Exact WKB Analysis for Continuous and Discrete Painlevé Equations

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Resurgence in Gauge and String Theories

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0 Introduction

Exact WKB analysis

= **WKB analysis based on the Borel resummation method**

(1) Schrödinger eq'ns case
(Voros, Pham-Delabaere, Aoki-Kawai-T., . . . ; 1980 –)

(2) (continuous) Painlevé eq'ns case
(Aoki-Kawai-T., . . . ; 1990 –)

(3) discrete Painlevé eq'ns case (joint research with N. Joshi)

In this talk we first review the exact WKB analysis for Schrödinger equations and continuous Painlevé equations, and then discuss our recent research on discrete Painlevé equations.

1 Schrödinger eq'ns case

$$\left(\eta^{-2}\frac{d^2}{dx^2} - Q(x)\right)\psi = 0 \qquad (\eta > 0 : \text{ large parameter})$$

$$\psi = \exp\left(\pm\eta\int^x \sqrt{Q(x)}\,dx\right)\sum_{n=0}^{\infty}\eta^{-(n+1/2)}\psi_{\pm,n}(x) \; : \text{ WKB solution}$$

Letting $s(x) = \int^x \sqrt{Q(x)}\,dx$**, we consider the Borel sum of** $\psi_{\pm}$**:**

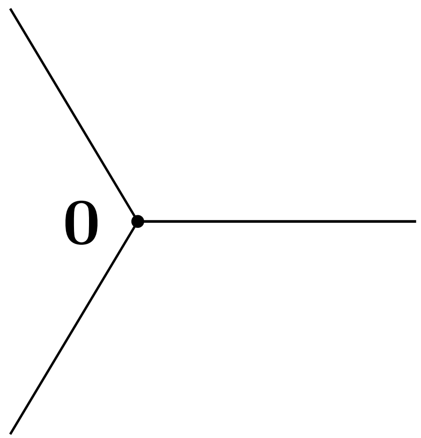
$$\psi_{\pm,B}(x,y) = \sum_n \frac{\psi_{\pm,n}(x)}{\Gamma(n+1/2)}(y \pm s(x))^{n-1/2} \; : \text{ Borel transform}$$

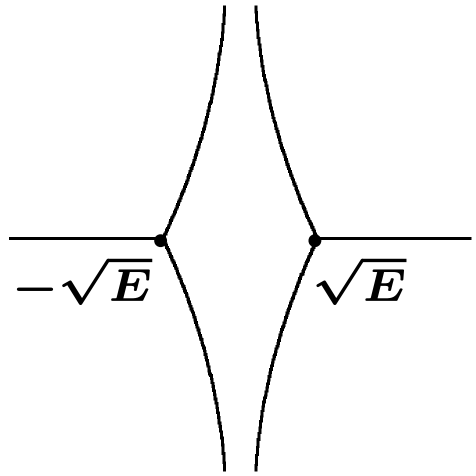
$$\Psi_{\pm}(x,\eta) = \int_{\mp s(x)}^{\infty} e^{-\eta y}\psi_{\pm,B}(x,y)\,dy \qquad : \text{ Borel sum}$$

Stokes geometry

$$\left(\begin{array}{lcl} \text{turning point} & \Longleftrightarrow & \text{zero of } Q(x) \\ \text{Stokes curve} & \Longleftrightarrow & \mathrm{Im}\ \eta \displaystyle\int_a^x \sqrt{Q(x)}\, dx = 0 \end{array} \right.$$

e.g., Airy : $Q(x) = x$ Weber : $Q(x) = x^2 - E$





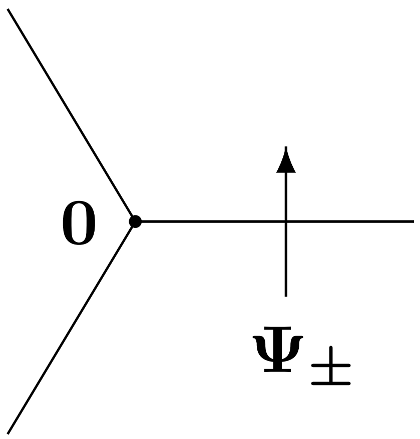
(cf. Gaiotto-Moore-Neitzke : "spectral network")

Fact 1 (Koike-Schäfke)

WKB sol'ns are Borel sum'ble except on Stokes curves.

Fact 2 Stokes phenomena occur on each Stokes curve.

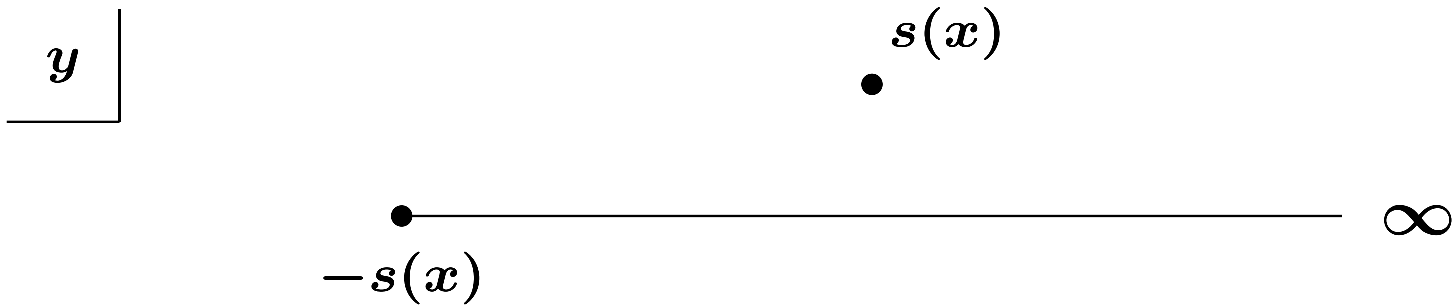
e.g., Airy



Connection formula (Voros, Pham-Delabaere-Dillinger, Aoki-Kawai-T.)

$$\begin{cases} \Psi_+ \rightsquigarrow \Psi_+ + i\Psi_- \\ \Psi_- \rightsquigarrow \Psi_- \end{cases} \tag{1}$$

Remark (1) is induced by movable singularity of $\psi_{+,B}(x,y)$.



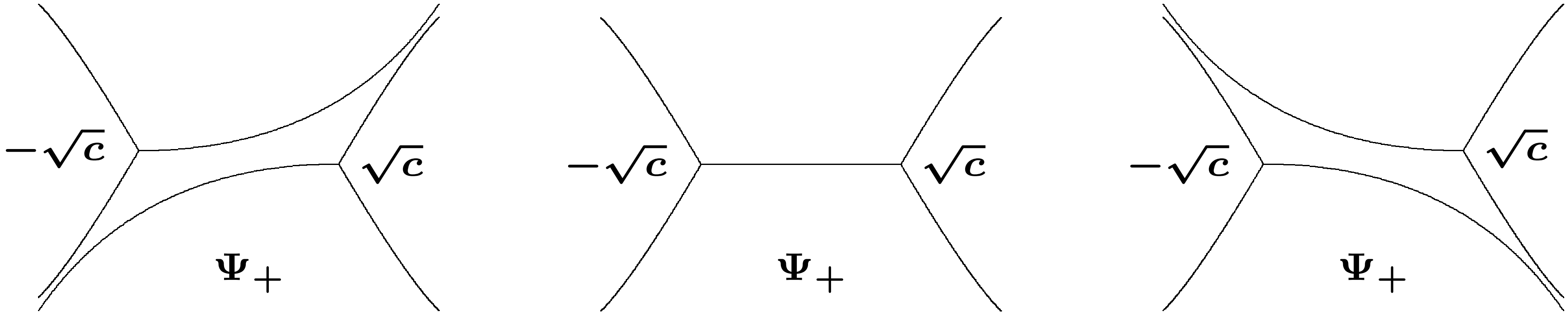
$$\Delta_{y=s(x)}\psi_{+,B}(x,y) = i\psi_{-,B}(x,y)$$

Fact 3 (Pham-Delabaere-Dillinger, Aoki-Kawai-T.)

Wall-crossing phenomena occur when a parameter changes.

e.g., $Q(x) = c - x^2/4$

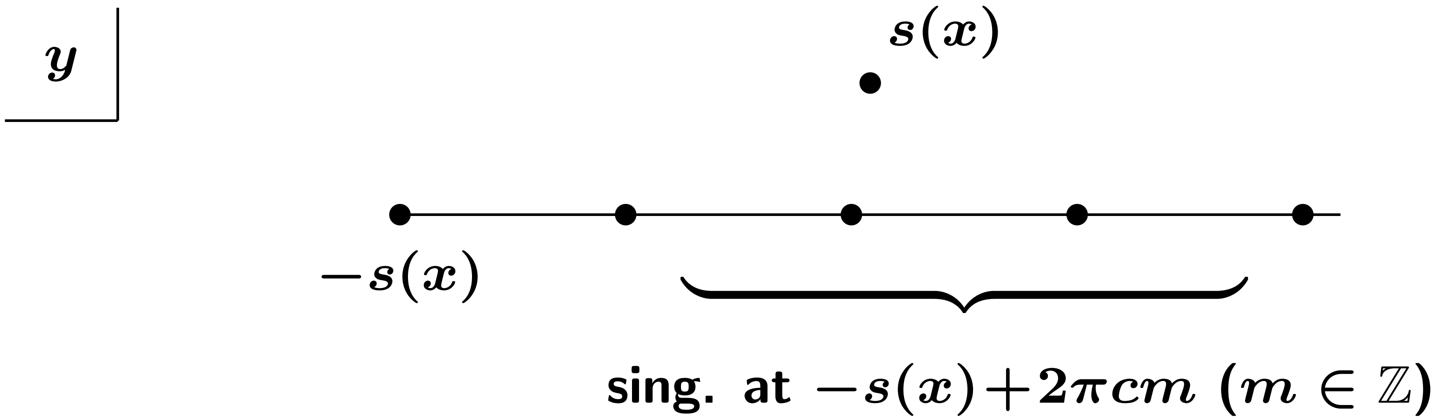
$(\mathrm{Im}\, c < 0)$ $(c > 0)$ $(\mathrm{Im}\, c > 0)$



$$\Psi_+ \quad \rightsquigarrow \quad (1 + e^{-2\pi c\eta})^{-1/2}\Psi_+ \tag{2}$$

(for $\mathrm{Im}\, c < 0$) (for $\mathrm{Im}\, c > 0$)

Remark (2) is induced by fixed singularity of $\psi_{+,B}(x,y)$.



$$\Delta_{y=-s(x)+2\pi cm}\,\psi_{+,B} = \frac{(-1)^m}{2m}\,\psi_{+,B}$$

2 (Continuous) Painlevé eq'ns case

Painlevé equations (Painlevé-Gambier)

(PJ) $$\frac{d^2\lambda}{dt^2} = \eta^2 F_J(\lambda, t) + G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) \qquad (J = \mathrm{I}, \ldots, \mathrm{VI})$$

e.g., **(PI)** $$\lambda'' = \eta^2(6\lambda^2 + t)$$

(PII) $$\lambda'' = \eta^2(2\lambda^3 + t\lambda + \zeta) \qquad (\zeta \in \mathbb{C})$$

- **Painlevé property, i.e., all movable singular points are poles.**
- **Related to isomonodromic deformations (Fuchs, . . .), that is, (PJ) describes the isomonodromic condition for**

(SLJ) $$\left(\eta^{-2}\frac{d^2}{dx^2} - Q_J(x, t, \lambda, \nu, \eta)\right)\psi = 0.$$

Formal solutions of (PJ)

(1) formal power series solution

$$\lambda^{(0)}(t,\eta) = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \cdots$$

where $\lambda_0(t)$ is determined by $F_J(\lambda_0(t), t) = 0$.

(2) transseries solution

$$\lambda(t,\eta;\alpha) = \lambda^{(0)} + (\eta^{-1/2}\alpha)\lambda^{(1)} + (\eta^{-1/2}\alpha)^2\lambda^{(2)} + \cdots$$

where $\alpha \in \mathbb{C}$ is a free parameter and $\lambda^{(1)}$ satisfies the linearized equation of (PJ) at $\lambda^{(0)}$:

$$(\Delta\text{PJ}) \qquad \eta^{-2}\frac{d^2}{dt^2}\lambda^{(1)} = \frac{\partial F_J}{\partial \lambda}(\lambda^{(0)}, t)\,\lambda^{(1)} + \text{(l.o.t.)}$$

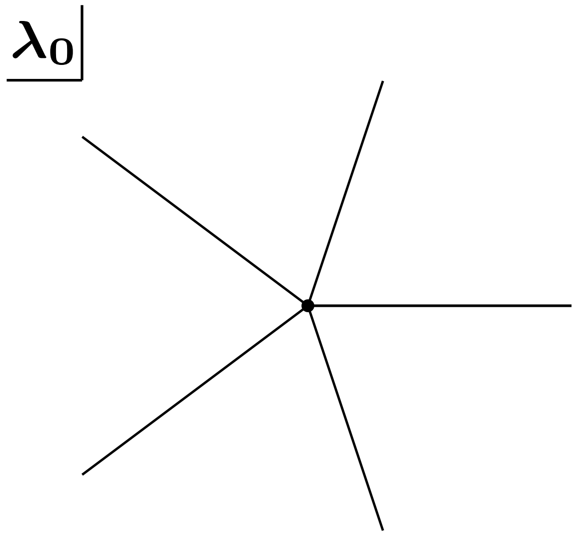
Stokes geometry

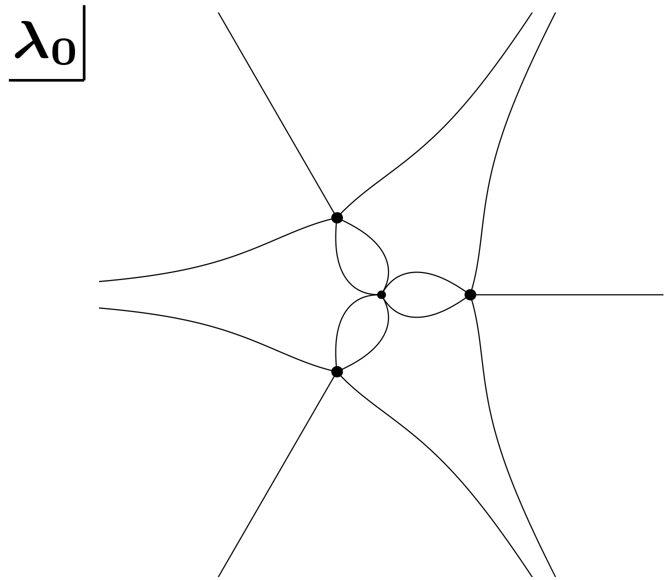
turning point & Stokes curve of (PJ)

$\Longleftrightarrow$ **turning point & Stokes curve of (ΔPJ)**

e.g., (PI) $\dfrac{\partial F_{\rm I}}{\partial \lambda} = 12\lambda_0 + \cdots$ with $6\lambda_0^2 + t = 0$

(PII) $\dfrac{\partial F_{\rm II}}{\partial \lambda} = (6\lambda_0^2 + t) + \cdots$ with $2\lambda_0^3 + t\lambda_0 + \zeta = 0$



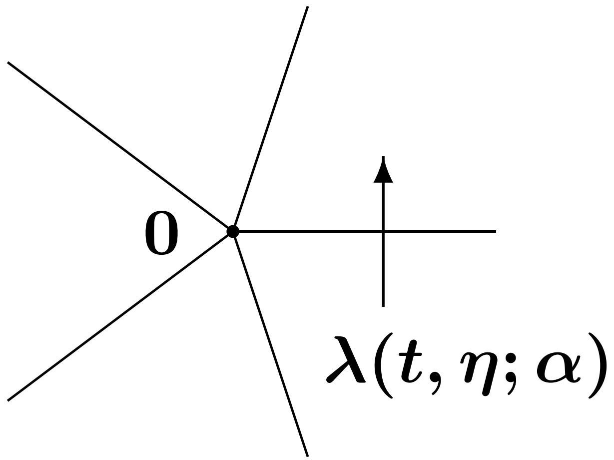


(when $\zeta > 0$)

Connection formula on a Stokes curve

(Kapaev, Kitaev, T., Costin, . . .)

e.g., **(PI)** $\lambda'' = \eta^2(6\lambda^2 + t)$



$$\lambda(t, \eta; \alpha) \rightsquigarrow \lambda\left(t, \eta; \alpha - \frac{i}{2\sqrt{\pi}}\right) \quad (3)$$

Remark **Formula** (3) **is obtained by using the isomonodromic deformation method, that is,**

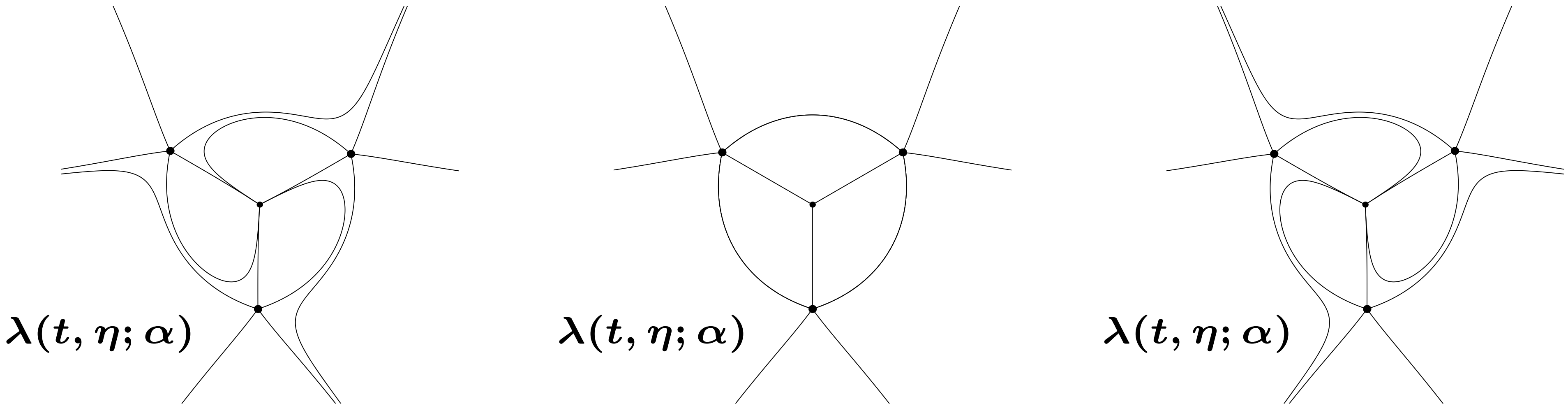
computatuin of monodromy data of (SLJ)

$+$ **isomonodromic property** $\longrightarrow$ (3)

Wall-crossing formula (Iwaki)

e.g., (PII) $\lambda'' = \eta^2(2\lambda^3 + t\lambda + \zeta)$

$(\arg\zeta < \pi/2)$ $(\arg\zeta = \pi/2)$ $(\arg\zeta > \pi/2)$



$$\lambda(t, \eta; \alpha) \quad \rightsquigarrow \quad \lambda(t, \eta; (1 + e^{2\pi i\zeta\eta})\alpha) \tag{4}$$

(for $\arg\zeta < \pi/2$) (for $\arg\zeta > \pi/2$)

3 Discrete Painlevé eq'ns case

(joint work with N. Joshi)

(PII) $$\eta^{-2}\lambda'' = 2\lambda^3 + t\lambda + \zeta$$

Solutions admit the Bäcklund transformation :

$$\sigma^{\pm 1}(\lambda) = \lambda\Big|_{\zeta \mapsto \zeta \pm \eta^{-1}} = -\lambda - \frac{\zeta \pm \eta^{-1}/2}{\lambda^2 \pm \eta^{-1}(d\lambda/dt) + t/2}$$

$\Longrightarrow$ **(alt-dPI)** $$\frac{\zeta + \eta^{-1}/2}{\lambda + \sigma(\lambda)} + \frac{\zeta - \eta^{-1}/2}{\lambda + \sigma^{-1}(\lambda)} + 2\lambda^2 + t = 0$$

Purpose : To analyze such discrete Painlevé equations from the viewpoint of exact WKB analysis.

- Replace

$$\sigma(\lambda) = \lambda\Big|_{\zeta \mapsto \zeta + \eta^{-1}} = \sum_{n \geq 0} \frac{\eta^{-n}}{n!} \frac{\partial^n \lambda}{\partial \zeta^n},$$

$\longrightarrow$ (alt-dPI) becomes an ∞-order singular-perturbative differential equation w.r.t ξ.

- In particular, we discuss

$$\begin{cases} \eta^{-2} \dfrac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \zeta, \\ \sigma(\lambda) = -\lambda - \dfrac{\zeta + \eta^{-1}/2}{\lambda^2 + \eta^{-1}(d\lambda/dt) + t/2}. \end{cases} \tag{5}$$

Formal solutions of (5)

(1) formal power series solution

$$\lambda^{(0)}(t,\eta) = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \cdots$$

with $2\lambda_0^3 + t\lambda_0 + \zeta = 0.$

(2) transseries solution

$$\lambda(t,\eta;\alpha) = \lambda^{(0)} + (\eta^{-1/2}\alpha)\lambda^{(1)} + (\eta^{-1/2}\alpha)^2\lambda^{(2)} + \cdots$$

where

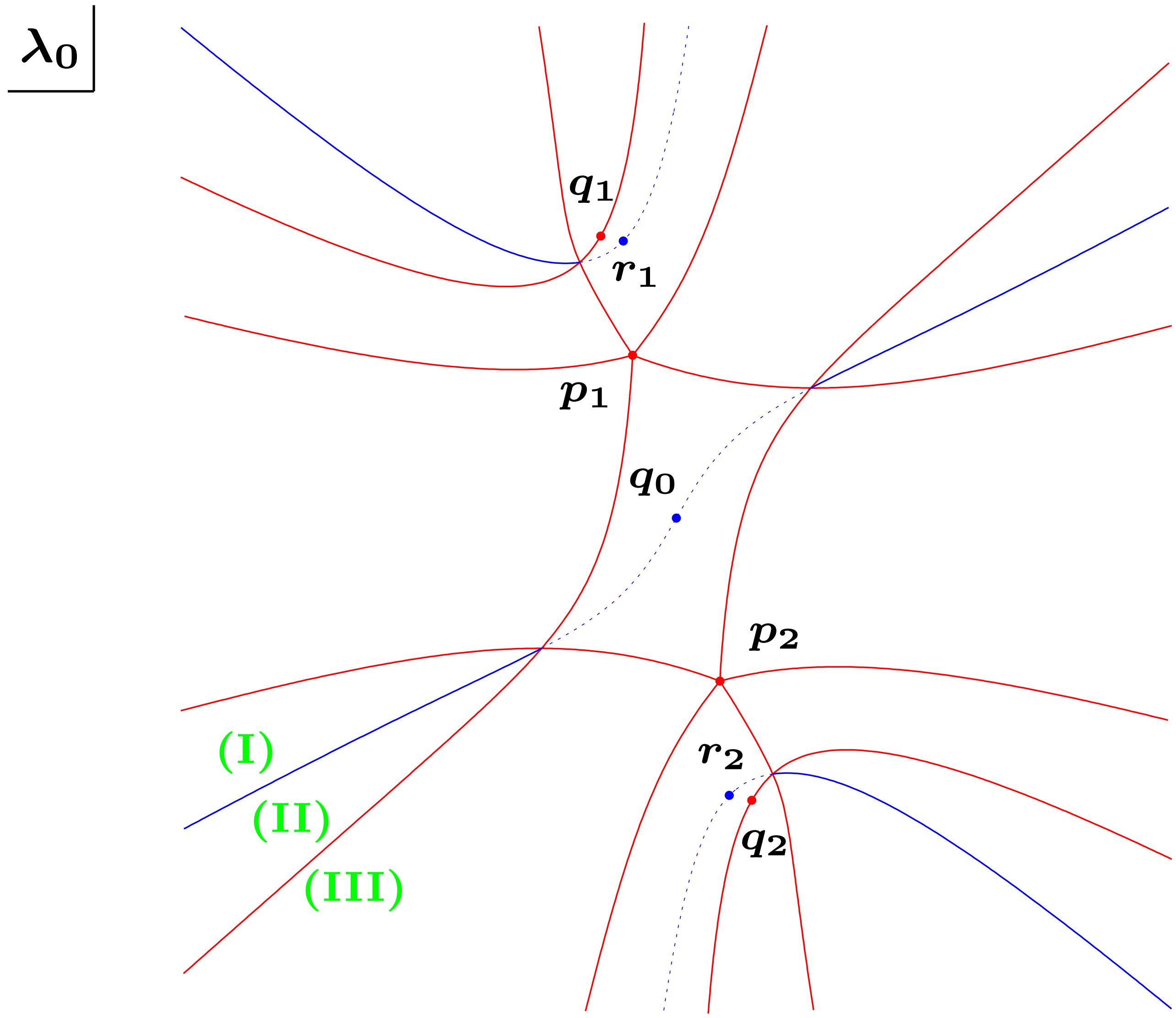
$$\lambda^{(k)} = \exp\left(k\eta\int^{(t,\zeta)}(T_{-1}dt + Z_{-1}d\zeta)\right)\sum_n \eta^{-n}\lambda_n^{(k)}(t,\zeta),$$

$$\left(\, T_{-1} = \sqrt{6\lambda_0^2 + t}, \quad Z_{-1} = \cosh^{-1}\left(8\lambda_0^3/\zeta - 1\right)\right),$$

$$\alpha = \sum_l \alpha_l e^{2\pi i l\eta\zeta},$$ **(** α_l **(**$l \geq 0$**) : free parameters).**

Stokes geometry of (alt-dPI) is defined in terms of Z_{-1}.

e.g., for $t = e^{\pi i/6}$



Connection formula for (alt-dPI)

$$\alpha_{\mathrm{II}} = \alpha_{\mathrm{I}}(1 + e^{2\pi i\eta\zeta}) \tag{6}$$

$$\alpha_{\mathrm{III}} = \alpha_{\mathrm{II}} + \frac{i}{2\sqrt{\pi}} e^{2\pi i\eta\zeta} \tag{7}$$

where α_J $(J = \mathrm{I}, \mathrm{II}, \mathrm{III})$ denote the free parameters when ζ belongs to Region (J).

Remark **The relation (6) is the same as the wall-crossing formula for (PII). Similarly, the relation (7) is the same as the connection formula for (PII), which is equivalent also to that for (PI).**

4 Concluding remarks

▶ **Exact WKB analysis is effective also for the discrete Painlevé equation**

(alt-dPI)
$$\frac{\zeta + \eta^{-1}/2}{\lambda + \sigma(\lambda)} + \frac{\zeta - \eta^{-1}/2}{\lambda + \sigma^{-1}(\lambda)} + 2\lambda^2 + t = 0.$$

▶ **In the discussion of (alt-dPI) the wall-crossing formula & connection formula for (PII) simultaneously appear in the same context.**

▶ **More global studies of (alt-dPI) and extension to the other discrete Painlevé equations are interesting future problems.**