Resurgence through Dyson-Schwinger equations

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Resurgence in Gauge and String Theories

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Outline

1. Review of Dyson-Schwinger equations
   - Schwinger's approach
   - Dyson's approach
2. Negative results in Y_4 and QED_4
   - Approximate fermion propagator in Yukawa theory Y_4
   - Photon propagator in QED_4
3. Signs of resurgence
   - Intersectorial communication
   - Resurgence through Dyson-Schwinger equations?
4. Renormalisation
   - Complications due to renormalisation
   - Class of transseries?
5. Nonperturbative transmonomials
   - Abelian vs. nonabelian theories

Dyson-Schwinger equations I: Schwinger's approach

Identities for correlators

Idea: derive from

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(x)} \left[e^{-S(\phi,\lambda)} \phi(x_1) \cdots \phi(x_n) \right] = 0$$

identities for correlators; $S(\phi, \lambda)$ euclidean action of scalar field

Example $n = 1$: relates different correlators

$$\int \mathcal{D}\phi \left[\frac{\delta S(\phi,\lambda)}{\delta\phi(x)} \phi(y) + \delta^d(x-y) \right] = 0,$$

where, eg $\frac{\delta S(\phi,\lambda)}{\delta\phi(x)} = (-\Delta + m^2)\phi(x) + 4\lambda\phi(x)^3$

Dyson-Schwinger equations I: Schwinger's approach, cont.

Reformulate using external current

$$\int \mathcal{D}\phi \, \left[\frac{\delta S(\phi,\lambda)}{\delta \phi(x)} \left(\frac{\delta}{\delta J} \right) \frac{\delta}{\delta J(y)} + \delta^d(x-y) \right] e^{-S(\phi,\lambda)+\int J\cdot\phi} = 0$$

or

$$\left[\frac{\delta S(\phi,\lambda)}{\delta \phi(x)} \left(\frac{\delta}{\delta J} \right) \frac{\delta}{\delta J(y)} + \delta^d(x-y) \right] \mathscr{Z}[J] = 0$$

with partition function

$$\mathscr{Z}[J] = \int \mathcal{D}\phi \; e^{-S(\phi,\lambda)+\int J\cdot\phi} = e^{-G[J]}$$

or write in terms of free energy $G[J] = -\log \mathscr{Z}[J]$

Example: QED fermion propagator

Feynman path integral of QED

same strategy applied to QED

$$\int \mathcal{D}(A,\psi,\overline{\psi}) \frac{\delta}{\delta\overline{\psi}(x)} e^{iS(A,\psi,\overline{\psi})+i\int[J_\mu \cdot A^\mu + \overline{\eta}\cdot\psi + \overline{\psi}\cdot\eta]} = 0$$

and after applying $\frac{\delta}{i\delta\eta(y)}$, this leads to

$$\left[\delta^d(x-y) + \left\{ i\partial\!\!\!/_x - m - e\gamma^\mu \frac{\delta}{i\delta J^\mu(x)} \right\} \frac{\delta}{i\delta\eta(y)} \frac{\delta}{i\delta\overline{\eta}(x)}\right] \mathscr{Z}[J,\eta,\overline{\eta}] = 0$$

where

$$\mathscr{Z}[J,\eta,\overline{\eta}] = \int \mathcal{D}(A,\psi,\overline{\psi})\; e^{iS(A,\psi,\overline{\psi})+i\int[J_\mu \cdot A^\mu + \overline{\eta}\cdot\psi + \overline{\psi}\cdot\eta]}$$

(partition function of QED)

Example: QED fermion propagator, cont.

Using

fermion propagator and self-energy

$$\left[i\partial\!\!\!/ - m - \Sigma(x,y)\right] S(x,y) = \delta^d(x-y)$$

with full fermion propagator

$$S(x,y) = \frac{\delta}{i\delta\eta(x)} \frac{\delta}{i\delta\eta(y)} \mathscr{Z}[J,\eta,\overline{\eta}]\Big|_{J,\eta,\overline{\eta}=0}$$

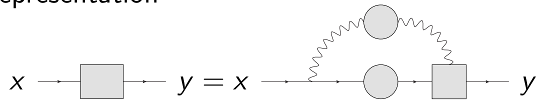
one ends up with

Dyson-Schwinger equation for fermion self-energy

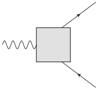
$$\Sigma(x,y) = ie \int d^d z\, d^d x'\; \gamma^\mu \Pi_{\mu\nu}(x,z) S(x,x') \Lambda^\nu(z,x',y)$$

Example: QED fermion propagator, cont.

Graphical representation



involves vertex function

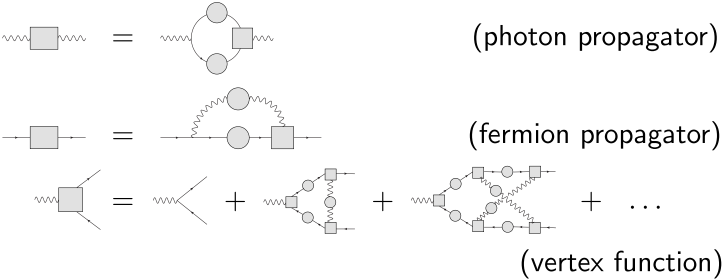
$$\Lambda_\nu(z, x', y) = \ \nu$$ 

We need additional Dyson-Schwinger equations (DSEs) for

- photon propagator
- vertex function (3-pt function)

Problem: DSE for vertex function involves 4-pt fermion function

DSEs in QED: coupled system of integral equations

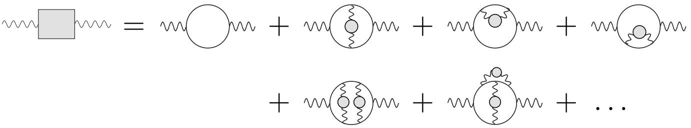


Combinatorial cheat to make them self-consistent:

- DSE for vertex function involves fermion 4-pt function
- Leads to infinite tower of coupled equations
- price to pay for truncation: infinite skeleton expansion

Single self-consistent DSE: combinatorial truncation

self-consistent DSE for photon propagator



Clever combinatorial truncation

- truncation leads again to combinatorially complete Feynman diagram series
- price again: infinite skeleton expansion (divergent)

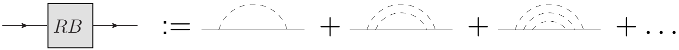
But: possible source of resurgence

since amenable to transseries analysis

Dyson-Schwinger equations II: Dyson's approach

Identities from self-similiarity of Feynman diagram series.

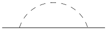
- example: rainbow approximation in Yukawa theory Y_4



stands for perturbative series

$$\Sigma = a\int \mathcal{K} + a^2 \int \mathcal{K} \int \mathcal{K} + a^3 \int \mathcal{K} \int \mathcal{K} \int \mathcal{K} + \dots$$

a coupling, K integral kernel of Feynman integral :



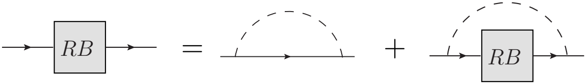
$$= \int \mathcal{K} = \int \frac{d^4k}{2\pi^2} \left\{ \frac{1}{k^2(q-k)^2} - \frac{1}{k^2(\tilde{q}-k)^2} \right\}$$

Self-similiarity of Feynman diagram series

- rainbow Dyson-Schwinger equation

$$\Sigma = a\int \mathcal{K}(1 + a\int \mathcal{K} + a^2 \int \mathcal{K} \int \mathcal{K} + \dots) = a \int \mathcal{K}(1+\Sigma)$$

- diagrammatically:



concretely, (euclidean, massless, renormalised)

$$\Sigma(q^2, a) = a \int \frac{d^4 k}{2\pi^2} \left\{ \frac{1}{k^2(q-k)^2} - \frac{1}{k^2(\tilde{q}-k)^2} \right\} \left[1 + \Sigma(k^2, a)\right]$$

Exact solution of rainbow DSE and transseries

Solution of rainbow DSE (massless case exactly solvable)

$$1 + \Sigma(q^2, a) = \left(\frac{q^2}{\mu^2}\right)^{-\gamma(a)}$$

where we call

$$\gamma(a) = \frac{1 - \sqrt{1 + 2a}}{2} = \sum_{n \geq 1} (-1)^n \frac{(2n-3)!!}{2 \cdot n!} a^n$$

the anomalous dimension.

Transseries viewpoint

Convergent series, ie trivial transseries

Self-consistent DSE in Yukawa theory Y_4

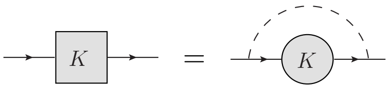
Kilroy DSE for self-energy (nontrivial)

$$\Sigma(q^2, a) = a \int d^4k\, \mathcal{K}(k, q, \tilde{q}) \left[1 - \Sigma(k^2, a)\right]^{-1}$$

where

$$\mathcal{K}(k, q, \tilde{q}) = \frac{1}{2\pi^2} \left\{ \frac{1}{k^2(q-k)^2} - \frac{1}{k^2(\tilde{q}-k)^2} \right\},$$

diagrammatically:



Anomalous dimension of Kilroy model

anomalous dimension

$$\gamma(a) = q^2 \frac{\partial}{\partial q^2} \Sigma(q^2, a)\Big|_{q^2=\mu^2} = \sum_{n\geq 1} c_n a^n$$

Numerical results (Broadhurst & Kreimer, 2009)

growth behaviour of perturbative coefficients ($n \leq 30$)

$$c_n \sim 2^{n-1}\Gamma(n+1/2)$$

model has renormalons

Question: perturbative solution known (in principle)

Transseries representation of $\gamma(a)$?

Higher RG functions

- perturbation theory suggests

$$\Sigma(q^2, a) = \sum_{k\geq 1} \frac{\gamma_k(a)}{k!} L^k$$

with momentum parameter $L = \ln(q^2/\mu^2)$ and $\gamma_1(a) = \gamma(a)$.

- Callan-Symanzik (renormalisation group) equation implies:

Recursion relation for higher RG functions

$$\gamma_n(a) = \gamma(a)[2a\partial_a - 1]\gamma_{n-1}(a)$$

and hence

$$\gamma_n(a) = (\gamma(a)[2a\partial_a - 1])^{n-1}\gamma(a)$$

Result Ia: fixed-point equation from DSE

Derive fixed point equation from Kilroy DSE

DSE for anomalous dimension

$$\gamma(a) = C_0 a + C_1 a\gamma(a) + a \sum_{r\geq 2} \sum_{n\geq r} C_n (\gamma_\bullet^{\star r})_n(a),$$

where $C_0, C_1, C_2, \ldots$ are constants and

$$(\gamma_\bullet^{\star r})_n(a) := \sum_{n_1+\ldots+n_r=n} \frac{\gamma_{n_1}(a)}{n_1!} \cdots \frac{\gamma_{n_r}(a)}{n_r!}$$

rhs of above DSE has an infinite # of differential operators!

Topical transseries ansätze

Currently used ansätze of the form

Height-1, depth-1 transseries

$$f = \sum_{\sigma \in \mathbb{N}_0^r} z^{c\cdot\sigma} e^{-(b\cdot\sigma)z} P_\sigma(\log z) \sum_{s\geq 0} c_{(\sigma,s)} z^{-s}$$

$c, b \in \mathbb{C}^r$, $P_\sigma(\log z) \in \mathbb{C}[\log z]$ polynomial, $z = (\text{coupling})^{-1}$

in QM, (toy model, SUSY) QFTs, string theories

Result Ib: ansatz wrong

Transseries ansatz: an ill fit

plug

$$\gamma(z) = \sum_{\sigma \geq 0} \sum_{s \geq 0} c_{(\sigma,s)} z^{\sigma c} e^{-\sigma(b_1 z + b_2 z^2)} z^{-s}$$

into fixed point equation and get $c_{(\sigma,s)} = 0$ for all $\sigma \geq 1$.

Kilroy ODE from DSE

insert ansatz with $b_1 z + \ldots + b_m z^m$ upstairs into

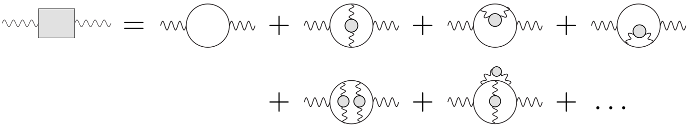
$$\gamma(a) + \gamma(a)[2a\partial_a - 1]\gamma(a) = a/2$$

and find the same for all $m \geq 1$.

Logarithmic transmonomials expedient? No.

Result II: photon DSE in QED

renormalised self-consistent DSE for photon propagator



leads to

$$\gamma = \alpha A_0 + \sum_{\ell\geq 1} \alpha^{\ell+1} \sum_{r_1\geq 0, n_1\geq r_1} \cdots \sum_{r_\ell\geq 0, n_\ell\geq r_\ell} C_{(n_1,\ldots,n_\ell)} (\gamma_\bullet^{\star r_1})_{n_1} \cdots (\gamma_\bullet^{\star r_\ell})_{n_\ell}$$

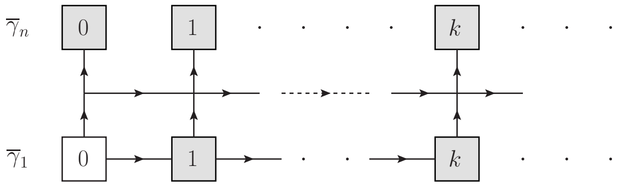
for anomalous dimension.

Result

transseries ansatz yet again ill fit

Sector cross-talk: tracking down coefficients

Finding: how sectors of RG functions communicate



should in principle be true as long as sectors are defined via exponential transmonomials

$$e^{-T(z)}$$

with $T(z)$ some large transseries, where $z = \alpha^{-1}$

Resurgence from sector cross-talk?

Idea

Use perturbative QFT and let DSEs make those series resurge.

Obstructions

- infinite tower of DSEs need to be truncated
- yet another truncation: truncate divergent skeleton series
- transseries unknown, renormalisation might complicate things

Conjecture for renormalisable theories

transseries not of the type currently topical with transmonomials

$$z^{-1}, e^{-z}, \log z$$

Renormalisation of $(\phi^4)_d$

Jump in complexity:

General form of (super)renormalised action in dimension d

$$\mathscr{R}_d[S](\phi,\lambda) = \frac{1}{2}\int \left\{\phi[Z(\lambda)(-\Delta) + m^2 Z_m(\lambda)]\phi + \lambda Z_v(\lambda)\phi^4\right\} d^d x$$

1. $d = 2$: $Z(\lambda) = 1$, $Z_m(\lambda) = 1 + c_1\lambda$, $Z_v(\lambda) = 1$
2. $d = 3$: $Z(\lambda) = 1$, $Z_m(\lambda) = 1 + c_1\lambda + c_2\lambda^2$, $Z_v(\lambda) = 1$
3. $d = 4$: asymptotic power series

$$Z(\lambda) = \sum_{s\geq 0} a_s\lambda^s, \quad Z_m(\lambda) = \sum_{s\geq 0} c_s\lambda^s, \quad Z_v(\lambda) = \sum_{s\geq 0} b_s\lambda^s$$

Formal semi-classical expansion of $(\phi^4)_d$

'usual' coupling dependence

partition function, rescaled

$$\mathscr{Z}(\mathcal{J}, \lambda) = \int \mathcal{D}\varphi \; e^{-\frac{1}{\lambda}S(\varphi,1)+\int \mathcal{J}\cdot\varphi}$$

Semi-classical expansion around critical points

$$\mathscr{Z}(\mathcal{J}, \lambda) \cong \underbrace{\sum_{\varphi_c} e^{-\frac{1}{\lambda}S(\varphi_c,1)} F_{\varphi_c}(\mathcal{J}, \lambda)}_{\text{transseries}} \in \sum_{\varphi_c} e^{-\frac{1}{\lambda}S_\Gamma(\varphi_c,1)} \mathbb{C}[[\lambda]]$$

connection to transseries: $z = \lambda^{-1}$

Semi-classical expansion of $(\phi^4)_d$, renormalised case

Rescaling hopeless for $d = 4$ due to Z factors:

partition function

$$\mathscr{Z}(J,\lambda) = \int \mathcal{D}\phi \; e^{-\mathscr{R}_d[S](\phi,\lambda) + Z(\lambda)^{1/2} \int_\Gamma J\cdot\phi}$$

Semi-classical expansion around critical points

$$\underbrace{\mathscr{Z}(J,\lambda) \cong \sum_{\phi_c} e^{-\mathscr{R}_d[S](\phi_c,\lambda)} F_{\phi_c}(J,\lambda)}_{\text{transseries?}} \in \text{transseries class ?}$$

Not grid-based? Or superexponentials like e^{-e^z}, $e^{-e^z(z^{-1}+z^{-2})}$?

Renormalisation of sine-Gordon model

sine-Gordon scalar field theory in $d = 2$

action with 2 coupling constants:

$$S(\phi, \alpha, \beta) = \int d^2x \left\{ \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - Z(\beta)\frac{\alpha}{\beta^2}[1 - \cos(\beta\varphi)] \right\}$$

Renormalisation Z factor (Faber & Ivanov, 2003)

$$Z(\beta) = \left(\frac{\Lambda^2}{\mu^2}\right)^{\frac{\beta^2}{8\pi}} = e^{\frac{\beta^2}{8\pi}\log(\Lambda^2/\mu^2)}$$ convergent series!

Λ: UV cutoff, μ: renormalisation subtraction point

nontrivial coupling dependence (transseries?)

Energy regimes and transmonomials

Logarithmic transmonomials $\mathfrak{l} = z^{-n}(\log z)^m$

contribute at weak coupling (large z)

- low-energy states (abelian gauge theories)
- high-energy states (nonabelian gauge theories?)

Exponential transmonomials $\mathfrak{m} = e^{-kz}z^{-n}$

contribute at strong coupling (small z)

- high-energy states (abelian gauge theories)
- low-energy states (nonabelian gauge theories)

Conclusion

1. renormalisation complicates matters by rendering coupling dependence of action nontrivial
2. for renormalised quantum field theories, we (probably) need fancier transseries ansätze,

$$\gamma(z) = \sum_{(\sigma,t,j)\geq(0,0,0)} c_{(\sigma,t,j)} z^{-\sigma c} e^{-\sigma(b_1 z + \ldots + b_m z^m)} z^{-t} (\log z)^j$$

   is not elaborate enough
3. future transseries may involve superexponentials $\mathfrak{m} = e^{-e^z}$
4. worth investigating DSEs for lower-dimensional models