A Smale-Barden manifold admitting K-contact but not Sasakian structure

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Smale-Barden manifold K-contact but not Sasakian

1. Kähler and symplectic structures

Main objective

Describe and classify (compact) manifolds with different geometrical structures.

Let M be a smooth manifold,
$J \in \mathrm{End}(TM)$ with $J^2 = -\mathrm{id}$,
J is called almost-complex structure,
(T_pM, J) are complex vector spaces.

Nijenhius tensor
$N_J(X, Y) := [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY]$

Theorem (Newlander-Niremberg)

If $N_J = 0$ then (M, J) has a complex atlas. We say that J is integrable.

Kähler manifolds

A $2n$-dimensional Kähler manifold (M, J, ω) consists of:

- (M, J) a complex manifold, $J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{id}$, $N_J = 0$,
- $h : T_pM \times T_pM \to \mathbb{C}$ a hermitian metric, $h = g + i\omega$,
  g is a riemannian metric, $\omega \in \Omega^2(M)$,
- $\omega(u, v) = g(u, J\,v)$ (compatibility of ω and J),
- $g(Ju, Jv) = g(u, v)$,
- $\omega \in \Omega^{1,1}(M)$ the Kähler form.
  Locally, $\omega = \frac{-i}{2} \sum h_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$, where $h = \sum h_{i\bar{j}}\, dz_i \cdot d\bar{z}_j$
- ω is non-degenerate, i.e. $\omega^n > 0$,
- $d\omega = 0$.

S is Kähler $\iff \nabla J = 0$,
i.e. S is a riemannian manifold with holonomy contained in $\mathrm{U}(n)$.

Symplectic manifolds

Definition

A symplectic manifold (M, ω) is a smooth $2n$-manifold with $\omega \in \Omega^2(M)$, with $\omega^n > 0$ and $d\omega = 0$.

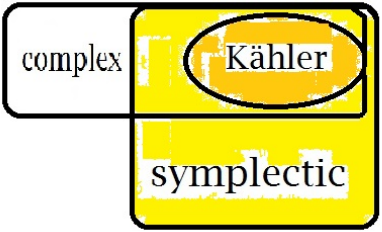
A symplectic manifold (M, ω) admits a compatible almost-complex structure J.
In general it is not integrable.
(M, J, ω) is called almost-Kähler.

We have the following inclusion

$$\{\text{Kähler manifolds}\} \subset \{\text{symplectic manifolds}\}$$

which has been largely studied (Thurston, Gompf, McDuff, Tralle-Oprea, Babenko-Taimanov, Fernández-M, Cavalcanti, Bazzoni, etc).

Kähler vs. symplectic



Main techniques to find manifolds which admit symplectic but not Kähler structures:

- Parity of odd-degree Betti numbers,
- hard Lefschetz property,
- Kähler fundamental groups,
- formality (rational homotopy theory),
- in dimension $2n = 4$, Enriques classification of complex surfaces.

2. Sasakian and K-contact structures

In odd dimensions, we have several types of geometric structures that parallel this situation:

Kähler	$\subset$	Symplectic
$\vert$		$\vert$
Sasakian	$\subset$	K-contact

Question

Determine which manifolds admit Sasakian and K-contact structures. Find manifolds which admit K-contact structures but not Sasakian structures.

Definition of Sasakian structure

Let M be a $(2n+1)$-dimensional manifold.
An *almost contact metric structure* is given by (η, ξ, ϕ, g), where:

- η is a 1-form, $\mathcal{D} = \ker \eta$ codimension one distribution.
- ξ is a nowhere vanishing vector field with $\eta(\xi) = 1$. So $TM = \mathcal{D} \oplus \langle \xi \rangle$.
- $\phi : TM \to TM$, $\phi^2 = -\mathrm{id} + \xi \otimes \eta$. So $\phi(\xi) = 0$ and $\phi|_{\mathcal{D}}$ is an almost-complex structure.
- g is a Riemannian metric with $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$. Thus $TM = \mathcal{D} \oplus \langle \xi \rangle$ is orthogonal, and ϕ is isometric on $\mathcal{D}$.

The fundamental 2-form is $F(X, Y) = g(\phi X, Y)$.
So $F(\phi X, \phi Y) = F(X, Y)$ and $\eta \wedge F^n \neq 0$.

Equivalently, M is almost contact if and only if TM has a reduction of structure group to $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n+1)$.

Definition of Sasakian structure

The almost contact structure (η, ξ, ϕ, g) is *contact metric* if $F = d\eta$ (so η is a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$).

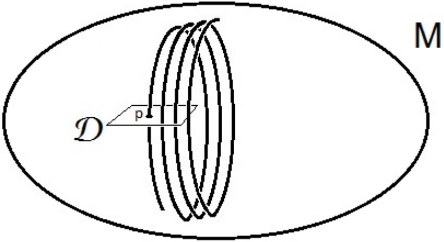
Definition

A *Sasakian structure* is a contact metric structure (η, ξ, ϕ, g) whose Nijenhuis tensor satisfies $N_\phi = -d\eta \otimes \xi$.

$$N_\phi(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

Definition of Sasakian structure

The vector field ξ is a Killing vector field, and the transversal structure is "Kähler".



Alternative definition

Let $X = C(M) = M \times \mathbb{R}$, $g_{C(M)} = t^2 g + dt^2$, be the *cone of M*.
Let $J : TX \to TX$, $J|_{\mathcal{D}} = \phi$, $J(\xi) = t\,\frac{\partial}{\partial t}$.
Then J is integrable, $N_J = 0 \iff N_\phi = -d\eta \otimes \xi$.
So M is Sasakian $\iff C(M)$ is Kähler.

K-contact structures

Definition

Let M be a $(2n+1)$-dimensional manifold.
A K-contact structure is a contact metric structure (η, ξ, ϕ, g) where ξ is a Killing vector field, i.e., $\mathcal{L}_\xi g = 0$.

The Killing condition makes sense of a “transversal structure”

The transversal structure to the Reeb foliation is “symplectic” (or more precisely, “almost-Kähler”).

Sasakian vs. K-contact structures

Question (Boyer-Galicki)

Are there (compact) manifolds with K-contact structure but with no Sasakian structures?

Main techniques:

- In dimensions $2n+1 \geq 7$. Topological properties:
  - parity of odd-degree Betti numbers,
  - hard Lefschetz property,
  - Sasaki fundamental groups,
  - formality.

  (Boyer-Galicki, Cappelletti Montano-de Nicola-Marrero-Yudin, Hajduk-Tralle, Biswas-Fernández-M-Tralle, etc).
- In dimension $2n+1=5$. Classification of complex surfaces (Boyer-Galicki, Kollár, M-Rojo-Tralle, Cañas-M-Rojo-Viruel).

3. Smale-Barden manifolds

Simply connected compact 5-dimensional manifolds were classified by Smale and by Barden.

Invariants:

- Second homology group

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus \big(\bigoplus_{p,i} \mathbb{Z}_{p^i}^{c(p^i)} \big),$$

  where $k = b_2(M)$. Moreover $c(p^i)$ is even except possibly for $c(2)$.

- The Barden invariant: $i(M) = j$, where the second Stiefel-Whitney class map

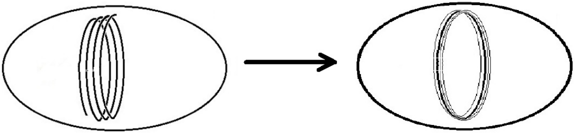
$$w_2 : H_2(M,\mathbb{Z}) \to \mathbb{Z}_2$$

  is zero on all but one summand $\mathbb{Z}_{2^j}$. If $c(2)$ is odd then $i(M) = 1$.

M is spin ($w_2 = 0$) if $i(M) = 0$.

Proposition (Rukimbira, M-Tralle)

Let M be a compact manifold with a Sasakian structure (resp. K-contact structure). Then M admits a Sasakian structure (resp. K-contact structure) foliated by circles.



Let $X = M/S^1$ be the space of orbits. Then

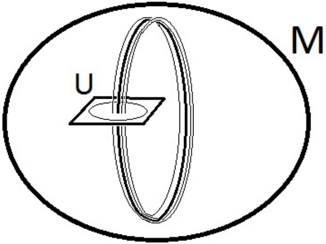
$$\pi : M \longrightarrow X.$$

is a Seifert bundle and X is a cyclic orbifold.

Seifert bundles

Locally we have (where $U = B_\epsilon(p) \cap \mathcal{D}_p$),

$$\pi : (S^1 \times U)/\mathbb{Z}_m \longrightarrow U/\mathbb{Z}_m$$



The local model around $x = \pi(p)$ is $\mathbb{C}^n/\mathbb{Z}_m$ with $\varepsilon = e^{2\pi i/m}$

$\varepsilon \cdot (s, z_1, \ldots, z_n) = (\varepsilon s, \varepsilon^{l_1} z_1, \ldots, \varepsilon^{l_n} z_n)$, for M

$\varepsilon \cdot (z_1, \ldots, z_n) = (\varepsilon^{l_1} z_1, \ldots, \varepsilon^{l_n} z_n)$, for X

where $\gcd(m, l_1, \ldots, l_n) = 1$.

Cyclic orbifolds

The orbifold singularities of a cyclic orbifold X consist of:

- Isotropy (smooth) surfaces D_i with isotropy coefficient m_i. Modelled on $\mathbb{C} \times \{0\} \subset \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_{m_i})$.
- If two isotropy surfaces intersect, they do in pairs, transversely, and the coefficients m_i, m_j satisfy $\gcd(m_i, m_j) = 1$.
- Isotropy points with link a lens space $S^3/\mathbb{Z}_d$. They can appear isolated, at an isotropy surface or at the intersection of two isotropy surfaces.

If the coefficients m_i, m_j are not coprime, then $D_i \cap D_j = \emptyset$.
The surfaces D_i intersect transversally.
Let $P \subset X$ be the singular points. Then at most two surfaces through each point of P.

Sasakian and K-contact Seifert bundles

- If M is Sasakian, then X is a Kähler cyclic orbifold. The isotropy locus are complex curves.
- If M is K-contact, them X is a symplectic cyclic orbifold. The isotropy locus are symplectic surfaces.

M is Sasakian (K-contact) when the (orbifold) Chern class of $\pi : M \to X$ is given by the (orbifold) Kähler (symplectic) form of (X, J, ω),

$$c_1(M) = [\omega] \in H^2_{orb}(X) = H^2(X, \mathbb{R}).$$

Note that always $c_1(M) \in H^2(X, \mathbb{Q})$.

Fundamental group

Let $M \to X$ be a Seifert bundle over a 4-orbifold X, and let D_i be the isotropy surfaces.

Definition

The orbifold fundamental group is

$$\pi_1^{orb}(X) = \pi_1(X - \cup D_i)/\langle \gamma_i^{m_i} = 1 \rangle,$$

where γ_i is a small loop around the isotropy surface D_i.

We have an exact sequence

$$\cdots \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(M) \to \pi_1^{orb}(X) \to 1$$

Topology of Seifert bundles

Proposition (Kollár)

Let $\pi : M \to X$ be a semi-regular Seifert circle bundle, with $D_i \subset X$ isotropy surfaces with coefficients m_i and genus g_i. Recall that $\gcd(m_i, m_j) = 1$ if $D_i \cap D_j \neq \emptyset$.

Then

$$H_1(M,\mathbb{Z}) = 0 \iff \begin{cases} H_1(X,\mathbb{Z}) = 0, \\ H^2(X,\mathbb{Z}) \twoheadrightarrow \oplus H^2(D_i, \mathbb{Z}/m_i), \text{ surjective} \\ c_1(M/\mathbb{Z}_m) \in H^2(X,\mathbb{Z}) \text{ is primitive}, m = \operatorname{lcm}(m_i). \end{cases}$$

Moreover, $H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus (\mathbb{Z}/m_i)^{2g_i}$, where $H_2(X,\mathbb{Z}) = \mathbb{Z}^{k+1}$.

We read the genus g_i, the isotropy coefficients m_i, and whether they are disjoint (if $\gcd(m_i, m_j) > 1$) from $H_2(M,\mathbb{Z})$.
We also read $b_2(X) = k + 1$.

5. Statement of main result

Main Problem (Boyer-Galicki)

Construct a Smale-Barden manifold that admits a K-contact structure but not a Sasakian structure.

Partial results:

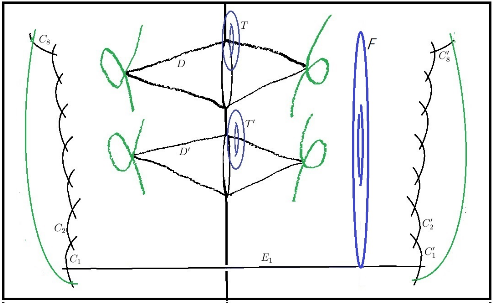
- M-Rojo.Tralle, Homology Smale-Barden manifolds with K-contact and Sasakian structures, Inter. Math. Research Notices, 2020, No. 21, 7397-7432. (homology Smale-Barden means $H_1(M, \mathbb{Z}) = 0$).
- Cañas-M-Rojo-Viruel, A K-contact simply connected 5-manifold with no semi-regular Sasakian structure, Publ. Mathemàtiques. (semi-regular means $P = \emptyset$).

Main Theorem (arxiv:2011.05783)

There exists a Smale-Barden manifold which admits a K-contact structure but does not admit a Sasakian structure.

6. Construction of K-contact manifold

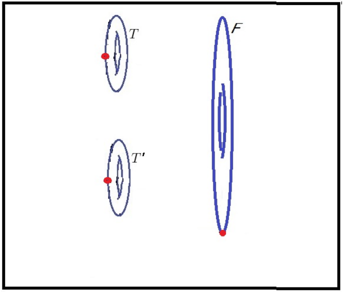
Take a rational elliptic surface with three nodal curves and one singular fiber of type I_9 (cycle of 9 rational curves of self-intersection -2).
Take a Gompf connected sum of two copies



$\chi = 24$ hence $b_2 = 22$.
Perturb ω to make the Lagrangian (-2)-spheres $\to$ symplectic.
Contract chain of 17 rational curves and the two (-2)-curves.

Construction of K-contact manifold

Get orbifold with 3 singular points, 3 disjoint symplectic tori, $b_2 = 3$.



Take the isotropy surfaces to be:

- $T_0 = T$, $T_1 = 2T + D$, $T_n = nT_1$, with $g_n = n^2 + 1$
- $T'_0 = T'$, $T'_1 = 2T' + D'$, $T'_m = mT'_1$, with $g_m = m^2 + 1$
- $A = 2F + 9E_1 + 8(C_1 + C'_1) + 7(C_2 + C'_2) + \ldots + (C_8 + C'_8)$, with $g_A = 10$.

Construction of K-contact manifold

Choose different primes p_{nm}, and isotropy coefficients (N large):

$$m_{T_n} = \prod_{m=0}^{N} p_{nm},$$

$$m_{T'_m} = \prod_{n=0}^{N} p_{nm}^2 ,$$

$$m_A = \prod_{n,m=0}^{N} p_{nm}^3 .$$

The Seifert bundle $M \to X$ is a K-contact Smale-Barden manifold which is spin and its homology is

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus \bigoplus_{n,m=0}^{N} \left(\mathbb{Z}_{p_{nm}}^{2n^2+2} \oplus \mathbb{Z}_{p_{nm}^2}^{2m^2+2} \oplus \mathbb{Z}_{p_{nm}^3}^{20} \right).$$

7. The Smale-Barden manifold is not Sasakian

If M admits a Sasakian structure, then it admits a Sasakian structure foliated by circles.
Hence $M \to Y$ is a Seifert bundle.
Y is a complex manifold with cyclic singularities $P \subset Y$.

As

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^2 \oplus \bigoplus_{n,m=0}^{N} \left(\mathbb{Z}_{p_{nm}}^{2n^2+2} \oplus \mathbb{Z}_{p_{nm}^2}^{2m^2+2} \oplus \mathbb{Z}_{p_{nm}^3}^{20} \right),$$

$b_2(Y) = 3$.
The isotropy locus: for each $n, m \in \{1, 2, \ldots, N\}$, there are $D_1^{nm}, D_2^{nm}, D_3^{nm}$, disjoint complex curves of genus g_n, g_m, g_A.

Step 1. $\#P$ universally bounded

- $K + D_1 + D_2 + D_3$ effective,
  $\mathcal{O}(K) \to \mathcal{O}(K + D_1 + D_2 + D_3) \to \oplus \mathcal{O}_{D_i}(K_{D_i})$,
  $g_i \geq 1$ and $H^1 = 0$.
- $K + D_1 + D_2 + D_3$ log canonical.
- $K + D_1 + D_2 + D_3$ nef (this is tricky).

Then $e_{orb}(Y - (D_1 \cup D_2 \cup D_3)) \geq 0 \implies$ bound on $\#P$.

Step 2. Number of $\{g_n, g_m, g_A\}$ with curves through singular points $\to$ bounded.

Step 3. Let $D_1^2 = m_1$, $D_2^2 = -m_2$, $D_3^2 = -m_3$. They are integer if the D_i do not pass through singular points.

$$K^2 = \frac{(2g_n - 2 - m_1)^2}{m_1} - \frac{(2g_m - 2 + m_2)^2}{m_2} - \frac{(18 + m_3)^2}{m_3}$$

Then $-C_1 \leq K^2 \leq C_2$, universally bounded.

Case $K^2 \leq 0$

The canonical bundle is

$$K = \frac{2g_n - 2 - m_1}{m_1} D_1 - \frac{2g_m - 2 + m_2}{m_2} D_2 - \frac{18 + m_3}{m_3} D_3$$

It cannot be $K < 0$ since $K + D_2$ is effetive. Hence $0 < m_1 < 2g_n - 2 = 2n^2$.

We use $\{g_n, 1, g_A\}$ and using bounds of

$$K^2 = \frac{(2g_n - 2 - m_1)^2}{m_1} - m_2 - \frac{(18 + m_3)^2}{m_3},$$

we bound possible denominators of K^2

Changing numerator $2g_n - 2 = 2n^2$, we produce a diophantine equation with no solutions.

Case $K^2 > 0$

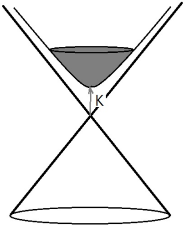
Can select sets $\mathcal{S}_a$ of fixed size for $\{g_n, g_a, g_A\}$, $n \in \mathcal{S}_a$, such that $(D_1^{na}, D_2^{na}, D_3^{na})$ are not projectively equivalent.

$(Q_n, R_n, S_n) = (D_1^{na}/\sqrt{(D_1^{na})^2}, D_2^{na}/\sqrt{(D_2^{na})^2}, D_3^{na}/\sqrt{(D_3^{na})^2})$ basis of hyperbolic space. They are ϵ-apart.

Recall $K^2 = \frac{(2g_n-2-m_1)^2}{m_1} - \frac{(2g_m-2+m_2)^2}{m_2} - \frac{(18+m_3)^2}{m_3}$.

$\frac{(2g_n-2-m_1)^2}{m_1}$ bounded $\implies K \cdot Q_n = \frac{K \cdot D_1}{\sqrt{D_1^2}}$ bounded (in grey region).

Bound in number of basis (Q_n, R_n, S_n) using hyperbolic area.



Q.E.D.