Borel plane analysis of hyperasymptotics
Resurgence of factorial series

M V Berry, O Costin, R D Costin & C Howls

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Hyperasymptotics

Hyperasymptotics is a powerful technique, yet far from having revealed its full potential, and I will describe a few steps towards pushing them further.

Discovered by Berry and developed by Berry, Howls, Olde Daalhuis and many others, hyperasymptotics is an important tool in dealing with divergent series occuring in applications.

Starting with summation to the least term it goes further in using the asymptotics of the remainder ε_1, summing its expansion to its least term, resulting in an ε_2 and so on indefinitely. Stage n asymptotic analysis yields the necessary information for step $n+1$ through resurgence relations going back to Dingle.

The process ends with nonzero errors, $\varepsilon_\infty = O(\varepsilon_1^{2\sqrt{2}})$: as it happens, the effective variable (singulant) is halved from one stage to the next. Is this a fundamental obstruction? (no), or can a detailed Borel plane analysis improve accuracy? (yes, substantially).

What is the sharp Borel plane structure of optimal truncation remainders? (meaning the exact physical plane-Borel plane duality of each ε_n and the complete structure of singularities.)

Can eventual convergence be achieved? Yes.

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Borel plane analysis of hyperasymptotic remainders

A *Borel $p-$plane resurgent function* is a function H analytic at zero, with a discrete set of singularities with convergent local expansions, and exponential bounds at infinity.

If H is resurgent, $\mathcal{L}H = h$, a function in the physical plane, is often also called resurgent. By Watson's lemma, the asymptotic series of h is $\tilde{h} = \sum \frac{H^{(k)}(0)}{x^{k+1}} =: \sum h_k x^{-k}$. By definition, such an $\tilde{h}$ is Borel summable.

Let H be resurgent. Rescale p so that the closest singularity $\omega \in S^1$. Then $H^{(k)} \propto k!$ and thus $|h_k/x^k| \searrow$ in k if $k < |x|$, $\propto e^{-|x|}$ when $k \sim |x|$ and $\nearrow$ thereafter.

Say $x > 0$. The least term truncation of $\tilde{h}$ is $h_T = \sum_{k=1}^{\lfloor x \rfloor} h_k x^{-k}$. If H is resurgent, then $h - h_T \propto e^{-x}$.

Write $h = h_T + e^{-x} r$, and if r is resurgent, repeat.

With $R = \mathcal{L}^{-1} r$, what we want to understand is the resurgence properties of the operator $H \to R = \mathcal{L}^{-1} r$.

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For simplicity, take $x = N \in \mathbb{N}$. Then $r = h - h_T = \mathcal{L}H(N) - \sum_{k=1}^{N} h_k N^{-k} =$

$$\frac{1}{N^N}\int_0^\infty e^{-Np} H^{(N)}(p)dp = \frac{N!}{2\pi i N^N}\int_0^\infty \mathrm{e}^{-Np}dp \oint_0 \frac{H(s+p)}{s^{N+1}}ds$$
$$= \frac{(-1)^N N!}{2\pi i N^N}\int_0^\infty \mathrm{e}^{-Np}\int_{-\omega}^{\infty}\frac{\Delta H(-t)}{(p+t)^{N+1}}dt$$

where ΔH is the branch jump of H. Change the order of integration

$$\underbrace{\frac{(-1)^N N!}{2\pi i N^N}}_{e^{-x}\text{-resurgent}}\int_{-\omega}^{\infty}dt\int_0^\infty \frac{\Delta H(-t)e^{-Np}}{(p+t)^{N+1}}dp$$

Change the variable in the innermost integral to

$$q = p + \ln(p+t);\, p(q,t) = -t + \Omega(q+t) \text{ where } \Omega(z) + \ln\Omega(z) = z \qquad (1)$$

(Ω is the Wright omega function) and change again the integration order.

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After some more algebra,

$$R = \frac{N! e^{-iN\varphi}}{2\pi i N^N} \int_0^\infty e^{-Nq} Q(q) dq$$

Thus in Borel plane, the transition operator is $H \mapsto Q$ where

$$\mathcal{T} H = \int_\omega^{e^{q+i\varphi}} \Delta H(t) [\Omega'(q - \ln(\omega) - \omega) - 1] dt$$

a resurgence-preserving operator, as seen next.

$$R = \frac{(-1)^N N!}{2\pi i N^N} \int_{-\omega}^{\infty} dt \int_0^{\infty} \frac{\Delta H(-t)}{(p+t)^{N+1}} e^{-Np} dp$$

Proposition

1. *The singular points q_s of $\mathcal{T}H$ are: the q_s s.t. $e^{q_s} = p_s$ p_s is a singular point for $\Delta H(z)$ ($\omega = -1 : q_s \in \pi i\mathbb{Z}$) and the q_s where $\Omega(q + \ln(-\omega) - \omega)$ is singular.*
2. *For $\omega \neq 1$, $\Omega(q + \ln(-\omega) - \omega)$ has only two singularities, at $\ln(-\omega) - \omega - 1 \pm i\pi$*

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Calculations for Airy Ai

The singularity of H is at -1, this implies that the smallest $|q_s|$ is π: the least term of the R series moves *farther* at $N = \pi x$:

F	new nr. of terms to least term	Absolute error after LTT+one stage
1	4	-0.0021
2	7	0.000037
3	12	3.1×10^{-7}
4	12	-3.9×10^{-9}
5	19	2.1×10^{-11}
6	24	3.9×10^{-13}
7	24	-4.0×10^{-15}
8	29	-5.5×10^{-17}
9	29	7.2×10^{-19}
10	29	-1.3×10^{-20}

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If the singularity of H is at -1, then the smallest $|q_s|$ is π times as far: the least term of the R series moves *farther*.

And, if $q_s = -1$, just one stage results in $\tilde{\varepsilon}_2 = \varepsilon_1^{\pi} \ll \varepsilon_1^{2\sqrt{2}} \ll \varepsilon_1$.

Now, each singularity of H results in infinitely many of $\mathcal{T}H$ (since Q is singular if $H(e^q)$ is singular).

Because of this, the *density* of singularities of R_n grows at the unexpected rate $(e^{e^{e^{\cdots}}})_n$.

Nonetheless, bafflingly, this whole zoo converges ends up in convergence of the expansion!

To understand the extravagant proliferation of singularities note that the hyperasymptotics of $\mathcal{T}H$ depends on the **Taylor coefficients** of H. If a function satisfies an ODE, then its Taylor coefficients satisfy a difference equation. Hence the dependence on e^p. This leads to part II, factorial series.

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Factorial series and their resurgence

Factorial expansions,

$$\sum_{k=1}^{\infty} \frac{c_k}{(x)_k}, \quad (x)_k := x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}$$

go back to Stirling and were developed by Jensen, Landau, Nörlund, Horn, Wasow. Since $(x)_{k+1} \sim k!$ for large k, the factorial series of a function may converge even when its *asymptotic series* in powers of x^{-1} has empty domain of convergence. Strangely perhaps, b/c $(x)_k$ is larger than x^k.

Their use in QM and QFT (Jentschura, Weniger) triggered considerable renewed interest and substantial literature.

However, typically, classical factorial series have two major limitations: slow convergence, at best power-like, and limited domain of convergence (a half plane which cannot be centered on the asymptotically important Stokes line).

However, for resurgent functions these limitations can be overcome. Ecalle-Borel summable series can be summed by rapidly convergent factorial series.

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$$\sum_{k=1}^{\infty} \frac{c_k}{(x)_k}, \quad (x)_k := x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}$$

go back to Stirling and were developed by Jensen, Landau, Nörlund, Horn, Wasow. Since $(x)_{k+1} \sim k!$ for large k, the factorial series of a function may converge even when its *asymptotic series* in powers of x^{-1} has empty domain of convergence. Strangely perhaps, b/c $(x)_k$ is larger than x^k.

Their use in QM and QFT (Jentschura, Weniger) triggered considerable renewed interest and substantial literature.

However, typically, classical factorial series have two major limitations: slow convergence, at best power-like, and limited domain of convergence (a half plane which cannot be centered on the asymptotically important Stokes line).

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By Watson's Lemma, the asymptotic series of the toy model

$$\mathcal{J} = \int_0^\infty \frac{e^{-xp}}{1+e^p}\,dp \left(= \tfrac{1}{2}\Psi(\tfrac{x}{2}+1) - \tfrac{1}{2}\Psi(\tfrac{x}{2}+\tfrac{1}{2})\right)$$

diverges like $k!\pi^{-k}x^{-k}$. However, changing variable to $t = e^{-p}$ and integrating by parts results in

$$\mathcal{J} = \int_0^1 \frac{t^x\,dt}{t+1} = \frac{1}{2(x+1)} + \frac{1}{x+1}\int_0^1 \frac{t^{x+1}\,dt}{(t+1)^2} = \cdots = \sum_{j=0}^{\infty} \frac{j!}{2^{j+1}(x)_{j+2}} \qquad (2)$$

which converges faster than 2^{-j} in $\mathbb{C} \setminus \overline{\mathbb{R}^-}$, since $(x)_j = O((j-1)!)$ for large j.

A change of variable in the classical Borel summation of the series $\sum_0^\infty \frac{k!}{(-x)^{k+1}}$

$$e^x \mathrm{Ei}(-x) = \int_0^\infty \frac{e^{-px}\,dp}{1+p} \overset{t=e^{-p}}{=} = \int_0^1 \frac{t^{x-1}\,dt}{1-\ln t}$$

the same procedure leads to an at-best power-like convergence (b/c of the singularity of ln at zero) at most in the right half plane, $\mathbb{H}$.

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Take now the prototypical example of **classically!** “non-summable” series:

$$\tilde{f} = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} \quad (x \to \infty) \tag{3}$$

Median Écalle-Borel summation $\mathcal{LB}$ gives

$$\mathcal{LB}\tilde{f} = e^{-x}\mathrm{Ei}(x) = PV \int_0^{\infty} \frac{e^{-xp}\,dp}{1-p}$$

The same substitution as before, $e^{-p} = t$ gives an integrand with pole at $1/e$,

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To understand the difference between these cases, the key element is the shape of the integrand. In the first example it was $(1+e^{-p})^{-1} \stackrel{e^{-p}=t}{\rightarrow} (1+t)^{-1}$, in the second it is $(1+p)^{-1} \rightarrow (1-\ln t)^{-1}$ whereas the third was plain singular.

To deal with these difficulties we need to represent Borel plane functions as sufficiently rapidly convergent combinations of analytic functions of the exponential $F_j(e^{-qp})$ where $F_j(z)$ are analytic in a disk of radius > 1 centered at one.

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First example: Ei in the Stokes ray sector

Let

$$e^{-x}\mathrm{Ei}^{+}(x) = \int_0^{\infty - i0} \frac{e^{-px}}{1-p}\,dp$$

(where $+$ refers to the intended direction of x, one in the upper half plane[1]).

The following identity holds in $\mathbb{C} \setminus \{1\}$:

$$\frac{1}{1-p} = -\frac{\pi i}{e^{-i\pi p}+1} + \pi i \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{e_k}{e^{-r_k p}+e_k}; \quad e_k = e^{-i\pi 2^{-k}},\ r_k = i\pi 2^{-k}$$

By integration we get

$$e^{-x}\mathrm{Ei}^{+}(x) = -i\int_0^{\infty-0i} \frac{e^{-px/\pi}}{e^{-ip}+1}\,dp + i\sum_{k=1}^{\infty}\int_0^{\infty-0i} \frac{e_k e^{-\frac{2^k px}{\pi}}}{e_k+e^{-ip}}\,dp$$

where the series converges at least as fast as 2^{-k} (Watson's lemma).

[1] Borel summation convention: the direction of integration is chosen s.t. $px > 0$.

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$$\mathrm{e}^{-x}\mathrm{Ei}^{+}(x) = -\sum_{m=1}^{\infty}\frac{\Gamma(m)}{2^m}\frac{1}{(y)_m} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Gamma(m)e_k}{(1+e_k)^m}\frac{1}{(2^k y)_m} \qquad (y = -ix/\pi) \tag{4}$$

(i) The double series (4) converges geometrically in $\mathbb{C}\setminus -i\overline{\mathbb{R}^{+}}$.
(ii) The difference between the k-th integral and the k-th term in the sum in (4) is of order $C\,2^{-k}m|1+e_k|^{-m}$ (note also that $e_k \to 1$).

There is a dense set of poles in (4) along $-i\mathbb{R}^{+}$ and there the expansion breaks down. This is to be expected b/c of $\neq$ behavior of Ei^{+} on the two sides of $-i\mathbb{R}^{+}$. Close to $-i\mathbb{R}$ but not on it we still have geometric convergence, but it deteriorates.

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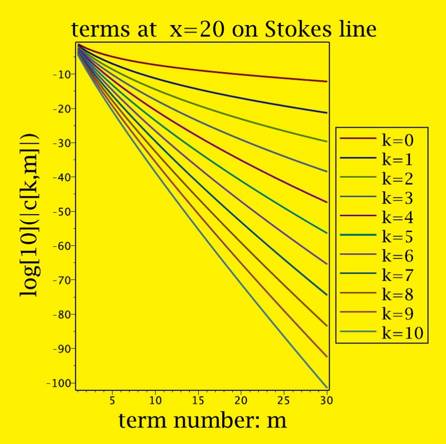

Figure: Size of terms in the successive series on the Stokes line $\mathbb{R}^+$ with the formula (4). This plot can be used to determine the number of terms to be kept for a given accuracy. To get 10^{-5} accuracy, 10 terms of the first series plus 5 from the second and so on, and all terms from the fifth series on can be discarded.

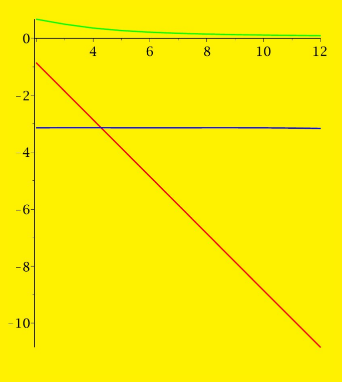

Figure: $f(x) = e^{-x}\mathrm{Ei}^{+}(x)$ on the Stokes line: $\mathrm{Re}\, f$ (green), $e^{x}\,\mathrm{Im}\, f$ (blue), $\ln(-\mathrm{Im}\, f)$ (red) from formula (4). We see that the small exponential is "born", with half of the residue, as expected by comparing with $\frac{1}{2}e^{-x}\left(\mathrm{Ei}^{+}(x) + \mathrm{Ei}^{-}(x)\right)$.

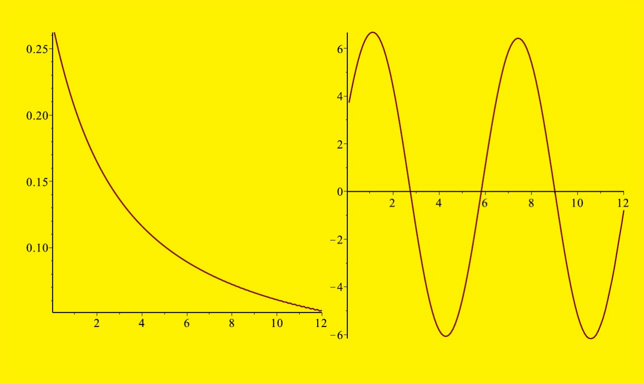

Figure: The antistokes transition of Ei^{+} from asymptotically decaying to oscillatory. Calculated at distance 0.3 from the two sides of the antistokes line.

Dyadic decompositions

Lemma (A strange dyadic decomposition)

The following identity holds in $\mathbb{C} \setminus \{0\}$:

$$\frac{1}{p} = \frac{1}{1 - e^{-p}} - \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + e^{-p/2^k}} \tag{5}$$

(The points $m\pi i$ are removable singularities of the right side.)

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Corollary (Dyadic decomposition of the Cauchy kernel)

$$\frac{1}{s-p} = -\frac{\beta e^{-\beta s}}{e^{-\beta s} - e^{-\beta p}} + \sum_{k=1}^{\infty} \frac{\beta 2^{-k} e^{-2^{-k}\beta s}}{e^{-2^{-k}\beta s} + e^{-2^{-k}\beta p}}$$

Proof.

$$\frac{1}{1-x}=\frac{2}{1-x^2}-\frac{1}{x+1}=\frac{4}{1-x^4}-\frac{2}{x^2+1}-\frac{1}{x+1}=\ldots=\frac{2^n}{1-x^{2^n}}-\sum_{j=0}^{n-1}\frac{2^j}{1+x^{2^j}}$$

which implies, with $x=\mathrm{e}^{-p/2^n}$, $\dfrac{1}{2^n(1-\mathrm{e}^{-p/2^n})}=\dfrac{1}{1-\mathrm{e}^{-p}}-\displaystyle\sum_{k=1}^{n}\frac{2^{-k}}{\mathrm{e}^{-p/2^k}+1}$ and equality (5) follows by passing to the limit $n\to\infty$. □

Generalization. Note that $1/(1-p)$ satisfies $\boxed{f(p)+f(-p)=2^{1-s}f(p^2)}$ for $s=0$. For general s, the solution is the polylog $\mathrm{Li}_s(p)$.

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$$\pi p^{s-1}=\Gamma(s)\sin(\pi s)\left[\mathrm{Li}_s\left(\mathrm{e}^{-p}\right)-\sum_{k=1}^{\infty}2^{-k(1-s)}\mathrm{Li}_s\left(-\mathrm{e}^{-2^{-k}p}\right)\right] \qquad (6)$$

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Lemma (A ramified generalization of (5))

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Example II $\Psi = \Gamma'/\Gamma$

Since

$$\frac{1}{p} - \frac{1}{e^{p}-1} = \sum_{k=1}^{\infty} \frac{e^{-\frac{p}{2^k}}}{2^k\left(e^{-\frac{p}{2^k}}+1\right)} \tag{7}$$

and

$$\Psi(x+1) = \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \ln x + \int_0^{\infty} \left(\frac{1}{p} - \frac{1}{e^{p}-1}\right) e^{-xp} dp \tag{8}$$

we have

$$\Psi(x+1) = \ln x + \sum_{k=1}^{\infty} \int_0^{\infty} \frac{e^{-2^k xp-p}}{1+e^{-p}} dp \tag{9}$$

Thus (Stirling's formula, factorially summed)

$$\Psi(x+1) = \ln x - \sum_{k=1}^{\infty} \Phi(-1,1,2^k x+1) = \ln x + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{(j-1)!}{2^j(2^k x+1)_j} \tag{10}$$

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Dyadic series of general resurgent functions

The dyadic expansion, used in the Cauchy formula,

$$F(p) = \frac{1}{2\pi i} \oint_{|p-s|<r} \frac{F(s)}{s-p} ds$$

allows for arranging the necessary analyticity in exponentials for quite general F. *Resurgent function* (in the sense of Écalle) is a function which is endlessly continuable and has suitable exponential bounds at infinity. The singularities are typically assumed to be regular, in the sense of having convergent local Puiseux series possibly mixed with logs.

We define *resurgent "elements" to be resurgent functions with only one singularity and algebraic decay.*

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Proof.

Decomposition in suitably modified Riemann-Hilbert problems. □

Let ω_k be the singularities of the resurgent function F of the type arising, say in nonlinear ODEs. Then:

1. Each ω_i is of the form $j\lambda_k$, with $j \in \mathbb{Z}^+$ and $\lambda_k \in \{\lambda_1, \dots, \lambda_n\}$ (the eigenvalues of the linearization at ∞ of the ODE assumed to be linearly independent over $\mathbb{Z}$ and of different complex arguments);
2. there is a ν s.t $\|F\|_\nu = \sup_{p \in \mathcal{A}} |F(p)e^{-\nu|p|}| < \infty$ where $\mathcal{A}$ is the complement of the union of thin strips S_i containing exactly one singularity ω_k; we let $C_i = \partial S_i$.

Let

$$F_i(p) = \frac{\exp(\mu_i p)}{2\pi i} \int_{C_i} \frac{F(s)\exp(-\mu_i s)}{s-p} ds \qquad (11)$$

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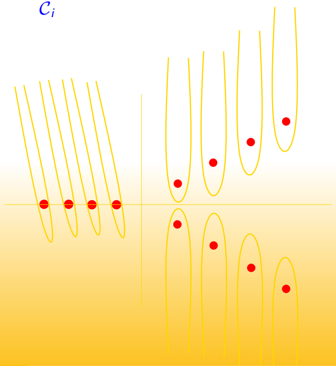

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For the **Proof**

Indeed, the contour can be deformed past p collecting a residue,

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In any compact set in $\mathcal{A}$, $\sum_{\omega_i} F_i$ converges at least as fast as $\sum_{j\in\mathbb{Z}^+,k=1,\dots,n} \mathrm{e}^{-j|\lambda_k|(\mu-\nu)}$.

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Ai (again!)

After normalization, the Airy function is brought to

$$h(x) = \int_0^\infty e^{-px} F(p)dp \tag{12}$$

where $F(p) = {}_2F_1(1/6, 5/6; 1, -p) = P_{-1/6}(1+2p)$ is analytic except for a logarithmic singularity at -1. To improve decay we integrate by parts

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$$F'(p) = \frac{1}{2\pi i}\oint_{|p-s|<r} \frac{F(s)}{(s-p)^2} ds = \frac{1}{2\pi i}\int_{-\infty}^{-1} \frac{\Delta F(s)}{(p-s)^2} ds \tag{14}$$

After the change of variables $s = -1 - t$ we get

$$h(x) = \frac{F(0)}{x} - \frac{1}{2\pi x}\int_0^\infty e^{-xp}\int_0^\infty \frac{F(t)}{(1+p+t)^2} dt \tag{15}$$

since $\Delta F(-1-t) = -iF(t)$.

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$$F'(p) = \frac{1}{2\pi i}\oint_{|p-s|<r} \frac{F(s)}{(s-p)^2} ds = \frac{1}{2\pi i}\int_{-\infty}^{-1} \frac{\Delta F(s)}{(p-s)^2} ds \tag{14}$$

After the change of variables $s = -1 - t$ we get

$$h(x) = \frac{F(0)}{x} - \frac{1}{2\pi x}\int_0^\infty \mathrm{e}^{-xp}\int_0^\infty \frac{F(t)}{(1+p+t)^2} dt \tag{15}$$

since $\Delta F(-1-t) = -iF(t)$.

Ai (again!)

After normalization, the Airy function is brought to

$$h(x) = \int_0^\infty e^{-px} F(p) dp \tag{12}$$

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The identity, differentiated:

$$\frac{1}{(s-p)^2} = \frac{\mathrm{e}^{-s-p}}{(\mathrm{e}^{-s}-\mathrm{e}^{-p})^2} + \sum_{k=1}^{\infty} 4^{-k} \frac{\mathrm{e}^{2^{-k}(-p-s)}}{(\mathrm{e}^{-2^{-k}s}+\mathrm{e}^{-2^{-k}p})^2}$$

Used with $s=-1-t$ this yields

$$\int_0^\infty \frac{F(t)dt}{(1+p+t)^2} = \int_0^\infty \frac{\mathrm{e}^{1-p+t}F(t)dt}{(\mathrm{e}^{1+t}-\mathrm{e}^{-p})^2} + \sum_{k=1}^{\infty} \int_0^\infty \frac{4^{-k}\mathrm{e}^{2^{-k}(1-p+t)}F(t)dt}{(\mathrm{e}^{2^{-k}(1+t)}+\mathrm{e}^{-2^{-k}p})^2}$$

The factorial expansion of h is

$$h(x) = \frac{F(0)}{x} - \sum_{m=2}^{\infty} \frac{(-1)^m\Gamma(m)}{2\pi(x)_m} d_m + \sum_{k=1}^{\infty} 2^{-k}\mathrm{e}^{2^{-k}} \sum_{m=2}^{\infty} \frac{(-1)^m\Gamma(m)}{2\pi(2^k x)_m} d_{km}$$

where

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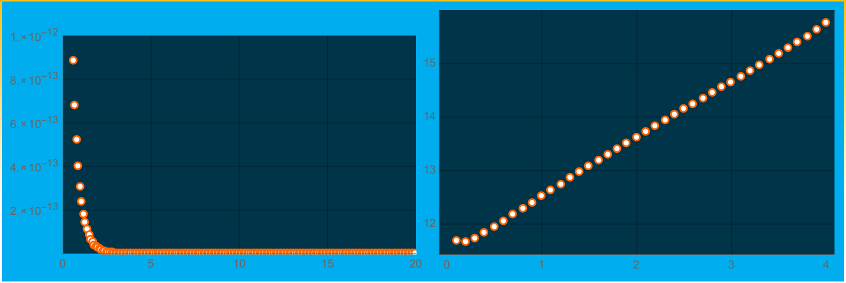

Figure: Relative accuracy for Ai (left), and number of exact digits (right) as functions of x. The total number of terms used in this calculation ranges from about 150 for small x to 30 terms at $x = 20$, found as explained in Fig. 3. The right graph plateaus at 16 digits for all $x \geq 4$, an artefact due to calculations being made in Mathematica's machine precision; thus the right graph was stopped at $x = 4$.

PDEs? Dyadic resolvent identities

Proposition

1. *Let A be self-adj., $0 \notin \sigma(A)$, $\lambda > 0$, $U_t := \mathrm{e}^{-itA}$ be the assoc. the unitary evolution.*

$$(A - i\lambda)^{-1} = i(1 - \mathrm{e}^{-\lambda} U_1)^{-1} - i \sum_{k=1}^{\infty} (1 + e^{-\lambda/2^k} U_{2^{-k}})^{-1}$$
$$= i \sum_{k=0}^{\infty} \mathrm{e}^{-k\lambda} U_k - i \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} 2^{-k} (-1)^j \mathrm{e}^{-j\lambda/2^k} U_{j2^{-k}} \quad (16)$$

and a similar sum for $\lambda < 0$. All sums are operator-norm convergent.

2. *Let A be positive, $0 \notin \sigma(A)$. Let $T_t = \mathrm{e}^{-tA}$ be the semigroup generated by A. Then*

$$A^{-1} = (1 - T_1)^{-1} - \sum_{k=1}^{\infty} 2^{-k} (1 + T_{1/2^k})^{-1} = \sum_{j=1}^{\infty} T_j - \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} 2^{-k} (-1)^j T_{j/2^k} \quad (17)$$

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Proposition

More generally, for $s < 1$,

$$\pi A^{s-1} = \Gamma(s)\sin(\pi s)\left[\mathrm{Li}_s\left(T_1\right) - \sum_{k=1}^{\infty} 2^{-k(1-s)}\mathrm{Li}_s\left(-T_{1/2^k}\right)\right] \tag{18}$$

where for $|z| < 1$ the polylog is defined by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} k^{-s} z^k \tag{19}$$

(More general identities can be obtained from the Cauchy kernel and analytic functional calculus.)

TBA and to be explored in the context of PDEs.

Conclusions: TBD

Thank you

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Thank you