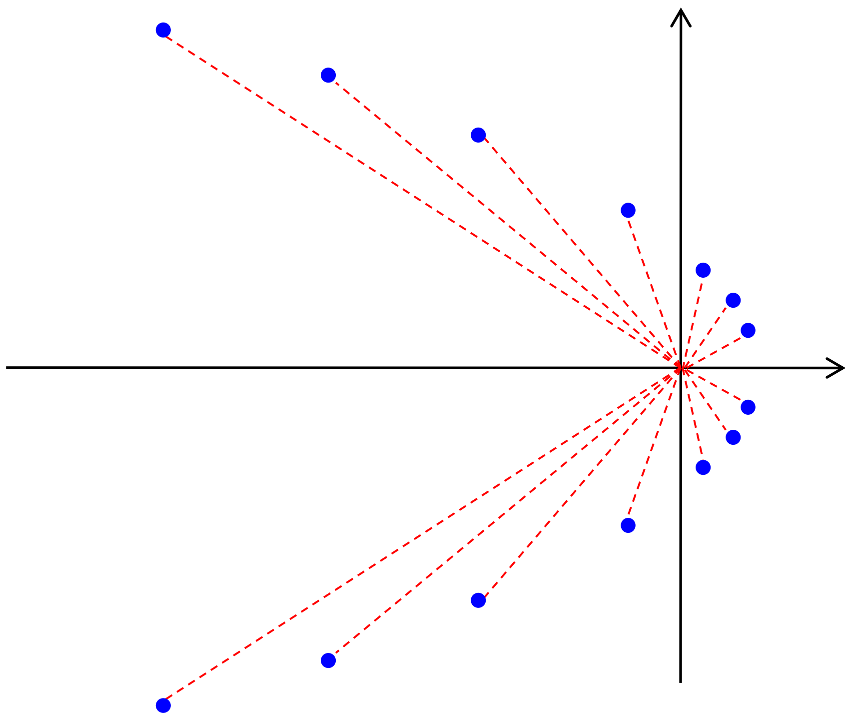
Padé approximants can be used to predict the value of a_{n+1} from the n-loop result with exponentially small error.

Massive gauge theories: The $\mathcal{N}$ = 2* case

This theory is obtained by a massive deformation of $\mathcal{N}$ = 4 theory.
For the SU(2) theory, singularities in the Borel plane now appear at complex values

$$s_{n,\pm} = \frac{1}{2} A (M \pm i n)^2 = \frac{1}{2} A (n^2 + M^2) e^{i\theta_{n,\pm}} \qquad , \qquad A = 8\pi^2$$

As in the $\mathcal{N}$ = 2 superconformal case, the singularities originate from the one-loop factor.
There are now infinite countable Stokes lines, each with one pole. Stokes discontinuities can be computed and the resurgent behavior of Z can be studied in a similar way as in the earlier example.



Decompactification limit and large N phase transitions

Many novel features of massive $\mathcal{N} = 2$ theories arise in the decompactification limit.
At infinite radius all singularities in the Borel plane move to infinity.
The weak coupling expansion becomes an expansion in Exp[- $8\pi^2/\lambda$]

The mass deformation leads to drammatic effects such as phase transitions [JR, Zarembo, 1302.6968, 1309.1004].
For $\mathcal{N} = 2^*$ at large N, they occur at:

$$\sqrt{\lambda} \approx n\pi \qquad , \qquad \lambda \approx 35.4, \quad 88.8, \quad 157.9, \quad 246.7, \ldots$$

Unlike $\mathcal{N} = 4$ theory, physical observables are not smooth as a function of the coupling.
[Similar transitions occur at finite N, i.e. in $\mathcal{N} = 2^*$ $SU(N)$ theory, see Hollowood, Kumar. arxiv:1509.00716]

Weak coupling expansion:

In the weak coupling phase, the free energy is given by the analytic formula

$$F=-\ln Z=-N^2M^2R^2 f(\lambda)\quad , \qquad f(\lambda)=2\sum_{n=1}^{\infty}\ln\left(1-(-1)^n e^{-\frac{8\pi^2 n}{\lambda}}\right)$$

$$\Rightarrow\quad f(\lambda)=2e^{-\frac{8\pi^2}{\lambda}}-3e^{-\frac{16\pi^2}{\lambda}}+\frac{8}{3}e^{-\frac{24\pi^2}{\lambda}}-\frac{7}{2}e^{-\frac{32\pi^2}{\lambda}}+\frac{12}{5}e^{-\frac{40\pi^2}{\lambda}}-4e^{-\frac{48\pi^2}{\lambda}}+\cdots=2\sum_{k=1}^{\infty}(-1)^k\sigma_{-1}(k)e^{-\frac{8\pi^2 k}{\lambda}}$$

The decompactification limit produced another drammatic effect: all perturbative contributions disappeared and only non-perturbative terms have been left.

These non-perturbative terms are *not* instanton contributions.
They represent the operator product expansion that arises after integrating out the massive hypermultiplet.
This leads to pure SYM with dynamically generated scale $\Lambda=M\,e^{-\frac{4\pi^2}{\lambda}}$

In ordinary QCD there are similar terms contributed by **renormalons** and implying large n! behavior.

A striking feature is that the coefficients of the OPE are simple rational numbers
More generally, one would expect an infinite perturbative series multiplying each exponential factor.
The OPE can thus be viewed as a transseries, where each sector has a single term.

The Free Energy From Seiberg-Witten

[J.R., 1411.2602]

SW computes the holomorphic prepotential $\mathcal{F}(a_k)$ in flat spacetime.

Write $$Z = \int Da \quad |\mathcal{Z}|^2$$

$$Nekrasov: \quad 2\pi i \mathcal{F}(a) = \lim_{\varepsilon_{1,2}\to 0} \varepsilon_1 \varepsilon_2 \ln \mathcal{Z}$$

or

For S^4, $\epsilon_1 = \epsilon_2 = 1/R$

$$2\pi i \mathcal{F}(a) = \lim_{R\to\infty} \frac{1}{R^2} \ln \mathcal{Z}$$

The identity includes the coupling of the scalar to the curvature –proportional to ***R²***
(it provides the classical contribution)

This gives:

$$Z = \int Da \quad \exp[R^2 (2\pi i \mathcal{F}(a_k) - 2\pi i \bar{\mathcal{F}}(a_k))]$$

The partition function Z can then be computed by saddle-point [J.R., 1411.2602]

Thus
$$\ln Z = -F = R^2 (2\pi i \mathcal{F}(a_k) - 2\pi i \bar{\mathcal{F}}(a_k))$$

Saddle-point equations are
$$\frac{\partial S}{\partial a_k} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{F}}{\partial a_k} = 0$$

But
$$\frac{\partial \mathcal{F}}{\partial a_k} \equiv a_{Dk}$$

Thus the saddle occurs at a particular degenerating point of the SW curve where all periods a_{Dk} vanish

Large *N*:

Since the integrand is exp (N^2 f(a)), the large N physics is also extracted from the condition $a_{Dk} = 0$

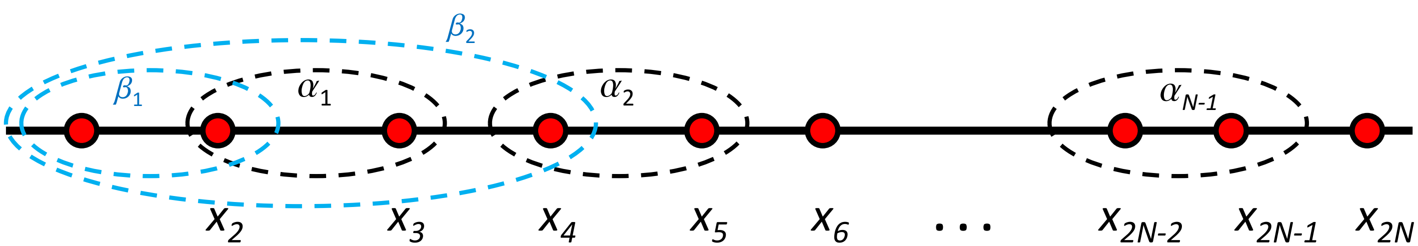
What does the condition $a_{Dk} = 0$ mean? (more generally, massless dyon singularity)

Consider a SW curve $y^2 = p(x)$, $p(x) = x^{2N} + \ldots$, $2N$ branch points $x_1, \cdots, x_{2N}$

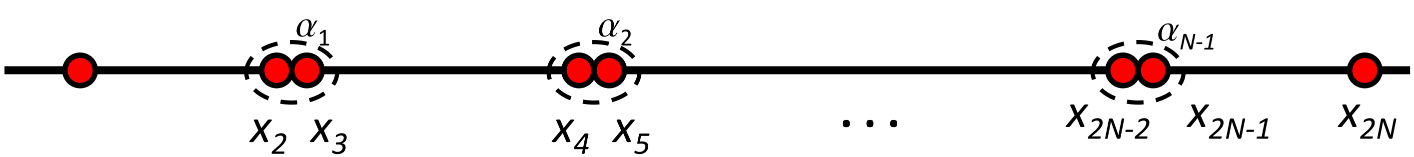
p(x) depends on:

- masses
- couplings
- moduli parameters $\{u_k\}$, $k = 1, \ldots, N-1$

Define homology cycles α_n and β_n



$a_k = \oint_{\beta_k} \lambda$, $a_{Dk} = \oint_{\alpha_k} \lambda$, λ = SW meromorphi c form



N-1 conditions for N-1 unknowns $\{u_k\}$

Substituting the solution for $\{u_k\}$ into the prepotential, we find the free energy $F(M,\lambda)$ at large N

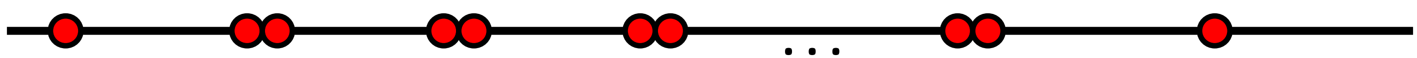
Example: SQCD [J.R., 1504.02958]

Consider the Seiberg-Witten curve that describes $\mathcal{N} = 2$ SU(N) gauge theory coupled to $2N_f$ fundamental hypermultiplets of mass M

$$y^2 = C(x)^2 - G(x) \equiv p(x)$$

$$C(x) = \prod_{i=1}^{N} (x - u_i) \quad , \quad \sum_{i=1}^{N} u_i = 0 \quad , \quad G(x) = \Lambda^{2N-2N_f} (x+M)^{N_f} (x-M)^{N_f}$$

We are interested in the degenerating limit



We must demand that N-1 roots of $p(x)$ are double roots, i.e. we must find the u_i for which $p(x)$ takes the form

$$p(x) = (x-a)(x-b) \prod_{i=1}^{N-1} (x - c_i)^2$$

The general condition is that $p'(x)$ shares the same roots c_i as $p(x)$

$$p'(x) = 2\prod_{i=1}^{N} (x-u_i)^2 \sum_{i=1}^{N} \frac{1}{x-u_i} - N_f \Lambda^{2N-2N_f} (x^2 - M^2)^{N_f} \left(\frac{1}{x+M} + \frac{1}{x-M} \right)$$

Using $p(x) = 0$

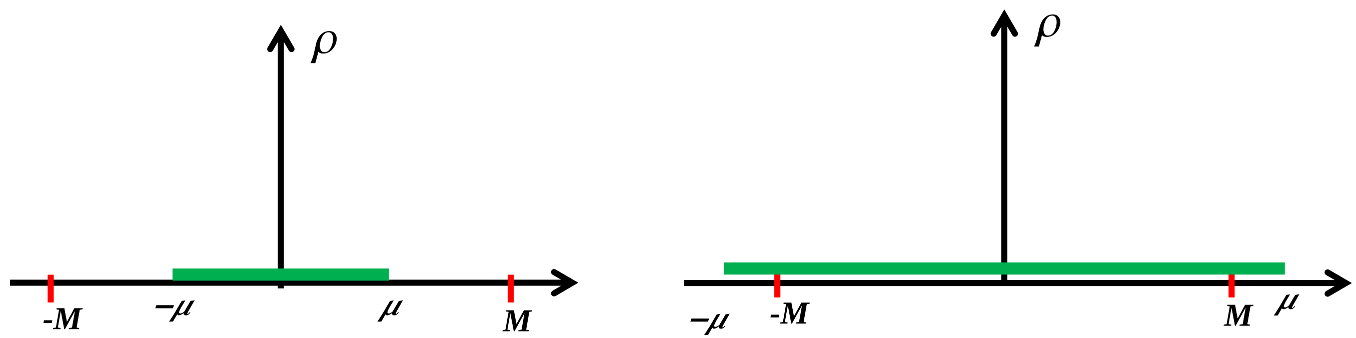
$$p'(x) = \Lambda^{2N-2N_f} (x^2 - M^2)^{N_f} \left(\sum_{i=1}^{N} \frac{2}{x-u_i} - \frac{N_f}{x+M} - \frac{N_f}{x-M} \right)$$

In the continuum, large *N* limit, the equation is transformed into an integral equation

$$-2\int_{-\mu}^{\mu} dy\ \frac{\rho(y)}{x-y} = \frac{\zeta}{x+M} + \frac{\zeta}{x-M} \qquad , \qquad \zeta \equiv \frac{N_f}{N}$$

reproducing exactly the same integral equation that one finds from localization.
This equation implies the existence of a phase transition, since the solution is different if x lies inside or outside the integration region (which in turns depends on the condition 2Λ/M> or < 1)

Two cases:



The parameter μ is determined by demanding that the roots also solve *p(x)* = 0

$$\prod_{i=1}^{N}(x-u_i)^2 = \Lambda^{2N-2N_f}(x^2-M^2)^{N_f}$$

Taking the logarithm and going to the continuum, we find

$$2\int_{-\mu}^{\mu} dy\ \ \rho(y)\ln\frac{(x-y)^2}{\Lambda^2} = \zeta\ln\frac{(x^2-M^2)^2}{\Lambda^4}$$

which reproduces the saddle-point equation derived from the localization partition function

Operator product expansion in $\mathcal{N} = 2$ SQCD at large *N*

Use exact results to compute non-perturbative physics.
Example: all-order OPE.

Consider dynamically generated scale $\Lambda_{\text{eff}} << M$. Then observables admit an expansion

$$O = (\Lambda_{eff})^{\Delta} \sum_{n=0}^{\infty} C_n \left(\frac{\Lambda_{eff}}{M} \right)^{2n}$$

-The mass M in the denominator arises from expanding the effective action in local operators.
-Powers of Λ_{eff} in the numerator come from the VEV of the local operators generated by the OPE.

These VEV involve non-perturbative physics and in ordinary QCD are difficult to calculate.

From the localization formula for SQCD with N_f flavors, we can now compute the OPE.

For M >> Λ, we can expand the free energy in inverse powers of the mass:

$$F = \Lambda_{eff}^2 \left(-2 + \zeta \frac{\Lambda_{eff}^2}{M^2} + \tfrac{2}{3} \zeta (1-2\zeta) \frac{\Lambda_{eff}^4}{M^4} + \tfrac{4}{3} \zeta (1-2\zeta)(5-8\zeta) \frac{\Lambda_{eff}^6}{M^6} + \ldots \right) \quad , \qquad \zeta = \frac{N_f}{N}$$

Where, $\Lambda_{eff} = M^{\zeta} \Lambda^{1-\zeta}$, $\quad \frac{\Lambda_{eff}^{2n}}{\mu^{2n}} = \exp\left(\frac{n}{\beta\, \alpha(\mu)} \right)$

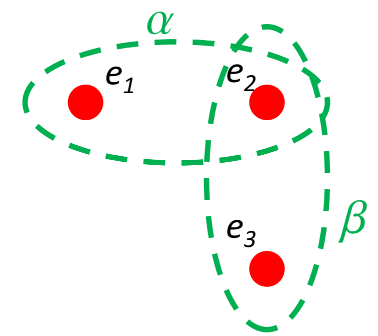
Does this imply the presence renormalon singularities in the Borel plane at *t* = n/β ?
Not in this case. No *n!* perturbative series

SQCD with SU(2) gauge group

Example: $\mathcal{N}$ = 2 SU(2) SYM with two flavors

$$y^2 = (x^2 - \tfrac{1}{64}\Lambda^4)(x-u) + \tfrac{1}{4}M^2\Lambda^2 x - \tfrac{1}{32}M^2\Lambda^4$$
$$= (x-e_1)(x-e_2)(x-e_3)\,,$$

$$\lambda = -\tfrac{\sqrt{2}}{4\pi}\frac{y\,dx}{x^2 - \tfrac{1}{64}\Lambda^4}$$



The branch points are at the three roots e_1, e_2, e_3 of the cubic polynomial.

- The cycle α defining a_D *surrounds* e_1 , e_2 .
- The cycle β defining a *surrounds* e_2 , e_3 .

At large R, the partition function is determined by a saddle point [J.R., 1411.2602]

$$Z = \int da\;\; e^{R^2(2\pi i\mathcal{F} - 2\pi i\bar{\mathcal{F}})}$$

Saddle points occur at singularities of the SW curve.
These are located at

$$u_1 = -M\Lambda - \frac{\Lambda^2}{8} \quad , \quad u_2 = M\Lambda - \frac{\Lambda^2}{8} \quad , \quad u_3 = M^2 + \frac{\Lambda^2}{8}$$

At $u \to u_3$, and $\boldsymbol{M < \Lambda/2}$, one finds $e_1 \to e_2$ and $\quad a_D = \dfrac{\partial\mathcal{F}}{\partial a_k} = 0$

If $\boldsymbol{M > \Lambda/2}$, then $e_2 \to e_3$, the cycle α does not shrink. Then a_D is different from 0 in the whole complex u-plane.

At $\boldsymbol{M = \Lambda/2}$, all e_1 , e_2 , e_3 branch points collapse. At this point $a \to M$ and the hypermultiplet becomes massless. It is an *Argyres-Douglas* point, first found in [Argyres, Plesser, Seiberg Witten], where mutually non-local states become massless. *Thus this point represents the critical point of our phase transitions.*

The Argyres-Douglas point is also a special point in the resurgent properties of the associated quantum mechanical system [Demulder, Dorigoni and Thompson, 1604.07851]

Summarizing:

The saddle-point a_D=0 occurs in the strong coupling phase **Λ>2M**

It lies at the singularity u = u_3

Is there any saddle-point in the weak coupling phase **Λ<2M ?**

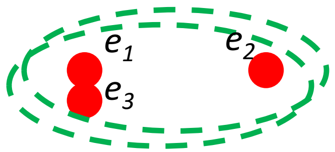
Let us look for complex saddle-points. More generally, the condition is

$$S = R^2 (2\pi i \mathcal{F}(a_k) - 2\pi i \bar{\mathcal{F}}(a_k)) , \qquad \frac{\partial S}{\partial a_k} = 0 \quad \Rightarrow \quad \mathrm{Im}[\frac{\partial \mathcal{F}}{\partial a_k}] = 0$$

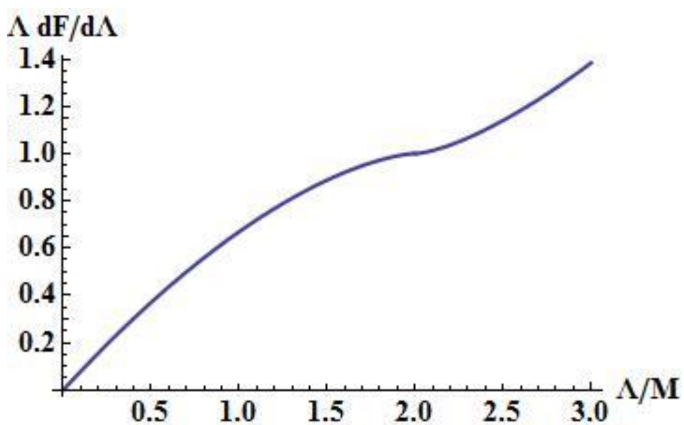
Hence, Im[a_D]=0.

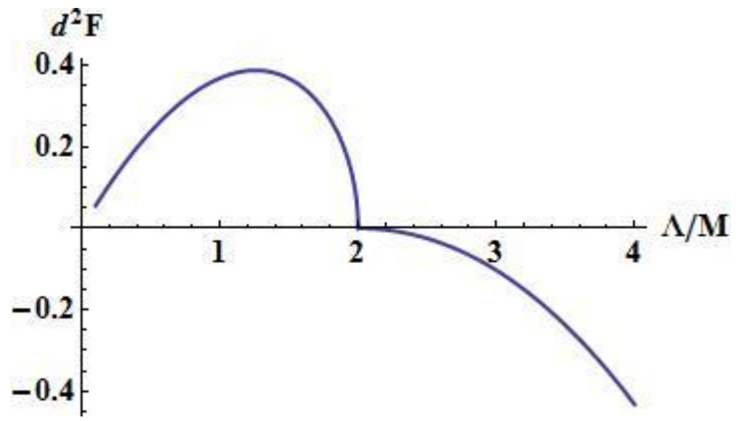
In the weak coupling phase, **Λ<2M**, one finds that this is satisfied at the singular point

$$u = u_2 = M\Lambda - \frac{\Lambda^2}{8}$$



This represents the dyon singularity $\qquad a - a_D = 0$





THUS THE ARGYRES-DOUGLAS POINT REPRESENTS THE CRITICAL POINT OF A 3rd-ORDER PHASE TRANSITION

Nekrasov-Shatashvili limit and quantum mechanics

Consider Nekrasov partition function with equivariant parameters $\varepsilon_1, \varepsilon_2$

Nekrasov-Shatashvili limit: $\varepsilon_2 \to 0\,, \qquad \varepsilon_1 \equiv \varepsilon = \text{fixed}$

In this limit, the supersymmetric vacua are related to the eigenstates of a quantum integrable system.
The non-zero deformation parameter ϵ plays the role of $\hbar$
The Seiberg-Witten prepotential can be constructed from the Bohr-Sommerfeld periods of quantum mechanical systems:

Mathieu: $\mathcal{N}=2:\text{pure SYM}: \quad V(x)=\sin^2 x$

Lamé: $\mathcal{N}=2^*: \quad V(x)=\nu(\nu+1)k^2\,\text{sn}(z,k^2)$

Whittaker-Hill: $\mathcal{N}=2\ \text{SQCD with}\ N_f=2: \quad V(x)=\sin^2 x+\eta^2\sin^4 x$

- Nekrasov, Shatashvili, 0908.4052
- Mironov , Morozov, 0910.5670

The WKB expansion is asymptotic and requires the construction of the associated transseries.
Discussions on resurgence and Stokes phenomena can be found in

- Dunne, Ünsal, 1306.4405, 1603.04924; Başar, Dunne, 1501.05671
- Kashani-Poor, Troost, 1504.08324
- Piatek, Pietrykowski, 1604.03574
- Ashok, Jatkar, John, Raman, Troost, 1604.05520
- Demulder, Dorigoni and Thompson, 1604.07851

Ω deformation and Super Yang-Mills theory on the Ellipsoid

Some curved spaces different from round spheres still admit rigid supersymmetry.
One example is the 4d ellipsoid [Hama-Hosomichi, 1206.6359]

$$\frac{x_0^2}{R_0^2}+\frac{x_1^2+x_2^2}{R_1^2}+\frac{x_3^2+x_4^2}{R_2^2}=1$$

The partition function Z for SYM depends on the gauge coupling, on masses, and on the squashing parameter $Q = b+1/b$, $b = \mathrm{sqrt}[R_1/R_2]$

The ellipsoid is particularly interesting for the AGT connection: Z is connected with correlation functions of 2D Liouville (or Toda CFTs) with central charge $c = 1+6Q^2$.

The one-loop determinant is given in terms of the double infinite product

$$\Upsilon(x)=\prod_{m,n\geq 0}(x+mb+nb^{-1})(x-mb-nb^{-1}-Q)$$

The contribution of localized instantons near the poles is described by Nekrasov's instanton partition function with equivariant parameters

$$\epsilon_1 = 1/R_1 \quad , \qquad \epsilon_2 = 1/R_2$$

Can we describe the Nekrasov – Shatashvili limit as a squashing limit of the ellipsoid?

The one-loop factor of Hama-Hosomichi looks different from the Nekrasov's one-loop factor
However, the key is the **equivariant mass parameter** [Okuda, Pestun, arXiv:1004.1222] :

$$iM = iM_0 + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) = iM_0 + \frac{1}{2R_1} + \frac{1}{2R_2} \qquad or \qquad iM\sqrt{R_1R_2} = iM_0\sqrt{R_1R_2} + \tfrac{1}{2}Q$$

In studying massless theories, Hama-Hosomichi set to zero M_0 . But the physical mass is M.

Example: $\mathcal{N}$ =2* theory on the ellipsoid

The partition function can be written as by

$$Z^{\mathcal{N}=2^*}(g) = \int d^N a \ \mathrm{e}^{-\frac{8\pi^2}{g^2}R_1R_2\sum_i a_i^2} \prod_{i<j} \frac{(a_i - a_j)^2 f(a_i - a_j)^2}{f(a_i - a_j + M) f(a_i - a_j - M)} \ |z_{inst}(a)|^2$$

$$f(x) = \prod_{\substack{m,n=0 \\ (m,n)\neq(0,0)}}^{\infty} \left(1 + \frac{x^2}{(n/R_1 + m/R_2)^2}\right) e^{-\frac{x^2}{(n/R_1+m/R_2)^2}}$$

$z_{inst}(a)$ is given by the Nekrasov partition function with parameters $\epsilon_1 = 1/R_1$, $\epsilon_2 = 1/R_2$
For example, for SU(2)

$$z_{inst}^{(k=1)}(a) = \frac{(4M^2 - (\varepsilon_1 - \varepsilon_2)^2)(4M^2 + 3(\varepsilon_1 + \varepsilon_2)^2 - 16a^2)}{8\varepsilon_1\varepsilon_2((\varepsilon_1 + \varepsilon_2)^2 - 4a^2)}$$

The one-loop factor exactly matches the one-loop term in the Ω deformation (c.f. Billo et al, 1302.0686)

Squashing limit:

At large R_2, the partition function takes the form

$$Z = \int da \quad \exp\left[-R_2\left(S_{cl}(a) + S_1(a) + S_{inst}(a)\right)\right]$$

$$S_{cl}(a) = \frac{16\pi}{g^2} R_1 a^2$$

$$S_1(a) = -2ia\ln\frac{\Gamma(1+2iR_1a)^2\Gamma(1-2iR_1(a+M))\Gamma(1-2iR_1(a-M))}{\Gamma(1-2iR_1a)^2\Gamma(1+2iR_1(a+M))\Gamma(1+2iR_1(a-M))} + \frac{1}{R_1}\ln\frac{H(2aR_1)^2}{H(2(a+M)R_1)H(2(a-M)R_1)}$$

The instanton contribution exponentiates,

$$z_{inst}(a) = \exp\left[-R_2 S_{inst}(a)\right] \to \exp\left[-R_2(c_1(a)q + c_2(a)q^2 + ...)\right]$$

$$c_1 = \frac{(4a^2R_1^2+3)}{8R_1(a^2R_1^2+1)}, \quad c_2 = \frac{48a^4R_1^4(4a^4R_1^4+26a^2R_1^2+53)+a^2R_1^2 2107+631}{256R_1(a^2R_1^2+4)(a^2R_1^2+1)^3}$$

Z is determined by a saddle-point. The solution can be obtained by expanding at large $a\,R_1$ the Gamma and the Barnes G-function. This produces an *asymptotic* expansion in powers of

$$\left(\frac{1}{R_1 a}\right)^{2n} = \left(\frac{\hbar}{a}\right)^{2n}$$

It is in correspondence with the WKB expansion of the Lamé quantum mechanical system.

Concluding remarks

1. SU(2) SQCD with $N_f = 2$

a) Decompactification limit of $\mathbf{S}^4$ leads to the theory on $\mathbf{R}^4$ which has two phases separated by Argyres-Douglas point.
b) Nekrasov-Shatashvili limit leads to a theory which is in correspondence with Whittaker-Hill quantum mechanical system.
The theory on $\mathbf{R}^4$, with phase transitions, is seen when also R_1 goes to infinity,
in the classical limit $\hbar \to 0$
(which is the leading term in the asymptotic semiclassical expansion)

2. Nekrasov equivariant partition function admits an interpretation as $\mathcal{N}=2$ gauge theory on an ellipsoid with $\epsilon_1 = 1/R_1$, $\epsilon_2 = 1/R_2$

Squashing limit of $\mathcal{N}=2$ gauge theories on ellipsoids (R_2 infinity , R_1 fixed) coincides with Nekrasov-Shatashvili limit.

The resulting partition function can be expanded in a/R_1 .
This expansion is in correspondence with the WKB of a quantum mechanical system.
It would definitely be interesting to exploit the ellipsoid interpretation, in particular consider other possible squashing limits.