Asymptotic Calculus for Combinatorial Dyson-Schwinger Equations

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Motivation

- For most systems, perturbation theory is necessary to compute physical quantities.
- Often the perturbation expansions turn out to have **vanishing** radius of convergence!
- Many of the expansions diverge **factorially**, i.e. $a_n \approx CA^n\Gamma(n+\beta)$ for large n.
- These expansions often have a **combinatorial interpretation**.

$\Rightarrow$ Analyse factorially divergent power series from a combinatorial perspective.

- Treat factorially divergent power series analogous to the powerful framework of **analytic combinatorics**. Flajolet and Sedgewick [2009]

- Suppose a power series behaves asymptotically as $A^n \Gamma(n+\beta)$ in contrast to, e.g. e^{n^2}, $\Gamma(\sqrt{n}+\beta)$, $\Gamma(n+\beta)^2$, etc.
- In the $A^n \Gamma(n+\beta)$ case, knowledge of the asymptotic behaviour of **one** observable is enough to obtain knowledge of the asymptotic behaviour of **all** derived quantities.
- This can be made quantitative by studying the **ring of factorially divergent power series**.

Factorially divergent power series

- Consider the class of **formal** power series $\mathbb{R}[[x]]^{\alpha}_{\beta} \subset \mathbb{R}[[x]]$ which admit an asymptotic expansion for large n of the form,

$$f_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{n+\beta} + \frac{c_2}{(n+\beta)(n+\beta-1)} + \ldots\right)$$

  including power series with

$$\lim_{n\to\infty} \frac{f_n}{\alpha^n\Gamma(n+\beta)} = 0$$
$$\Rightarrow c_k = 0 \text{ for all } k \geq 0.$$

- Note, that the type of the asymptotic expansion is heavily restricted!

- Consider a power series $f(x) \in \mathbb{R}[[x]]^{\alpha}_{\beta}$ for large n:

$$f_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{n+\beta} + \frac{c_2}{(n+\beta)(n+\beta-1)} + \ldots\right)$$

- Idea: Interpret the coefficients c_k of the **asymptotic** expansion as a new power series.

Definition

$\mathcal{A}$ maps a power series to its asymptotic expansion:

$$\begin{array}{cccc} \mathcal{A} \quad : & \mathbb{R}[[x]]^{\alpha}_{\beta} & \rightarrow & \mathbb{R}[[x]] \\ & f(x) & \mapsto & \gamma(x) = \sum_{k=0}^{\infty} c_k x^k \end{array}$$

Theorem 1

$\mathcal{A}$ is a derivation on $\mathbb{R}[[x]]^\alpha_\beta$:

$$(\mathcal{A}f \cdot g)(x) = f(x)(\mathcal{A}g)(x) + (\mathcal{A}f)(x)g(x)$$

- Follows from the *log-convexity* of Γ.

$\Rightarrow$ $\mathbb{R}[[x]]^\alpha_\beta$ is a subring of $\mathbb{R}[[x]]$.

Proof sketch

With $h(x) = f(x)g(x)$,

$$h_n = \underbrace{\sum_{k=0}^{R-1} f_{n-k}g_k + \sum_{k=0}^{R-1} f_k g_{n-k}}_{\text{High order times low order}} + \underbrace{\sum_{k=R}^{n-R} f_k g_{n-k}}_{\mathcal{O}(\alpha^n \Gamma(n+\beta-R))}$$

.

- What happens for **composition** of power series $\in \mathbb{R}[[x]]^\alpha_\beta$?

Theorem 2 Bender [1975]

If $|f_n| \leq C^n$ then, for $g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$:

$$f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$$
$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x)$$

- Bender considered much more general power series, but this is a direct corollary of his theorem in 1975.

Theorem 3 MB [2016a]

More general for $f \in \mathbb{R}\{y_1, \ldots, y_L\}$ and $g^1, \ldots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$:

$$(\mathcal{A}(f(g^1(x), \ldots, g^L(x)))(x) = \sum_{l=1}^{L} \frac{\partial f}{\partial y_l}(y_1, \ldots, y_L)\Big|_{\substack{y_m = g^m(x) \\ \forall m \in \{1,\ldots,L\}}} (\mathcal{A}g^l)(x).$$

- What happens if $f \notin \ker \mathcal{A}$, i.e. f does not have a finite radius of convergence.
- $\mathcal{A}$ fulfills a general 'chain rule':

Theorem 4 MB [2016a]

If $f, g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$ and $g_1 = 1$:

$$f \circ g \in \mathbb{R}[[x]]^\alpha_\beta$$

$$(\mathcal{A} f \circ g)(x) = f'(g(x))(\mathcal{A} g)(x) + \left(\frac{x}{g(x)} \right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A} f)(g(x))$$

$\Rightarrow$ We can solve for asymptotics of implicitly defined power series.

- The factor $e^{\frac{g(x)-x}{\alpha x g(x)}}$ generates typical prefactors of the form

$$e^{\frac{g_2}{\alpha}}$$

in asymptotic expansions.

Example: Chord diagrams

- A chord diagram is the same as a single closed fermion loop with arbitrary photon interactions.
- A connected diagram is the same as such a diagram without fermion self energy insertions.
- Let $I(x) = \sum_{n=0}^{\infty}(2n-1)!!x^n$ be the ordinary generating function of all chord diagrams and
- $C(x)$ the ordinary generating function of connected chord diagrams.
- They are related by $I(x) = 1 + C(xI(x)^2)$.

$$I(x) = 1 + C(xI(x)^2)$$
$$(\mathcal{A}I)(x) = (\mathcal{A}C(xI(x)^2))(x)$$
$$(\mathcal{A}I)(x) = 2xI(x)C'(xI(x)^2)(\mathcal{A}I)(x) + \left(\frac{x}{xI(x)^2}\right)^{\frac{1}{2}} e^{\frac{xI(x)^2 - x}{2x^2 I(x)^2}} (\mathcal{A}C)(xI(x)^2)$$

$I(x)$ is given by

$$\begin{aligned} I(x) &= \sum_{n=0}^{\infty} (2n-1)!! x^n \\ &= \sum_{n=0}^{\infty} \frac{2^{n+\frac{1}{2}}}{\sqrt{2\pi}} \Gamma(n + \frac{1}{2}) x^n \in \mathbb{R}[[x]]^2_{\frac{1}{2}} \end{aligned}$$

- Using the chain rule for $\mathcal{A}$, we can solve for $(\mathcal{A}C)(x)$:

$$(\mathcal{A}C)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x) + C(x)^2)}$$

$$(\mathcal{A}C)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}$$

$\Rightarrow$ Generating function of the full asymptotic expansion of

$$C_n = (2n-1)!!e^{-1} \left(1 - \frac{5}{2}\frac{1}{2n-1} - \frac{43}{8}\frac{1}{(2n-1)(2n-3)} + \ldots \; C_n =$$

Applications

Action on Dyson-Schwinger-Equations

Let $p, g, f \in \mathbb{R}[[x]]^{\alpha}_{\beta}$ and $p \in \ker \mathcal{A}$, then the functional equation,

$$p(g(x)) = x + f(g(x))$$

implies $$(\mathcal{A}g)(x) = g'(x) \left(\frac{x}{g(x)} \right)^{\beta} e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}f)(g(x))$$

and $$(\mathcal{A}f)(x) = {g^{-1}}'(x) \left(\frac{x}{g^{-1}(x)} \right)^{\beta} e^{\frac{g^{-1}(x)-x}{\alpha x g^{-1}(x)}} (\mathcal{A}g)(g^{-1}(x)).$$

where $g(g^{-1}(x)) = x$.

$\Rightarrow$ Solving the DSE 'perturbativly' to n terms gives an asymptotic expansion up to order $n-2$!

- $\mathcal{A}$ maps low order expansions to high order expansions.
- Asymptotic expansion independent of p.

Example: Simple permutations

- Let $\pi \in S_n^{\text{simple}} \subset S_n$ such that $\pi([i,j]) \neq [k,l]$ for all $i,j,k,l \in [0,n]$ with $2 \leq |[i,j]| \leq n-1$, then π is a simple permutation, which does not map an interval to another interval.
- With $S(x) = \sum_{n=0}^{\infty} |S_n^{\text{simple}}| x^n$ and $F(x) = \sum_{n=1}^{\infty} n! x^n$:

Albert et al. [2003]

$$\frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x))$$

- $F(x) \in \mathbb{R}[[x]]_1^1$ and $(\mathcal{A}F) = 1 \Rightarrow$ even though $S(x)$ is only given implicitly, we have an asymptotic expansion.

- Generating function for asymptotic coefficients of $S(x)$:

$$(\mathcal{A}S)(x) = \frac{1}{1+x}\frac{1-x-(1+x)\frac{S(x)}{x}}{1+(1+x)\frac{S(x)}{x^2}}e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}}$$

$$s_n = e^{-2}n!\left(1 - 4\frac{1}{n} + 2\frac{1}{n(n-1)} - \frac{40}{3}\frac{1}{n(n-1)(n-2)} + \ldots\right)$$

- Generating function for asymptotic coefficients $\Rightarrow$ can analyze asymptotics of asymptotics.

The ring of factorially divergent power series

- $\mathbb{R}[[x]]^\alpha_\beta$ forms a subring of $\mathbb{R}[[x]]$ **closed under multiplication, composition, differentiation and integration**.
- $\mathcal{A}$ is a **derivation** on $\mathbb{R}[[x]]^\alpha_\beta$ which can be used to obtain asymptotic expansions of **implicitly defined power series**.
- Nice closure properties under asymptotic derivative $\mathcal{A}$.
- Generalizations possible to multiple $\alpha_1, \ldots, \alpha_l \in \mathbb{C}$ with $|\alpha_i| = \alpha$.
- Question: Which classes of power series are **closed** under the operation of the asymptotic derivative?

Some power series closed under the 'asymptotic' derivative

- A huge set of examples for factorially divergent power series is given by the following **formal** integral:

$$Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)}$$

- This is to be interpreted as the power series given by,

$$Z(\hbar) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{-\frac{x^2}{2\hbar}} x^n [y^n] e^{\frac{F(y)}{\hbar}}$$

$$= \sum_{n=0}^{\infty} (2n-1)!! \hbar^n [y^{2n}] e^{\frac{F(y)}{\hbar}}$$

which gives a valid power series expansion in $\mathbb{R}[[\hbar]]$ for $F(x) \in x^3\mathbb{R}[[x]]$.

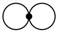
$$Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)}$$

- Expansion of a **zero-dimensional** QFT. Cvitanović et al. [1978], Argyres et al. [2001], Hurst [1952], Molinari and Manini [2006]
- Also the **combinatorial** generating function of the Feynman graphs contributing to the QFT with the interaction given by $F(x)$.
- Maps from power series with non-vanishing radius of convergence to factorial growth power series.

⇒ Perfect ground to study the divergence of the perturbation expansion in general QFTs!

$$Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+\sum_{k\geq 3}\lambda_k \frac{x^k}{k!}\right)}$$

- Combinatorial interpretation:

$Z(\hbar) = 1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \ldots$

$$= 1 + \hbar\left(\frac{1}{8}\lambda_3^2 + \frac{1}{12}\lambda_3^2 + \frac{1}{8}\lambda_4 \right) + \ldots$$

- Z counts **graphs** with weights λ assigned to each vertex. $\hbar$ counts the **Euler characteristic** of the graph (i.e. $\#$loops $-$ $\#$components)

Example

$$Z^{\text{stir}}(\hbar) := \frac{\Gamma\left(\frac{1}{\hbar}\right)}{\sqrt{2\pi\hbar}\left(\frac{1}{\hbar}\right)^{\frac{1}{\hbar}} e^{-\frac{1}{\hbar}}} = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}-(e^x-1-x-\frac{x^2}{2})\right)}$$

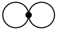
- **Combinatorial integral** representation of Stirling's famous (asymptotic) expansion of the Gamma-function.
- Counts the (orbifold) Euler characteristic of the moduli space of (stable) open curves Kontsevich [1992],

$$\log Z^{\text{stir}}(\hbar) = \sum_{\substack{g,n \\ n+2g-2\geq 0}} \frac{\chi(\mathcal{M}_{g,n})}{n!} \hbar^{n+2g-2}$$

Example

$$Z^{\text{stir}}(\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} - (e^x - 1 - x - \frac{x^2}{2})\right)}$$

- Set $F(x) = -(e^x - 1 - x - \frac{x^2}{2})$. Combinatorial: All vertices are allowed and $\lambda_k = -1$.
- Diagrammatically:

$Z^{\text{stir}}(\hbar) = 1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \ldots$

$$= 1 + \hbar \underbrace{\left(\frac{1}{8}(-1)^2 + \frac{1}{12}(-1)^2 + \frac{1}{8}(-1)\right)}_{=\frac{1}{12}} + \ldots$$

$$= 1 + \hbar\frac{1}{12} + \hbar^2\frac{1}{288} - \hbar^3\frac{139}{51840} - \hbar^4\frac{571}{2488320} + \ldots$$

Computation

$$\mathcal{F}[F](\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)} = \sum_{n=0}^{\infty} (2n-1)!!\hbar^n [y^{2n}] e^{\frac{F(y)}{\hbar}}$$

- Defines a map $\mathcal{F} : x^3\mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]$.
- Efficient calculation is possible using,

Bivariate power series diagonalization

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)} = \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi\hbar}} e^{-\frac{y^2}{2\hbar}} G'(y),$$

where $G(y)$ is the power series solution of $\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))$.

$$\mathcal{F}[F](\hbar) = \sum_{n=0}^{\infty} (2n-1)!![y^{2n}]G'(y)$$

where $G(y)$ is the (positive) solution of $\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))$.

- The implicit equation $\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))$ defines a **complex curve** in $\mathbb{C}^2$.
- The asymptotics of $\mathcal{F}[F](\hbar)$ is governed by the asymptotics of the **convergent** power series $G(y)$.
- Asymptotics of $G(y)$ can be calculated using methods of analytic combinatorics (Flajolet-Salvy algorithm). Banderier and Drmota [2015]

$$\mathcal{F}[F](\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)}$$

- For minor restrictions on F, the mapping $\mathcal{F} : x^3\mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]$ behaves nicely under the asymptotic derivative:

Asymptotics of combinatorial integrals MB [2016b]

$$\mathcal{A}\mathcal{F}[F](\hbar) = \frac{1}{2\pi\sqrt{(F''(\tau)-1)}}\mathcal{F}[\widetilde{F}]\left(-\frac{\hbar}{F''(\tau)-1}\right)$$

where $\alpha = \frac{1}{\frac{\tau^2}{2}-F(\tau)}$, $\beta = 0$,
$\widetilde{F}(x) = -\frac{F(x+\tau)-F(\tau)-xF'(\tau)-\frac{x^2}{2}F''(\tau)}{F''(\tau)-1} \in x^3\mathbb{R}[[x]]$ and τ is the dominant (branch cut) singularity associated to the curve $\frac{y^2}{2} = \frac{x^2}{2} - F(x)$.

- $\mathcal{AF}[F](\hbar)$ is given by the expansion of the 'combinatorial integral' shifted to the 'nearest' saddle-point of the exponent,

$$Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}H(x)}$$

  with $H(x) = -\frac{x^2}{2} + F(x)$.

- As a mnemonic (not well-defined!)

$$\mathcal{A}Z(\hbar) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}(H(x+\tau)-H(\tau))}.$$

  with τ the position of the 'nearest' saddle-point.

- That means with $Z(\hbar) = \sum_{n=0} z_n \hbar^n$ and $\alpha = \frac{1}{H(\tau)}$

$$z_n = \sum_{k=0}^{R-1} \alpha^{n-k}\Gamma(n-k)[\hbar^k]\mathcal{A}Z(\hbar) + \mathcal{O}(\alpha^n \Gamma(n-R))$$

Example

$$\widetilde{Z}^{\text{QED}}(\hbar) = \int_{\mathbb{R}} \frac{dA}{\sqrt{2\pi\hbar}} e^{-\frac{1}{\hbar}\frac{\sin^2(A)}{2}}$$

- Using the formalism we see that $\widetilde{Z}^{\text{QED}} \in \mathbb{R}[[\hbar]]_0^2$ and $\mathcal{A}\widetilde{Z}^{\text{QED}}(\hbar) = \frac{2}{2\pi}\widetilde{Z}^{\text{QED}}(-\hbar)$

- **Asymptotics of QED diagram counting MB [2016b]**

  Therefore with $\widetilde{Z}^{\text{QED}}(\hbar) = \sum_{n=0} z_n^{\text{QED}} \hbar^n$

$$z_n^{\text{QED}} = \frac{1}{\pi} \sum_{k=0}^{R-1} 2^{n-k} \Gamma(n-k) [\hbar^k] \widetilde{Z}^{\text{QED}}(-\hbar) + \mathcal{O}(2^n \Gamma(n-R))$$

- Full asymptotic expansion of $\widetilde{Z}^{\text{QED}}(\hbar)$.

- The computation of the asymptotic expansion is as ‘easy’ as the computation of the original expansion.
- From $\mathcal{A}\widetilde{Z}^{\text{QED}}(\hbar)$ asymptotic expansions of all derived quantities can be obtained using the algebraic properties of the ring of factorially divergent power series.
- Example:

$$\mathcal{A}\log\widetilde{Z}^{\text{QED}}(\hbar) = \frac{\mathcal{A}\widetilde{Z}^{\text{QED}}(\hbar)}{\widetilde{Z}^{\text{QED}}(\hbar)} = \frac{1}{\pi}\frac{\widetilde{Z}^{\text{QED}}(-\hbar)}{\widetilde{Z}^{\text{QED}}(\hbar)}$$

  This is the generating function of the asymptotic expansion of connected QED diagrams.
- Implicit functional relations can be solved using the generalized chain rule.
- This gives rise to the asymptotic expansions of ‘renormalized’ quantities. Combinatorially, these correspond to the number of skeleton or primitive diagrams.

- Combinatorial integrals

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)}$$

  with minor restrictions on $F(x)$ provide a large set of generating functions which are **algebraically closed** under composition, inversion, and the **asymptotic derivative**.
- Asymptotic expansions of arbitrary order can be obtained from a combinatorial integral as well as any implicitly given function of them.

Conclusions

- A direct application of the ring of factorially divergent power series is bringing the classical treatments of zero-dimensional QFT to the asymptotic level.
- Applications to graph-enumeration.
- ‘Canonical’ nature of combinatorial integrals, because of the resemblance to path integral formulations.

Further applications in QFT

- The divergence of the perturbation expansions in physical QFTs is believed to be governed by the growth of diagrams.
- In fact there are strong indications for this.
- Possible to give bounds on Feynman integrals at each loop order.

$\Rightarrow$ Formulate combinatorial models which encode these bounds in terms of combinatorial integrals and study their asymptotics.

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