Resurgence out of the (literal) box

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Resurgence for QFT

Belief: QFT observables are resurgent transseries in the couplings

$$\mathcal{O}(\lambda) \simeq \sum_n p_n \lambda^n + \sum_c e^{-\frac{S_c}{\lambda}} \sum_k p_{k,c} \lambda^k + \cdots$$

Lots of evidence:

Integrals with saddles	Stokes, Dingle, Berry, Howls …
matrix models	Marino, Schiappa, Weiss …
topological strings	Aniceto, Hatsuda, Marino, Schiappa, Vonk, …
QM (d=1 QFT)	Basar, Dunne, Kawai, Misumi, Nitta, Sakai, Takei, Unsal, Zinn-Justin …
some SUSY theories	Aniceto, Dorigoni, Hatsuda, Honda, Russo, Schiappa, …

More generic/realistic $d > 1$ QFTs, with asymptotic freedom?

Resurgence in asymptotically-free QFTs

Most explicit checks: 1+1D asymptotically-free QFTs

CP^{N-1} , principal chiral, O(N), and Grassmannian non-linear sigma models

To the extent it's been checked, resurgence works!

Dunne, Unsal, AC, Dorigoni; Fujimori, Misumi, Nitta, Sakai, …

Why the weasel words?

In d > 1 QFT, very difficult to precisely characterize large-order behavior

Strong coupling in IR in asymptotically-free theories

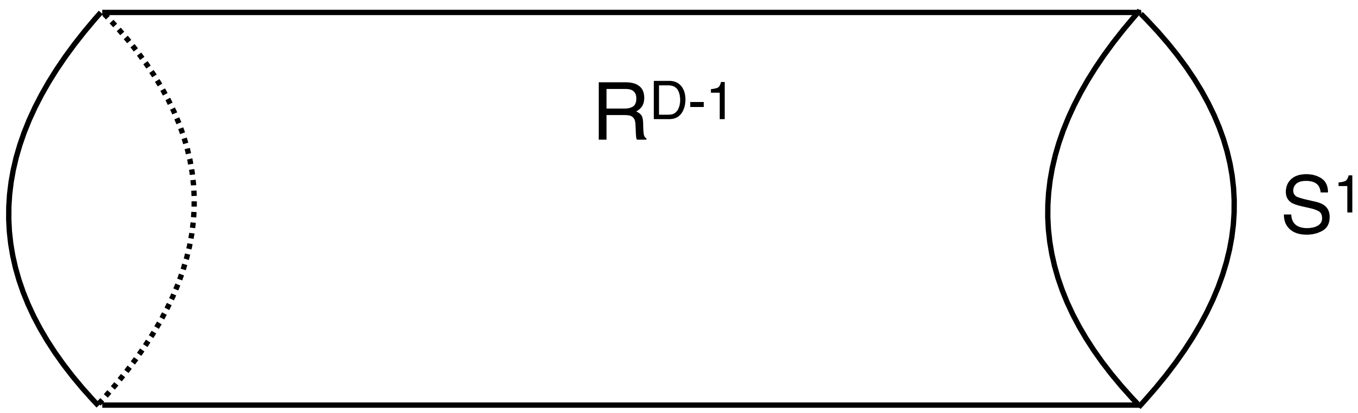
$$\Lambda \approx \mu\, e^{-c/\lambda}\,,\ \lambda = g^2 N$$

All work so far used idea of adiabatic
compactification from R^2 to RxS^1

Tiny boxes as tools

Compactify asymptotically-free QFT from R^D to $R^{D-1} x S^1$

Idea: when S^1 size $L << \Lambda^{-1}$, theory becomes ≈ weakly-coupled



Simplest circle is a thermal one. Trouble: physics at small-L and large-L can look totally different

Examples:

Large N phase transitions as a function of L

Dependence of gap Δ on 2D strong scale Λ is power law at large L, only logarithmic at small L.

Adiabatic small circle limit

For a smooth $L \ll \Lambda^{-1}$ limit, use special non-thermal boundary conditions.

Idea is actually quite general, very closely related to constructions in 4D gauge theory

Unsal and collaborators, 2012-onward

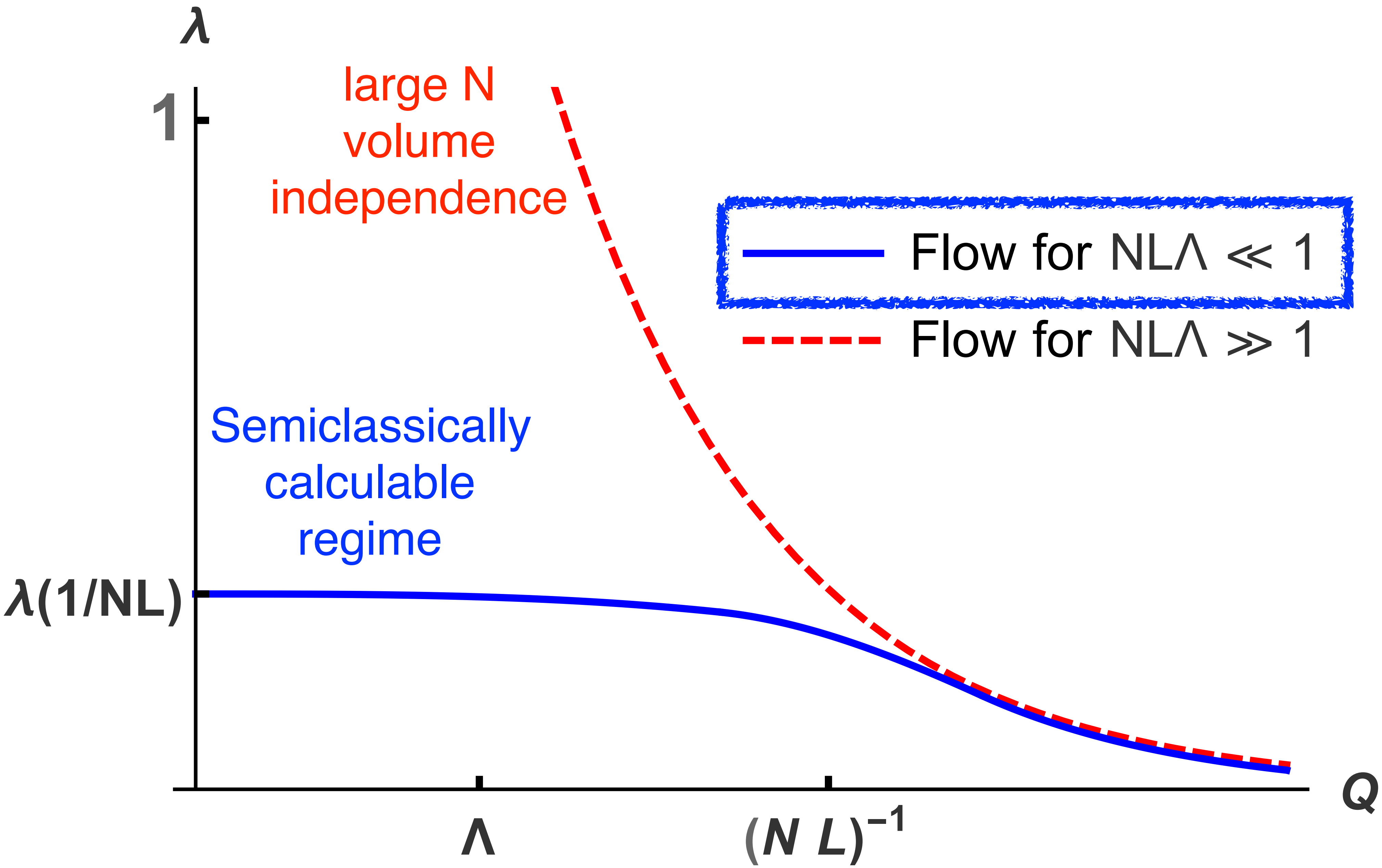
4D gauge theory: adiabatic small-L limit obtained with Z_N-invariant S^1 holonomy for the dynamical gauge field

2D sigma models: adiabatic small-L limit obtained with Z_N-invariant S^1 holonomy for the background `flavor' gauge field

With such compactifications, effective KK scale is 1/(NL), not 1/L.

Large N and small L limits do not commute
- tied to large N volume independence!

Coupling flow with adiabatic compactification



NLΛ >> 1 regime is strongly coupled

The NLΛ << 1 regime gives a weakly-coupled theory

Physics is very rich - mass gap, renormalons present at small N L!

Resurgence in a box

In perturbation theory 2D sigma models like O(N), CP^{N-1}, etc are gapless.

What about non-perturbatively, in the small NLΛ limit?

Need to know about non-perturbative saddle points!

The Z_N-invariant holonomies make instantons fractionalize into ~ N constituent ‘fractons’ (or ‘monopole-instantons’, etc.)

Without instantons, what fractionalizes are `unitons’ - finite-action, non-BPS saddle-point solutions.

Dabrowski, Dunne; AC, Dorigoni, Dunne, Unsal, Fujimori, Misumi, Nitta, Sakai,…

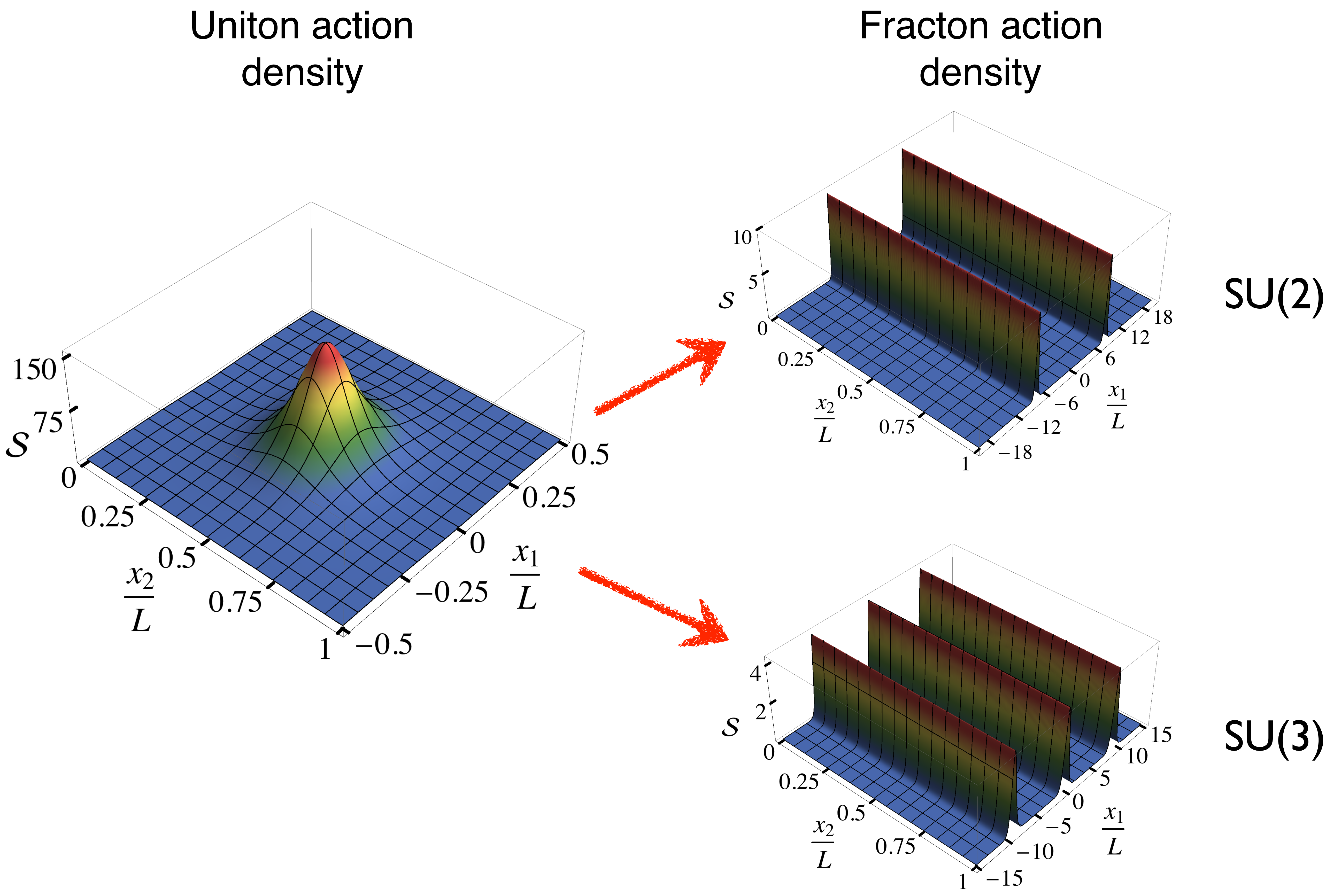
Very common in 2D: relevant homotopy group is π_2.

O(N) model: $\pi_2[O(N)] = 0$; SU(N) Principal chiral model $\pi_2[SU(N)] = 0$

The fractons, or composites built from them, drive appearance of mass gap!

$$[\mathcal{F}] \sim e^{-\frac{c}{\lambda}}, \ \lambda = g^2 N, c \sim \mathcal{O}(1)$$

Fractionalization of unitons



Resurgence in a box

To obtain results, use small NLΛ 1D effective field theory. EFT UV cutoff μ ~ 1/(NL).

At small NLΛ, mass gap ends up looking like

$$\Delta = \mu\, e^{-\frac{c}{\lambda}} \left(\sum_n p_n \lambda^n + e^{-\frac{2c}{\lambda}} \sum_m b_m \lambda^m + \cdots \right)$$

Fracton ($\mathcal{F}$) effect

$\mathcal{F}\,\mathcal{F}\bar{\mathcal{F}}$ effect

Fluctuations

Very schematic expression: really there's log(λ) factors, and sometimes gap starts with contributions from two fractons, etc

All existing studies indicate these series are resurgent transseries.

Resurgence in a box

So, seems resurgence applies to 2D QFTs.

But the explicit checks use the small-L EFT, which is QM.

Is this ok?

Could it be that QM is too special?

In QM $\exists$ powerful arguments for resurgence from e.g. ‘quantum geometry/exact WKB’. Generalization to QFT not obvious.

No regularization/renormalization needed in QM with non-singular potentials, but needed in $d > 1$ QFT

Interplay between resurgence, regularization and renormalization?

Resurgence for QFT?

(Witten 2009) Dunne, Unsal, AC, Dorigoni, Basar, …
2013-now

Why should the d = 1 results generalize to d > 1?

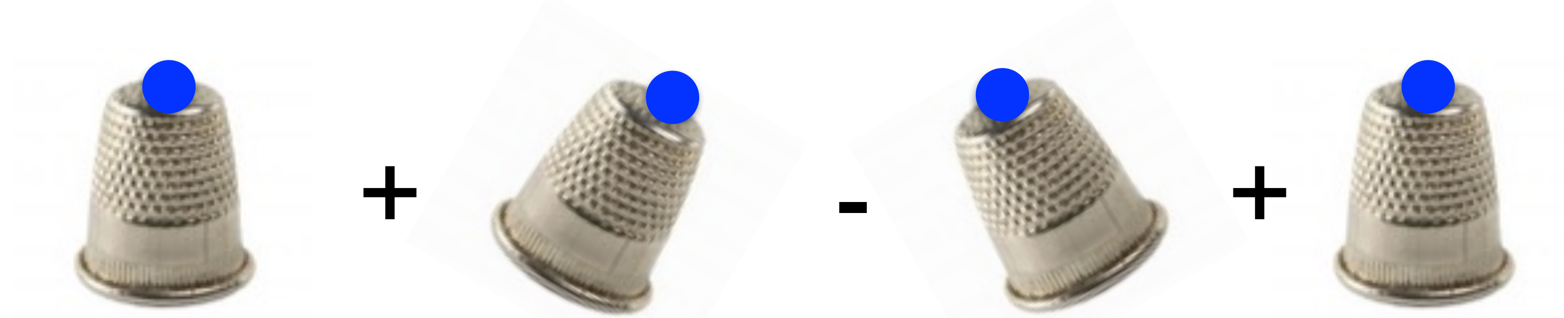
Path integral perspective?

$$Z(\lambda) = \sum_k C_k Z_{J_k}(\lambda)$$

"Lefshetz thimble" integration cycles

One 'thimble' per critical point of classical action, defined by steepest descent.

$Z(\lambda) =$



perturbation theory

non-perturbative contributions

{set of thimbles} = complete basis for
convergent QM phase-space path integrals

Resurgence relations = jumps in C_k as arg[λ] varies.

Resurgence for QFT?

Thimble perspective might sound taylor-made for generalization to QFT…

… but this isn't yet obvious!

Witten proved thimble decomposition works in d = 1

No proof that set of critical-point cycles is a basis if d > 1!

Several possibly-related issues.
What counts as a critical point? How to perform decomposition? …

Even in d = 1 discontinuous saddle-point-field configurations must be taken into account!

Behtash, Dunne, Schafer, Sulejmanpasic, Unsal, 2015

Construction in d > 1 may be sensitive to regularization of integral.

Shouldn't be too shocking: regularization always important in d > 1 !

Resurgence in full QFT

AC, Dorigoni, Unsal: coming soon

1980s: F. David; Novikov, Shifman, Vainshtein, Zakharov; Beneke,…

Use large N expansion to get around strong-coupling issues on R^2

Idea: work perturbatively in 1/N, but exactly in 't Hooft coupling, then explore 't Hooft coupling expansion structure.

Example for this talk: 2D O(N) model

$$S = \int_{\mathbb{R}^2} d^2x \, \partial_\mu n_a \partial^\mu n^a \, , \; n_a n^a = \frac{N}{4\pi\lambda}, a = 1, \cdots, N.$$

Model is asymptotically free, with dynamical mass gap $m \sim \mu \, e^{-1/2\lambda}$

Expectation: 'IR renormalon' ambiguities in resummation of perturbation theory

Size: $\pm i m^2, \pm i m^4, \pm i m^6, \ldots$

Borel plane singularities: $t_* = 1, 2, 3, \cdots$

Resurgence in large N O(N) model

Integrate in a Lagrange multiplier σ to make life easier:

$$S = \int_{\mathbb{R}^2} d^2x \left[\partial_\mu n^a \partial^\mu n_a - \sigma \left(n^a n_a - \frac{N}{4\pi\lambda} \right) \right]$$

Questions: what's the mass gap Δ? Resurgence as a function of λ?

Perturbation theory: theory of N - 1 massless particles, Δ = 0.

To define theory, must regularize UV. We'll use momentum cutoff μ .

$$\frac{d\lambda}{d\log\mu} = -2\lambda^2 \left[1 - \frac{2}{N} \right] + \mathcal{O}\left(\frac{\lambda^3}{N} \right)$$

$$\Lambda_{\mathrm{one-loop}} \sim \mu\, e^{-\frac{1}{2\lambda}}$$

Mass gap Λ ~ Δ far outside any semiclassical regime on R^2!

Resurgence in large N O(N) model

Large N solution is textbook material - see e.g. Peskin & Schoeder

Integrate out n^a fields, giving

$$S = N \int_{\mathbb{R}^2} d^2x \left[\frac{\sigma}{4\pi\lambda} - \frac{1}{2} \mathrm{Tr}\ \log(\partial^2 - \sigma) \right]$$

At large N, physics captured by saddle-point for σ, which satisfies

$$\frac{\partial S}{\partial \sigma} = 0 \ \Rightarrow \int^{|p|<\mu} \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \sigma} = \frac{1}{4\pi\lambda}$$

Want σ in terms of μ and λ.
Non-zero σ is a mass-squared for n^a fields!

Resurgence in large N O(N) model

$$\frac{\partial S}{\partial \sigma} = 0 \;\Rightarrow\; \int^{|p|<\mu} \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+\sigma} = \frac{1}{4\pi\lambda}$$

The textbooks (e.g. Peskin and Schroeder) give

$$\int^{|p|<\mu} \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+\sigma} = \frac{1}{4\pi}\log\left(\frac{\mu^2}{\sigma}\right)$$

$$\Rightarrow \quad \sigma = \mu^2 e^{-1/\lambda}$$

Spectrum has N massive particles, with $m^2 = \sigma$

$$\frac{\partial \sigma}{\partial \log \mu} = 0 \;\Rightarrow\; \frac{\partial \lambda}{\partial \log \mu} = -2\lambda^2$$

Celebrated result: O(N) beta function is one-loop exact at large N

Resurgence in large N O(N) model

The textbooks (e.g. Peskin and Schroeder) give

$$\int^{|p|<\mu} \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+\sigma} = \frac{1}{4\pi} \log\left(\frac{\mu^2}{\sigma}\right)$$

Bizarre fact: the equal sign above is wrong.

$$\int^{|p|<\mu} \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+\sigma} = \frac{1}{4\pi} \log\left(\frac{\mu^2+\sigma}{\sigma}\right)$$

Consequences:

$$\sigma = \mu\, e^{-1/\lambda} \frac{1}{1-e^{-1/\lambda}}$$

$$\frac{\partial\lambda}{\partial \log \mu} = -2(1-e^{-1/\lambda})\lambda^2$$

non-perturbative corrections!

Coupling constant flow

One-loop coupling diverges at $\Lambda = \mu\, e^{-1/2\lambda}$:

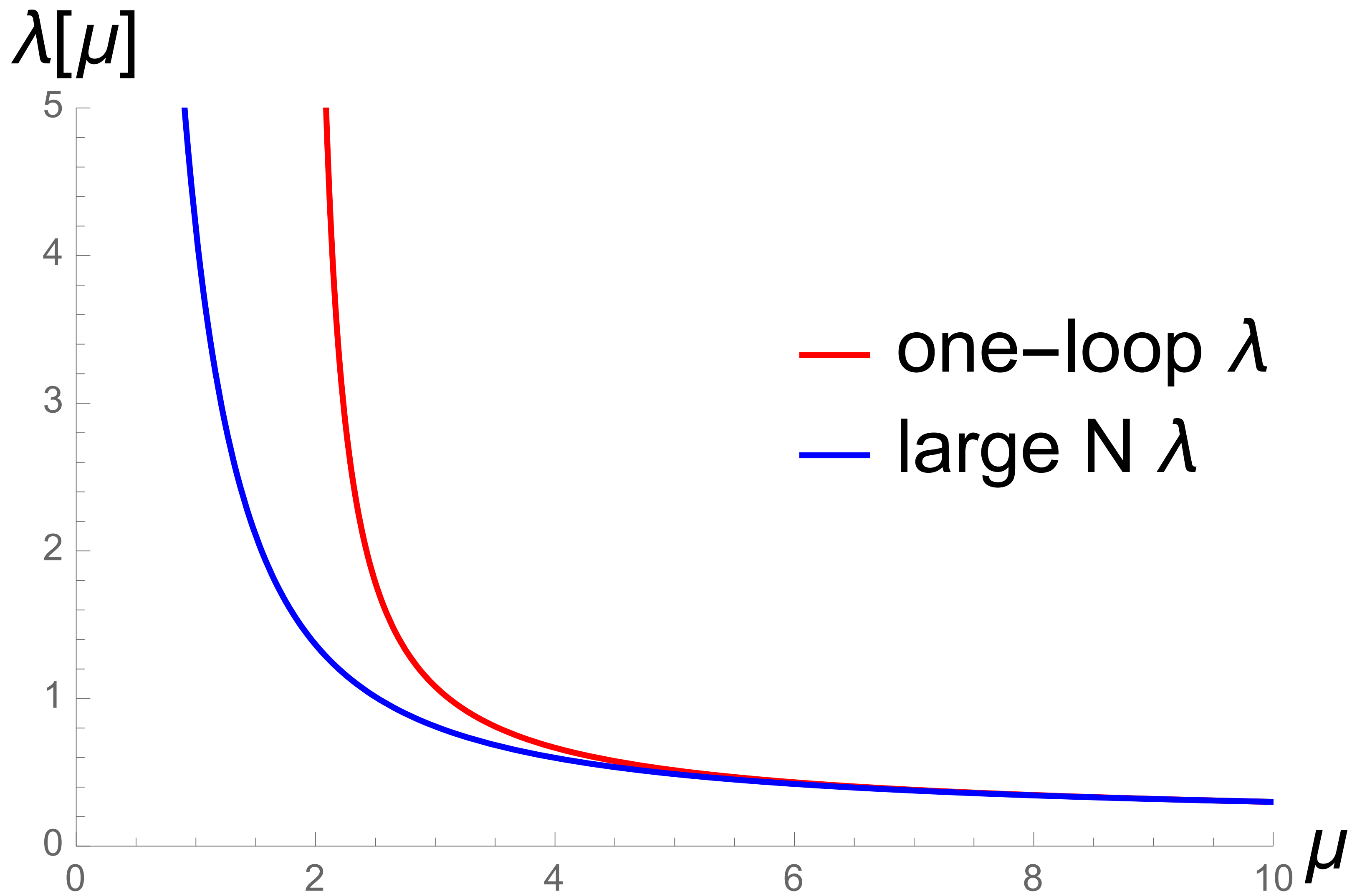
$$\lambda_{\mathrm{P}}[\mu] = \frac{\lambda_0}{2\lambda_0 \log\left(\frac{\mu}{\mu_0}\right) + 1}$$

Exact large N coupling only diverges at $\mu = 0$:

$$\lambda(\mu) = \frac{1}{\log\left(1 + \frac{\mu^2}{\mu_0^2}\left(e^{+1/\lambda_0} - 1\right)\right)}$$

Coupling constant flow

AC, Dorigoni, Unsal coming soon



The large N coupling is infrared-finite.

In QCD literature, phenomenological construction of “IR-finite couplings” is well-explored.

Dokshitzer, Webber, … 1990s

Here exact large N solution gives such a coupling automatically.

Exact large N mass gap & coupling

Only first two coefficients of series expansion of beta functions invariant under renormalization scheme changes.

Perturbative coupling and large N coupling related by non-perturbative scheme change:

$$\lambda = f_P(\lambda_P)$$

one-loop coupling

$$f_P(\lambda_P) = \frac{\lambda_P}{1 + \lambda_P \log(1 + e^{-1/\lambda_P})}$$

$$= \lambda_P \left[1 - e^{-1/\lambda_P} (\lambda_P) + e^{-2/\lambda_P} \left(\tfrac{1}{2}\lambda_P + \lambda_P^2 \right) + \cdots \right]$$

Still exploring: is there some extra universal data in beta functions non-perturbatively?

O(N) model at large N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

The QFT is giving a transseries but no resurgence, due to suppression of fluctuations by large N

$$\Delta\big|_{N=\infty} = \mu\, e^{-\frac{1}{2\lambda}} \left(1 + \frac{1}{2} e^{-\frac{2}{2\lambda}} + \frac{3}{8} e^{-\frac{4}{2\lambda}} + \cdots \right)$$

To see resurgent behavior, need to look at 1/N corrections.

To be specific, we'll continue to examine $< \sigma >$

$$\langle \sigma \rangle = \left\langle \frac{4\pi\lambda}{N} \partial_\mu n_a \partial^\mu n^a \right\rangle = \Delta^2$$

O(N) model at order 1/N

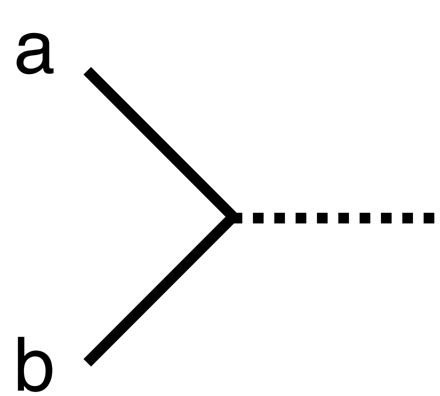
Large N theory consists of N massive fields with mass m = Δ

$$G^{ab}(p) = \frac{\delta^{ab}}{p^2 + m^2}$$

and a field `σ' describing fluctuations around VEV, $\sigma \to \langle \sigma \rangle + \sigma/N^{1/2}$

$$G_\sigma(p) = \frac{-4\pi\sqrt{p^2(p^2+4m^2)}}{\log\left[\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}\right]}$$

with an interaction vertex



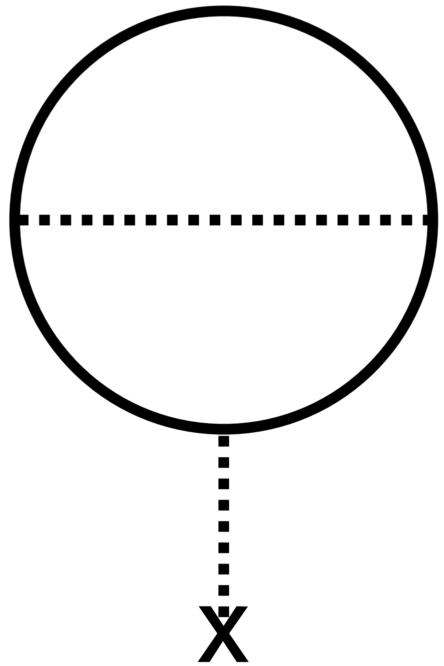
$$\frac{1}{\sqrt{N}}\delta^{ab}$$

Dependence on λ only enters through m!

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

Leading correction to < σ > comes from



$$\langle \sigma \rangle = m^2 + \frac{1}{N} I(\mu, m) + \mathcal{O}(1/N^2)$$

The 1/N correction is UV-divergent. Put cutoff at μ, assume $\mu \sim N^0$

$$I(\mu, m) = \frac{1}{2} G_\sigma(0) \int^{|p| \leq \mu} \frac{d^2 p}{(2\pi)^2} \int \frac{d^2 k}{(2\pi)^2} G_\sigma(p) G_{ab}(k) G^{bc}(k) G^a_c(p+k).$$

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon;
also F. David 1984

Evaluating the integrals, get ugly but (eventually!) instructive result:

$$I(\mu, m) = m^2 \left(-E_i \left[\frac{1}{2} \log A(\mu, m) \right] + E_1 \left[\frac{1}{2} \log A(\mu, m) \right] + 2\gamma_E + 2 \log \left[\frac{1}{2} \log A(\mu, m) \right] - 2 \log \left[1 + \frac{\mu^2}{4m^2} \right] \right)$$

$$A(\mu, m) = \left(\sqrt{1 + \frac{\mu}{4m^2}} + \sqrt{\frac{\mu}{4m^2}} \right)^4$$

The 1/N correction is entirely unambiguous at this stage. Statement almost trivial: Given a regulator, path integral will be unambiguous.

Where's the resurgence?

O(N) model at order 1/N

AC, Dorigoni, Unsal
coming soon

Interested in resurgence properties in λ - so define

$$\tilde{\lambda} = \frac{\lambda}{1 - \lambda \sum_{n=1}^{\infty} c_n e^{-n/\lambda}}$$

c_n = central trinomial coefficients'; series converges.

$$\frac{1}{\tilde{\lambda}} = \frac{1}{2} \log A(\mu, m)$$

Expansions of the exponential-integral functions in $\tilde{\lambda}$ are asymptotic:

$$E_i\left(\frac{1}{\tilde{\lambda}}\right) = \begin{cases} -i\pi + e^{1/\tilde{\lambda}} \sum_{n=0}^{\infty} n! \tilde{\lambda}^{n+1} , & 0 < \arg(\tilde{\lambda}) < \pi , \\ +i\pi + e^{1/\tilde{\lambda}} \sum_{n=0}^{\infty} n! \tilde{\lambda}^{n+1} , & -\pi < \arg(\tilde{\lambda}) < 0 , \end{cases}$$

$$E_1\left(\frac{1}{\tilde{\lambda}}\right) = e^{-1/\tilde{\lambda}} \sum_{n=0}^{\infty} (-1)^n n! \tilde{\lambda}^{n+1} .$$

O(N) model at order 1/N

Plug these expansions back into < σ >, to find

$$I(\mu, \tilde{\lambda}) \simeq \mu^2 \left(\sum_{n=0} n! \tilde{\lambda}^{n+1} \mp i \pi e^{-\frac{1}{\tilde{\lambda}}} + \cdots \right)$$

Factorial growth leads to renormalon ambiguity, which is cancelled by non-perturbative contribution.

$$\text{size} \sim \mu^2 e^{-1/\tilde{\lambda}} \sim m^2$$

Are there further renormalon ambiguities?

Behavior in terms of the standard perturbative coupling?

Renormalon ambiguities

AC, Dorigoni, Unsal
coming soon

Full expression for spin-wave condensate:

$$\langle\alpha\rangle - \langle\alpha\rangle_{N=\infty} = \mu^2 \Bigg[-\sum_{n=0}^{\infty} n!\tilde{\lambda}^{n+1} + e^{-1/\tilde{\lambda}} \Bigg(\pm i\pi + 2\gamma - 2\log\tilde{\lambda} - \frac{2}{\tilde{\lambda}} + 4\log 2 - 2\sum_{n=0}^{\infty} n!\tilde{\lambda}^{n+1} \Bigg)$$

$$\sum_{k=2}^{\infty} e^{-k/\tilde{\lambda}} \Bigg(-(k+1)\sum_{n=0}^{\infty} n!\tilde{\lambda}^{n+1} + (k-1)\sum_{n=0}^{\infty} (-1)^n n!\tilde{\lambda}^{n+1} +$$

$$+k(2\gamma - 2\log\tilde{\lambda} \pm i\pi\,) + (1-(-1)^k)\left\{ \frac{k+1}{2\tilde{\lambda}} + (k+1)\log(2) \right\} + 4p_k \Bigg) \Bigg]$$

complicated rational numbers

Full set of IR renormalon ambiguities of size $e^{-1/\lambda}$, $e^{-2/\lambda}$, $e^{-3/\lambda}$, …

UV renormalons also present!

Only singularities in Borel plane t conjugate to λ are at t = -1, +1.

Probably a large N accident, but we do not know for sure.

Behavior in perturbative coupling

AC, Dorigoni, Unsal
coming soon

Pass to perturbative one-loop coupling

$$\tilde{\lambda} = \lambda_{\rm P}\left[1 - 2\lambda_{\rm P} e^{-1/\lambda_{\rm P}} + \lambda_P e^{-2/\lambda_P}(3+4\lambda_P) - \frac{4}{3}\lambda_P\left(5+9\lambda_P+6\lambda_P^2\right)e^{-3/\lambda_P} + \cdots\right].$$

Then transseries looks like

$$\langle\alpha\rangle - \langle\alpha\rangle_{N=\infty} = \frac{1}{N}\mu^2\left[\sum_{n=0}^{\infty} n!\lambda_P^{n+1} + e^{-1/\lambda_P}\left(\mp i\pi - 2\left[\gamma_E + \log\left(\frac{1}{\lambda_P}\right)\right] + 10\sum_{n=0}^{\infty} n!\lambda_P^{n+1}\right) + \cdots\right]$$

Key point: all perturbative coefficients are rational!

With a canonically-normalized 't Hooft coupling, rational*$(4\pi)^{-(n+1)}$

But this is sum of Feynman diagrams to all loop orders. Diagram by diagram, increasing transcendentality with loop order

Transcendentals all cancel. Consequence of integrability?

$\mathcal{N}$=4 SYM: Kotikov, Lipatov; Bern, Kosower…; Beisert,…; …

ladder diagrams: Kreimer,….

O(N) model at order 1/N

AC, Dorigoni, Unsal coming soon;

At this point you could ask, if < σ > = m^2 + I(μ,λ)/N + …, and

$$I(\mu,\lambda) \simeq \mu^2 \left(\sum_{n=0} n!\,\lambda^{n+1} \mp i\,\pi\, e^{-\frac{1}{\lambda}} + \cdots \right)$$

(1) What happens if we subtract `all' divergences by counter-terms?
Does < σ > then become ambiguous?

No: counter-terms pick up ambiguities, but < σ > stays unambiguous.

(2) If dim-reg is used, no power divergences. Ambiguous result? David; Beneke; yes.

No. "Dimensional regularization" is not
a valid regulator non-perturbatively.

Dimensional regularization

Idea of dim-reg:

$$\int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^a}{(p^2+m^2)^b} \longrightarrow \mu^{d-n} \int \frac{d^n p}{(2\pi)^n} \frac{(p^2)^a}{(p^2+m^2)^b}$$

(1) Find `n' where integral from |p|=0 to |p| = ∞ converges, then do it:

$$\frac{1}{(4\pi)^{d/2}} \frac{\mu^{d-n}}{\Delta^{b-a-d/2}} \frac{\Gamma(a+d/2)\Gamma(b-a-d/2)}{\Gamma(b)\Gamma(d/2)}$$

(2) Expand near desired dimension d, discard poles like $1/(n-d) = 1/\epsilon$

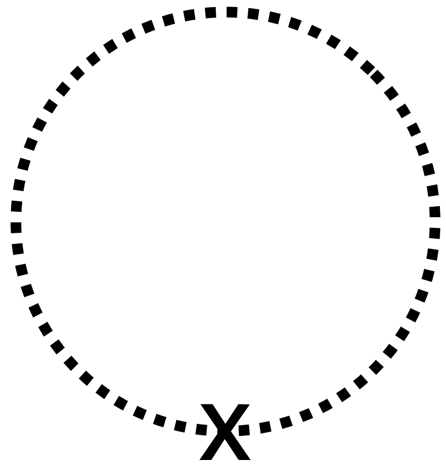
(3) Profit from remaining $\log(m^2/\mu^2)$ terms!

No explicit power divergences.

Recipe works to any fixed order in perturbation theory.

Failure of dimensional regularization

In the large N O(N) model, dim-reg fails at step 1. Example:

$$\langle \sigma^2 \rangle - \langle \sigma^2 \rangle_{N=\infty} =$$  $$= \frac{1}{N} G_\sigma(0)^2 \int \frac{d^2 p}{(2\pi)^2} G_\sigma(p)$$

$$p \to \infty, \ G_\sigma \sim \frac{p^2}{\log(p^2/m^2)}$$

$$p \to 0, \ G_\sigma \sim m^2$$

(Using $G_\sigma(p,n)$ doesn't help!)

In dimension n, need Re[n] < -3 in UV and Re[n] > 0 in IR for convegence.

No choice of n gives finite result.
'Dimensional regularization' is not a regularization non-perturbatively.

Perhaps not so shocking, but amusing to see explicit illustration.

(Problem persists in correlation functions.)

Conclusions

Not obvious that resurgence should apply in d > 1.

But it **does**, as illustrated using large N solution of 2D models!

“We know much more than we can prove…”

Peculiarity of vector-like QFTs: need 1/N effects to see resurgence.
Expect resurgence at leading order in matrix-like QFTs.

Large N β-function of 2D sigma models is **not** one-loop exact - there are non-perturbative corrections.

1/N: Full set of renormalons, but only a couple Borel singularities.

Interesting rationality of all-loop perturbation theory.

Regularization is subtle at non-perturbative level.
Dimensional regularization isn’t regularization.

Privileged role for cut-off regulators?

The end