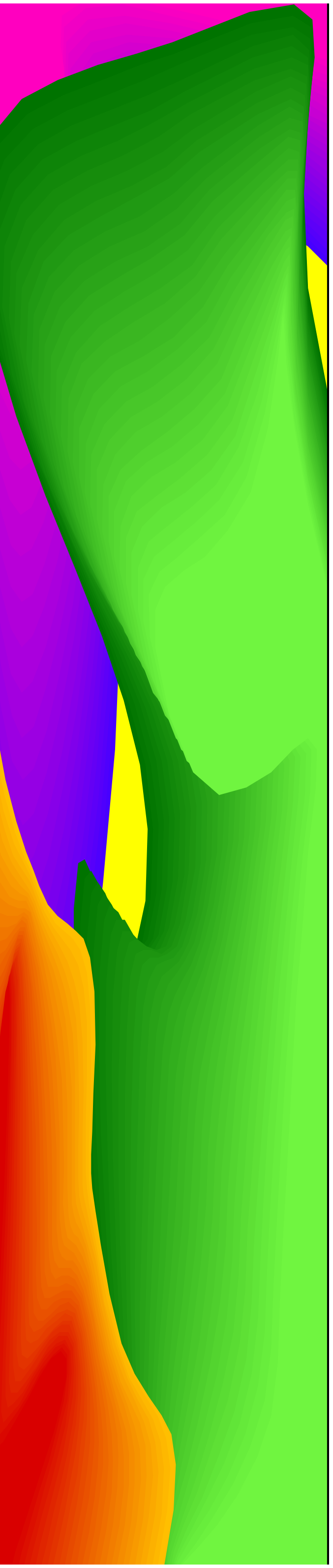
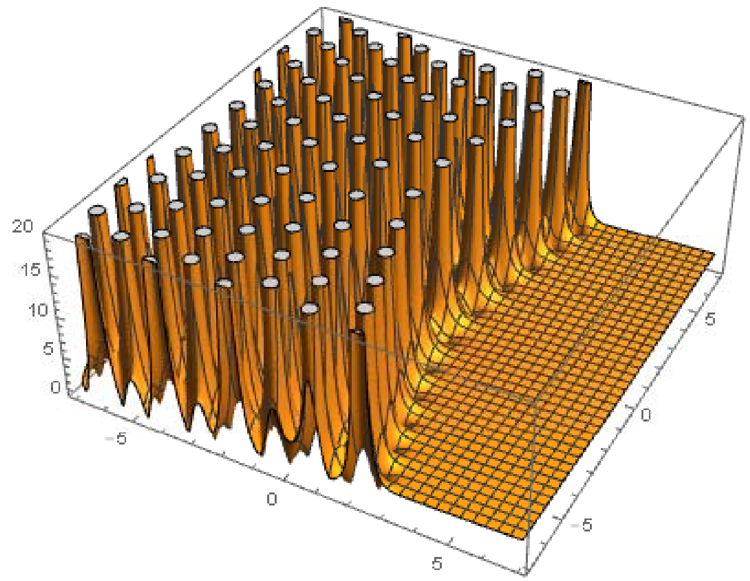


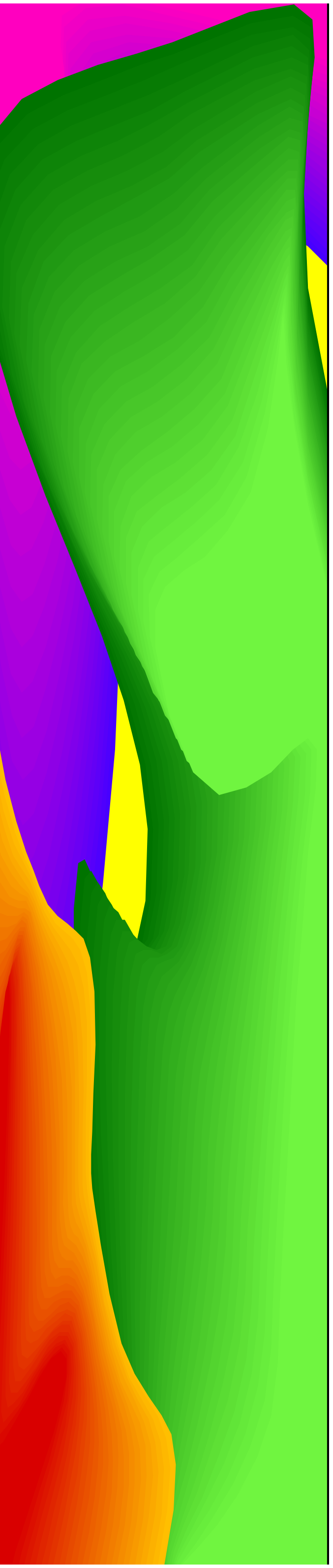
Two-parameter transseries for Painlevé I



Marcel Vonk (University of Amsterdam)

Resurgence in Gauge and String Theories

Lisbon, 21 July 2016



Very much work in progress!

160x.xxxx *On the modularity of Painlevé Resurgent Transseries* (R. Schiappa, MV)

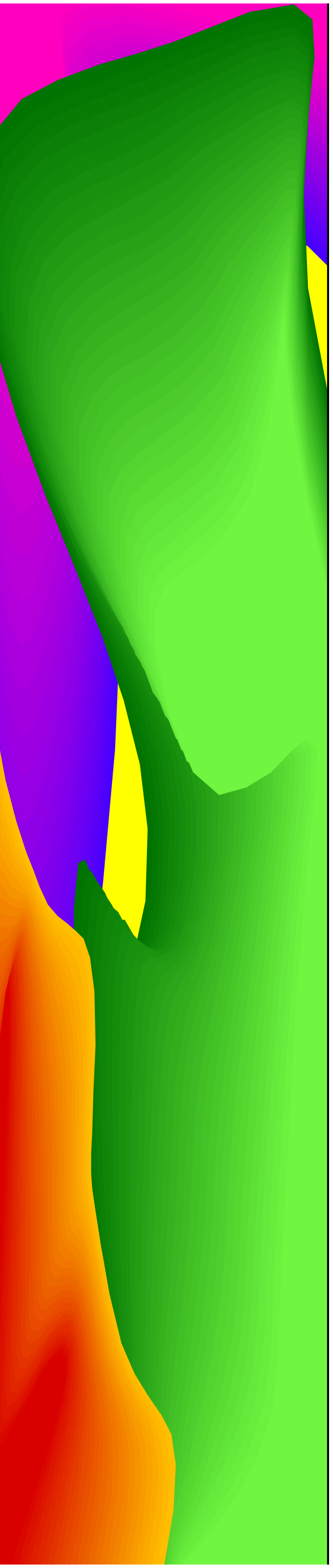
Our earlier work on resurgence, transseries and (among other things) Painlevé I:

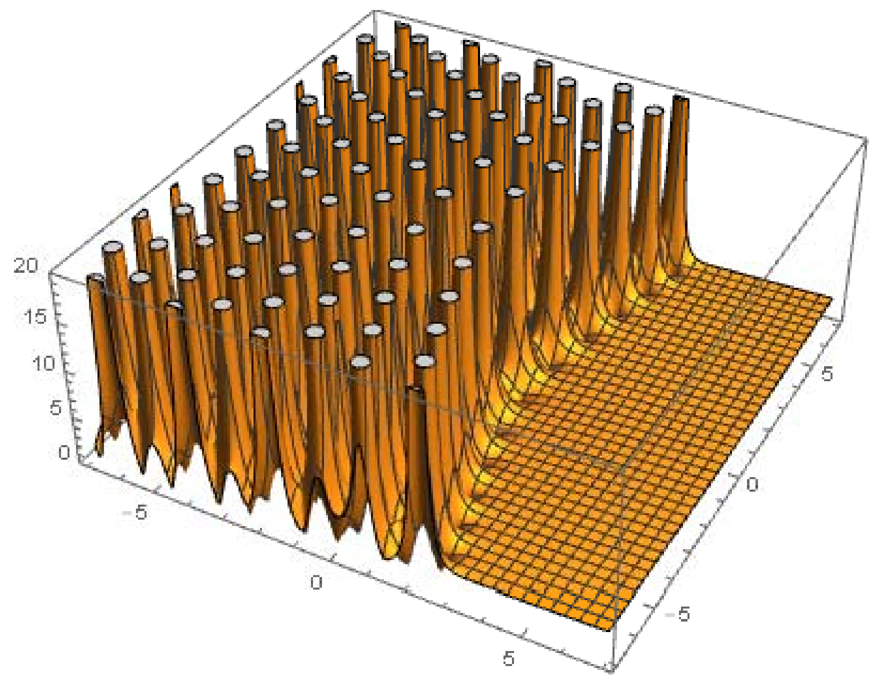
1106.5922 *The resurgence of instantons in string theory* (I. Aniceto, RS, MV)

Influenced by work by Costin et al. (transasymptotics), David / Eynard / Mariño et al. (theta functions) and many others.

Outline

1. The Painlevé I equation
2. Numerical solutions
3. Transseries solution
4. Transasymptotics: beyond first poles
5. The second parameter
6. Modularity



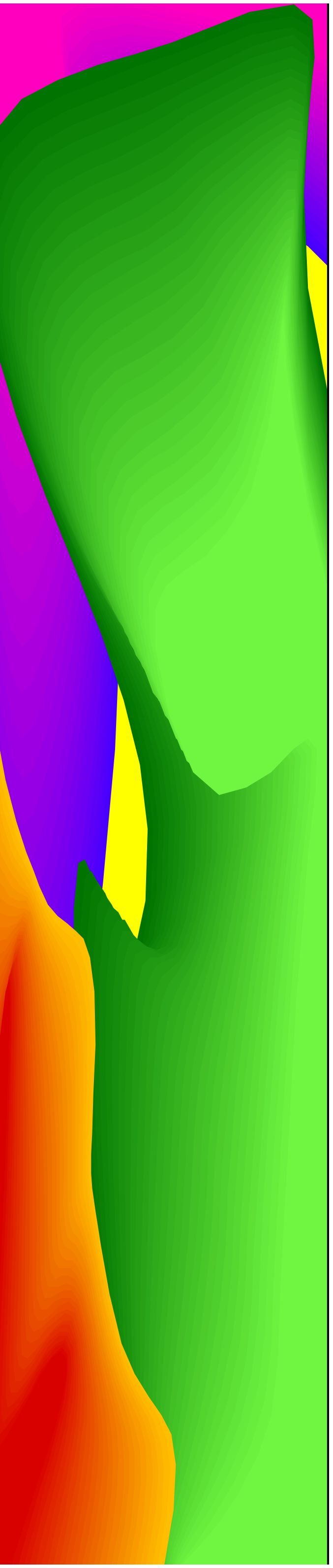


1. The Painlevé I equation

The Painlevé I equation

Paul Painlevé (1863-1933) studied second order ODEs whose only moveable singularities are poles.

6 classes found: Painlevé transcendants.



The Painlevé I equation

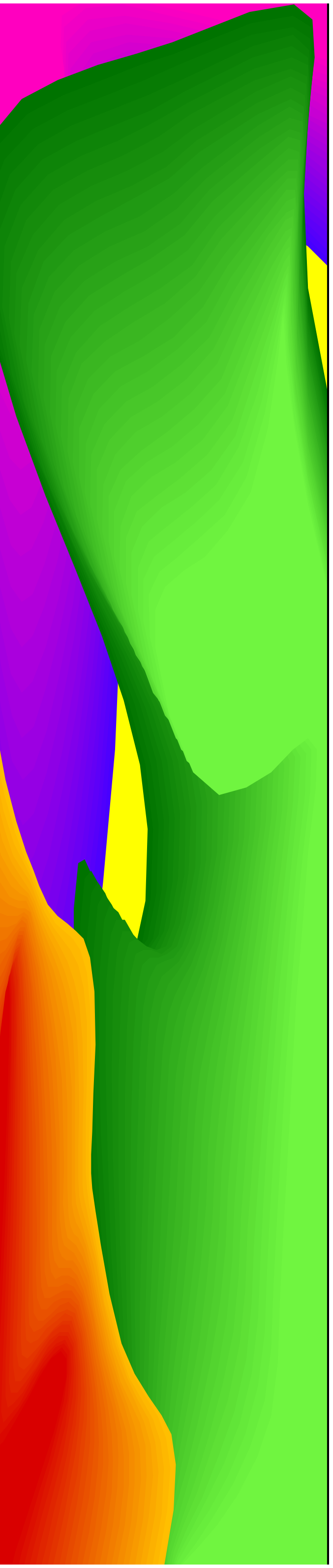
We study the Painlevé I equation:

$$u^2(z) - \frac{1}{6}u''(z) = z$$

Studied extensively by mathematicians.

Many applications in physics:

- 2d quantum gravity
- Minimal string theories
- Double scaling limits of matrix models
- …



The Painlevé I equation

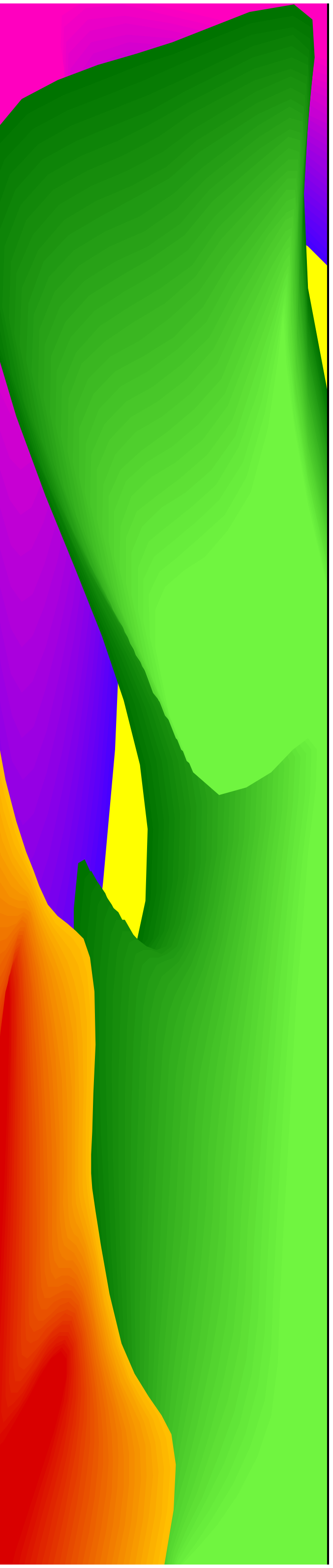
Boutroux investigated solutions in detail.

Some properties: $$u^2(z) - \frac{1}{6}u''(z) = z$$

1) The equation has the symmetry

$$z \to e^{2\pi i/5} z, \qquad u \to e^{-4\pi i/5} u$$

As a result, there is a $\mathbf{Z}_5$-action on the space of solutions. Moreover, the *z*-plane can be divided into five sectors where the solutions may have different asymptotics.



The Painlevé I equation

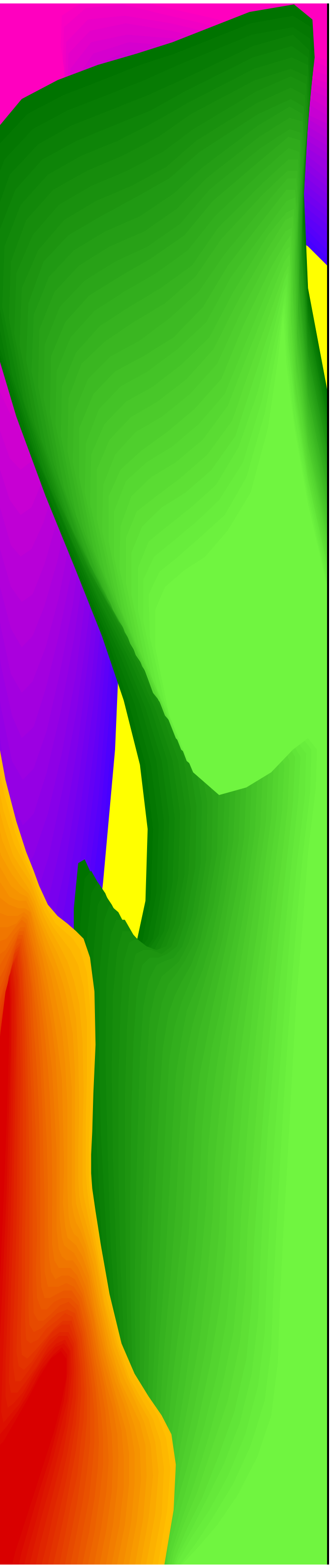
$$u^2(z) - \frac{1}{6}u''(z) = z$$

2) All poles are double poles with the same leading coefficient:

$$u(z) = \frac{1}{(z-z_0)^2} + \frac{3z_0}{5}(z-z_0)^2 + (z-z_0)^3 + h(z-z_0)^4 + \mathcal{O}\left((z-z_0)^5\right)$$

Note the second parameter, h.

Generic solution has infinitely many poles throughout the complex *z*-plane.



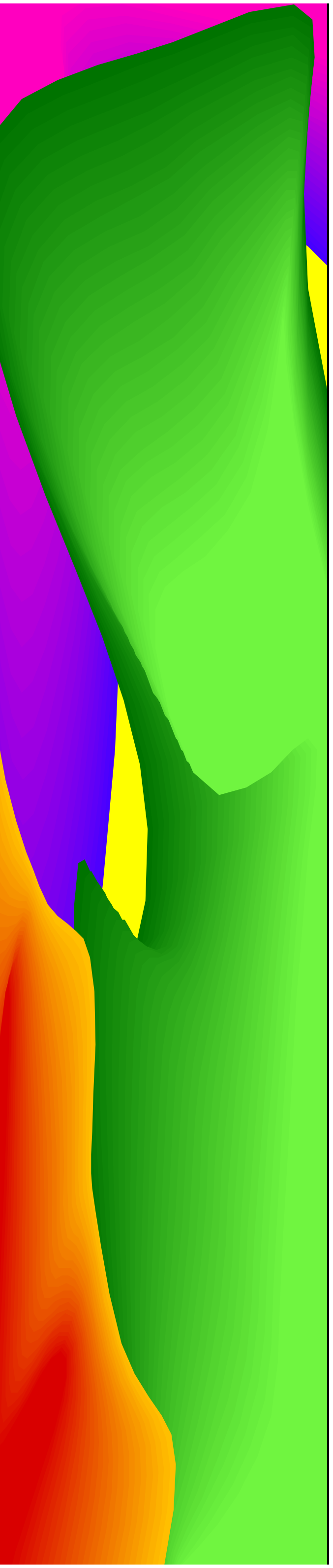
The Painlevé I equation

$$u(z) = \frac{1}{(z-z_0)^2} + \frac{3z_0}{5}(z-z_0)^2 + (z-z_0)^3 + h(z-z_0)^4 + \mathcal{O}\left((z-z_0)^5\right)$$

In physics, one is often interested in the associated free energy and partition function:

$$F''(z) = -u(z), \qquad Z(z) = e^{F(z)}$$

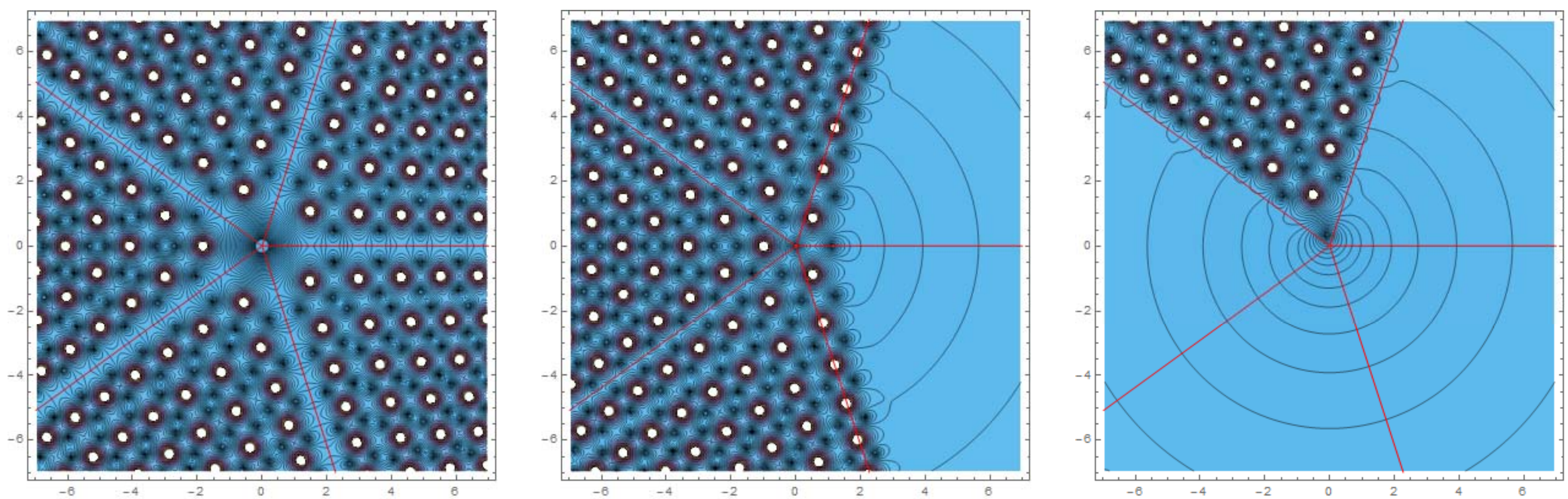
Note: double pole of $u \leftrightarrow$ zero of Z.

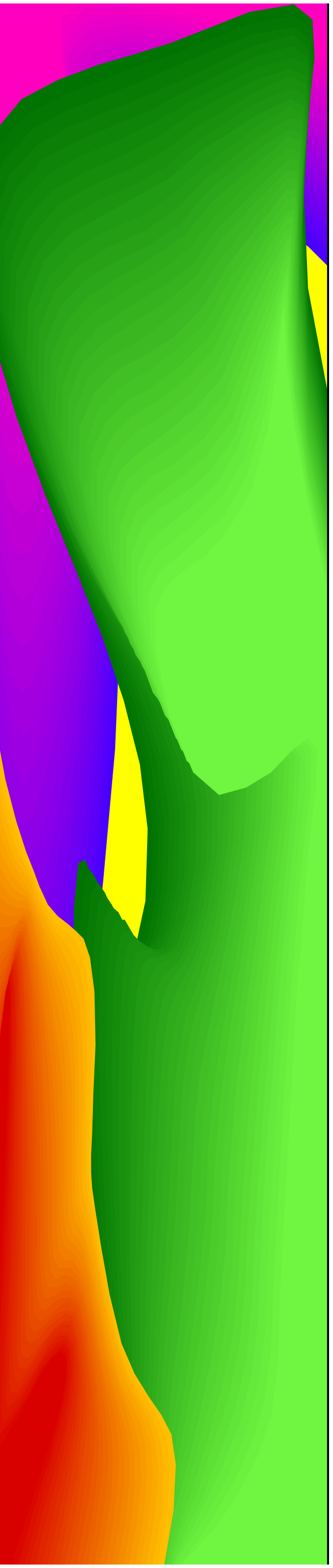


The Painlevé I equation

$$u^2(z) - \frac{1}{6}u''(z) = z$$

3) Special solutions: tronquées and tritronquée.

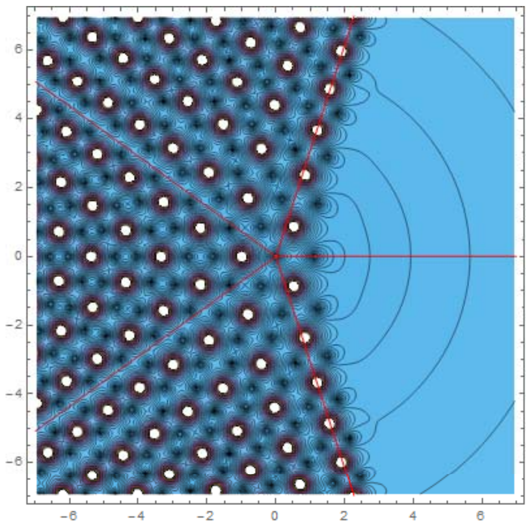


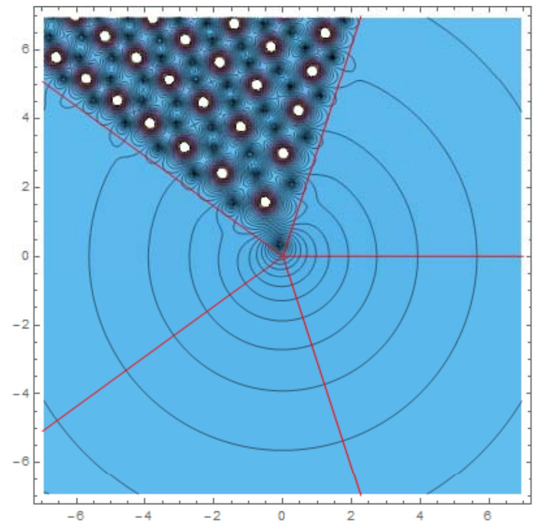


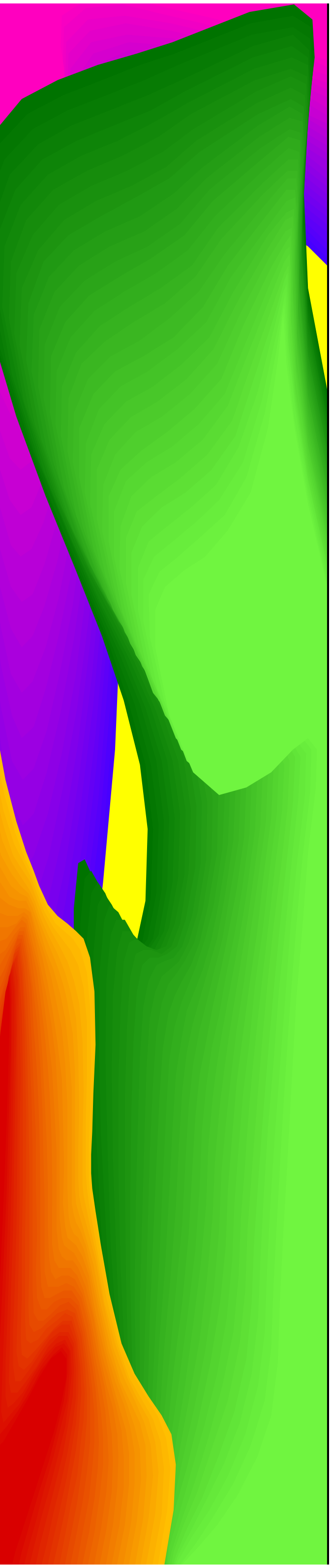
The Painlevé I equation

$$u^2(z) - \frac{1}{6}u''(z) = z$$

4) In the pole-free sectors, the solutions behave asymptotically as u ~ √z







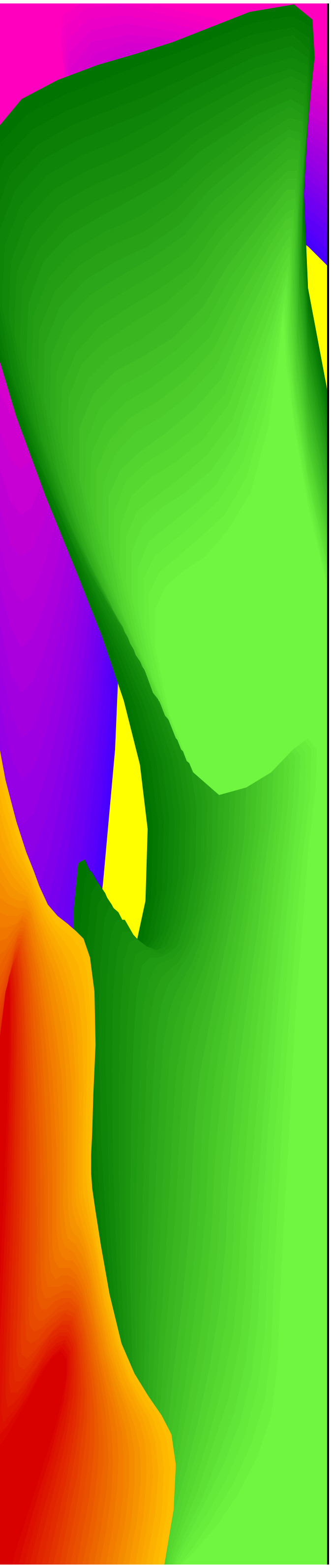
The Painlevé I equation

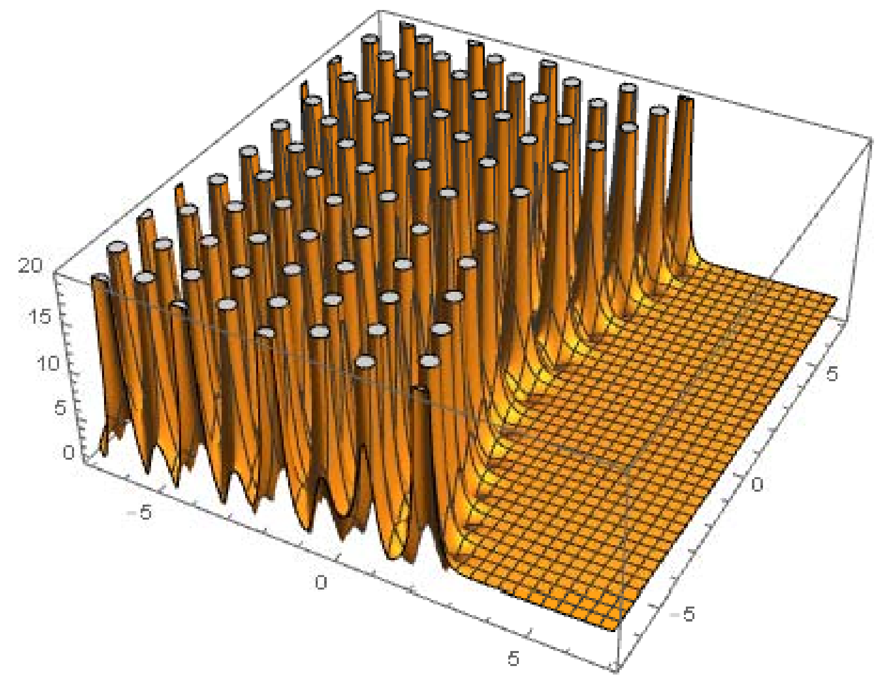
So far for generics – how do we construct specific solutions?

Host of methods:

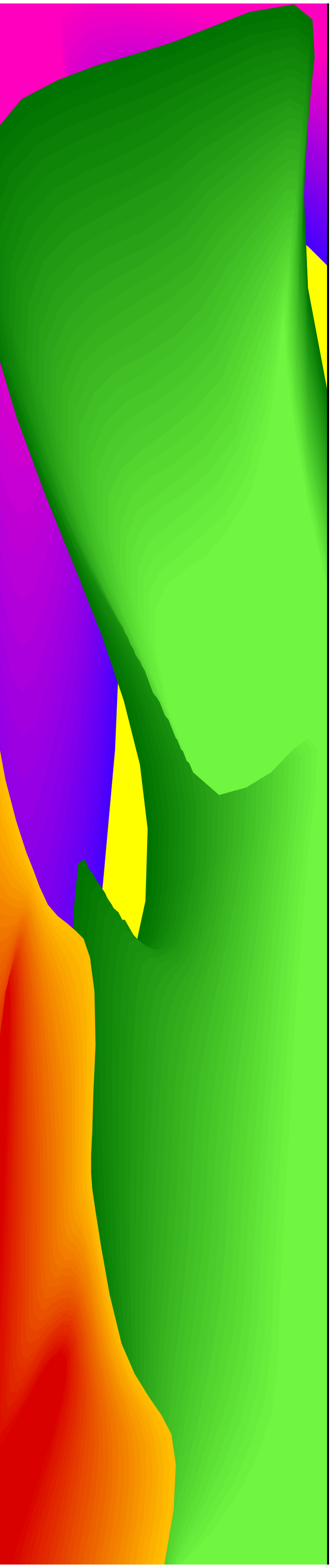
- Numerical
- Transseries
- Transasymptotics
- …

Can we relate those, and how do we incorporate the second parameter?





2. Numerical solutions



Numerical solutions

Standard methods:

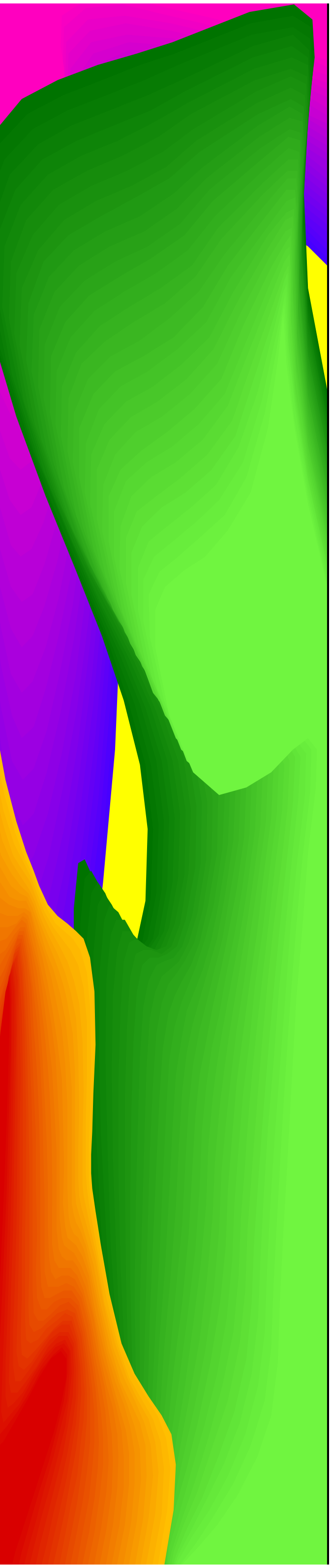
- **Forward Euler**: calculate u(z+ε) using

$$
\begin{aligned}
u(z+\epsilon) &= u(z) + \epsilon\, u'(z) \\
u'(z+\epsilon) &= u'(z) + \epsilon\, u''(z)
\end{aligned}
$$

Note: can get u''(z) from Painlevé I.

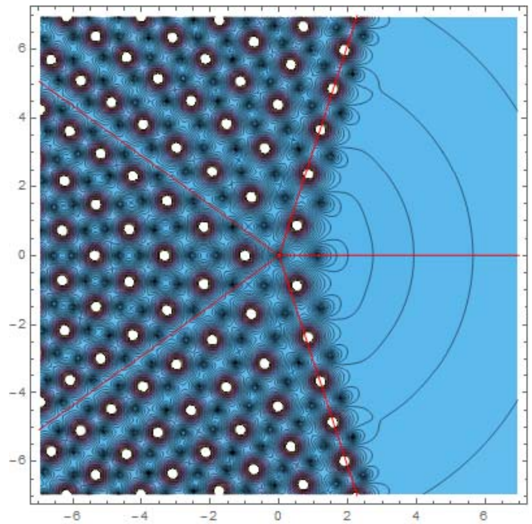
- **Taylor method**: obtain further derivatives of *u* by taking further derivatives of PI, use

$$u(z+\epsilon) = u(z) + \epsilon u'(z) + \frac{1}{2}\epsilon^2 u''(z) + \dots$$

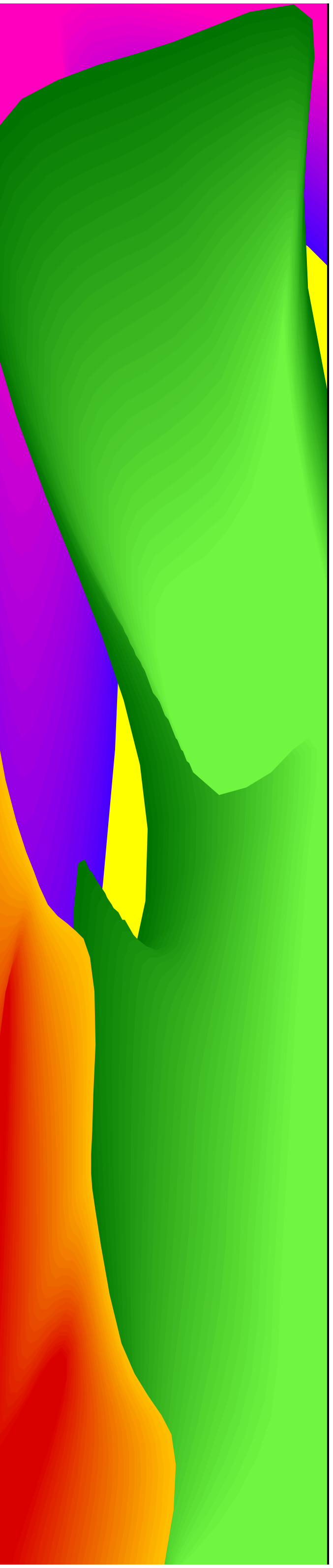


Numerical solutions

Problem with these: don’t work well in the pole fields.



We use a two-trick procedure found by Fornberg and Weideman (2011).



Numerical solutions

Trick 1: instead of Taylor series, use Padé approximants.

$$u(z+\epsilon) = \frac{a_0 + a_1\epsilon + \ldots + a_n\epsilon^n}{1 + b_1\epsilon + \ldots + b_n\epsilon^n} + \mathcal{O}\left(h^{2n+1}\right)$$

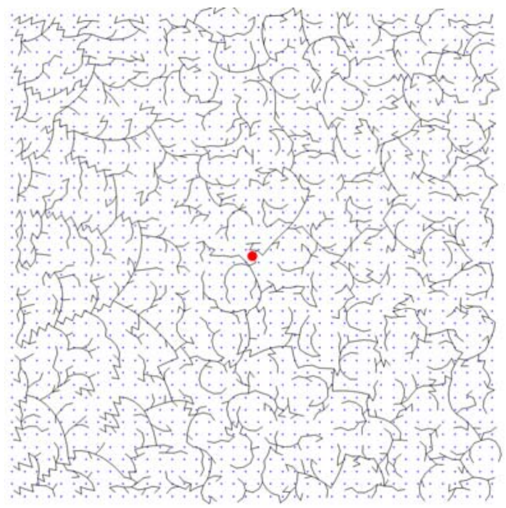
Plug into Painlevé I in the form

$$u^2(z+\epsilon) - \frac{1}{6}u''(z+\epsilon) = z + \epsilon$$

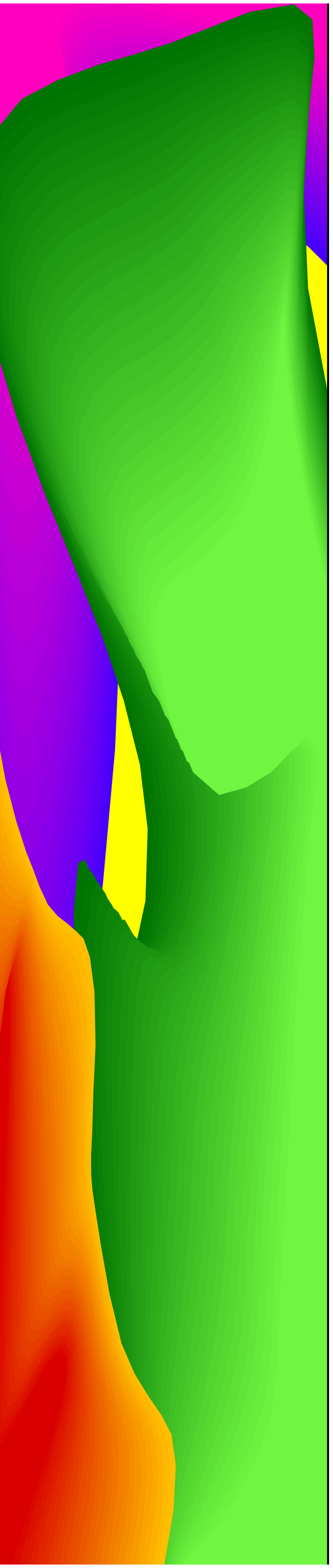
and equate powers of ε to find the coefficients in terms of *u(z)* and *u’(z)*.

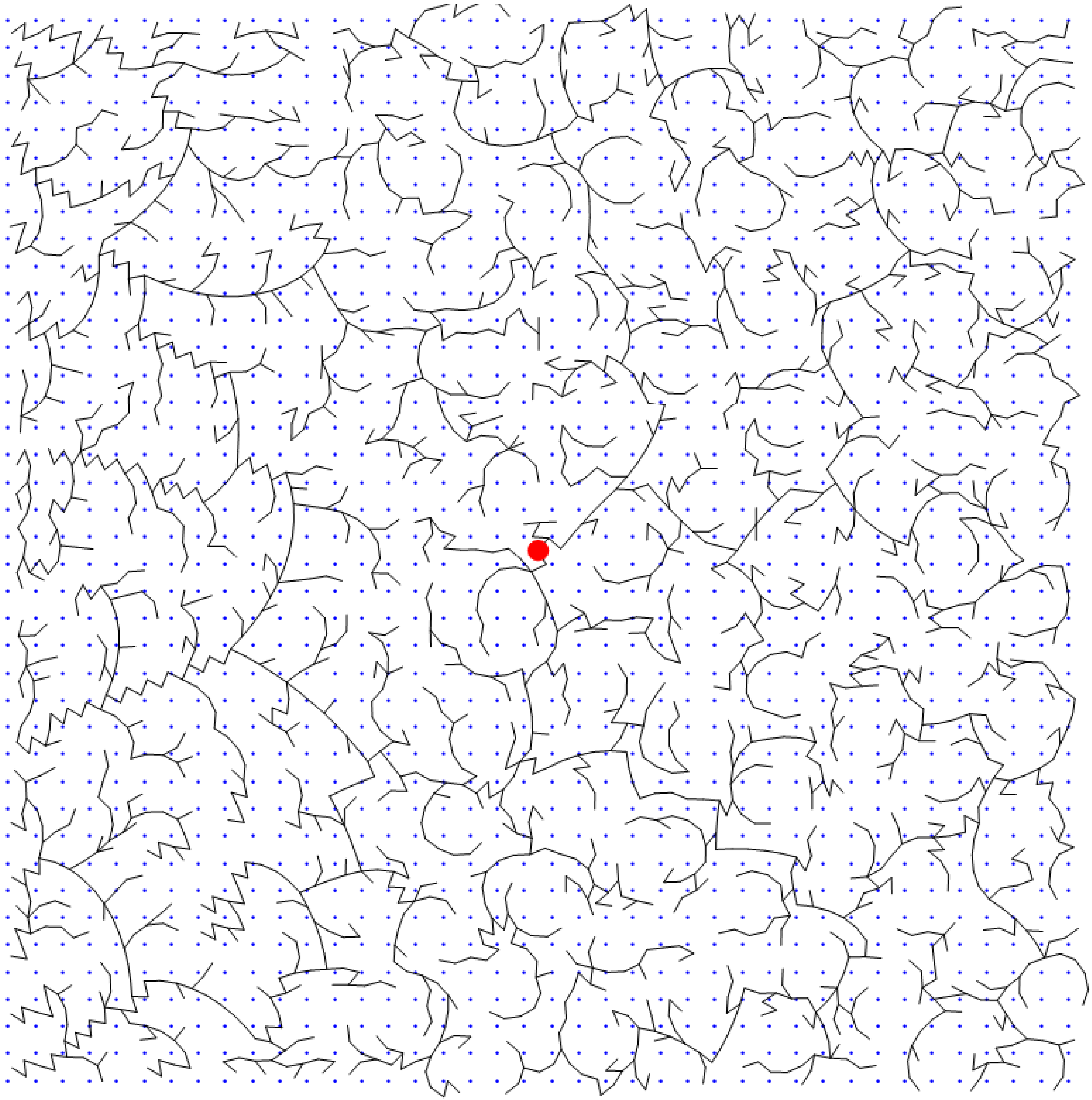
Numerical solutions

Trick 2: Follow paths in the complex *z*-plane that initially stay away from the poles. (Coarse-grained solution.)



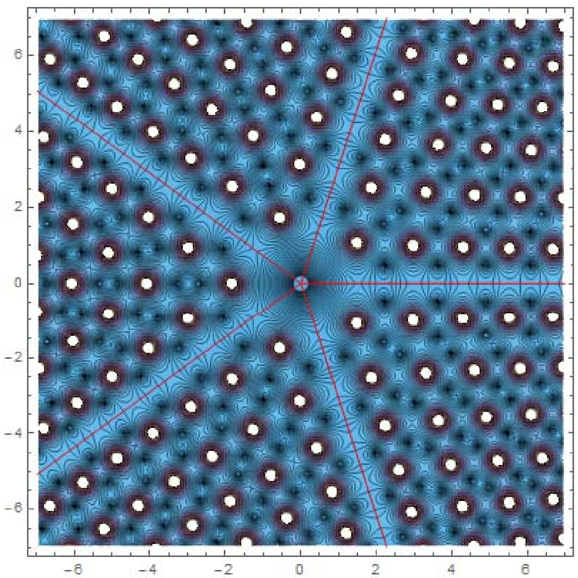
Finally, do a fine-grained approximation around each point using the Padé approximant.

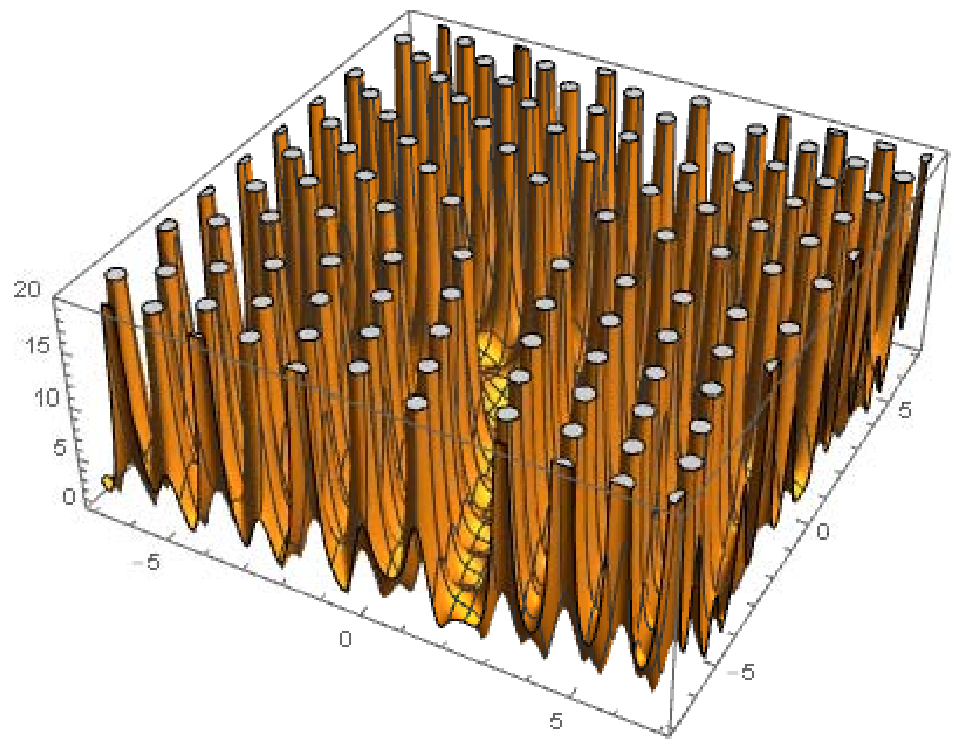




Numerical solutions

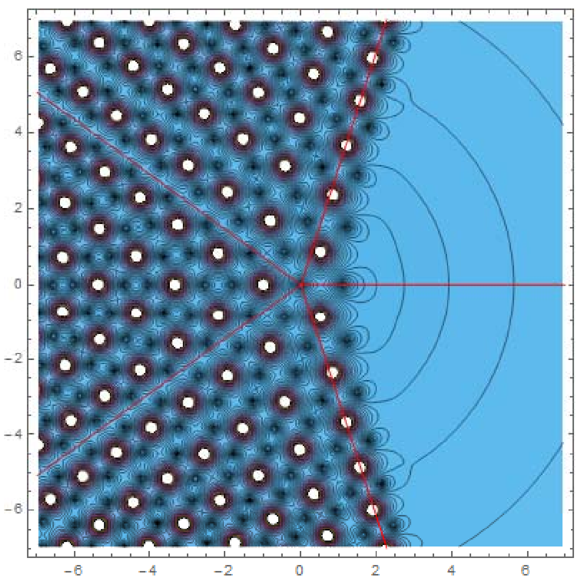
Results (1): generic solution

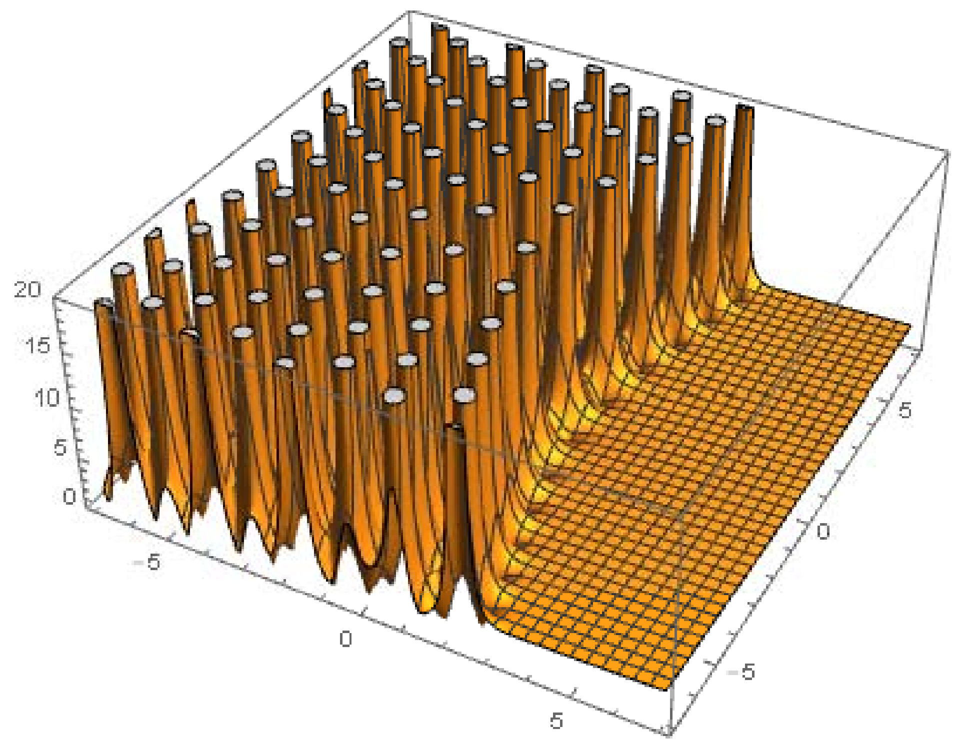




Numerical solutions

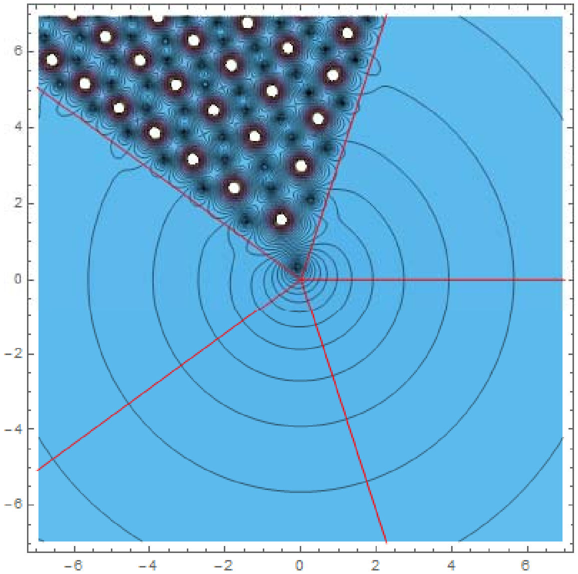
Results (2): tronquée solution

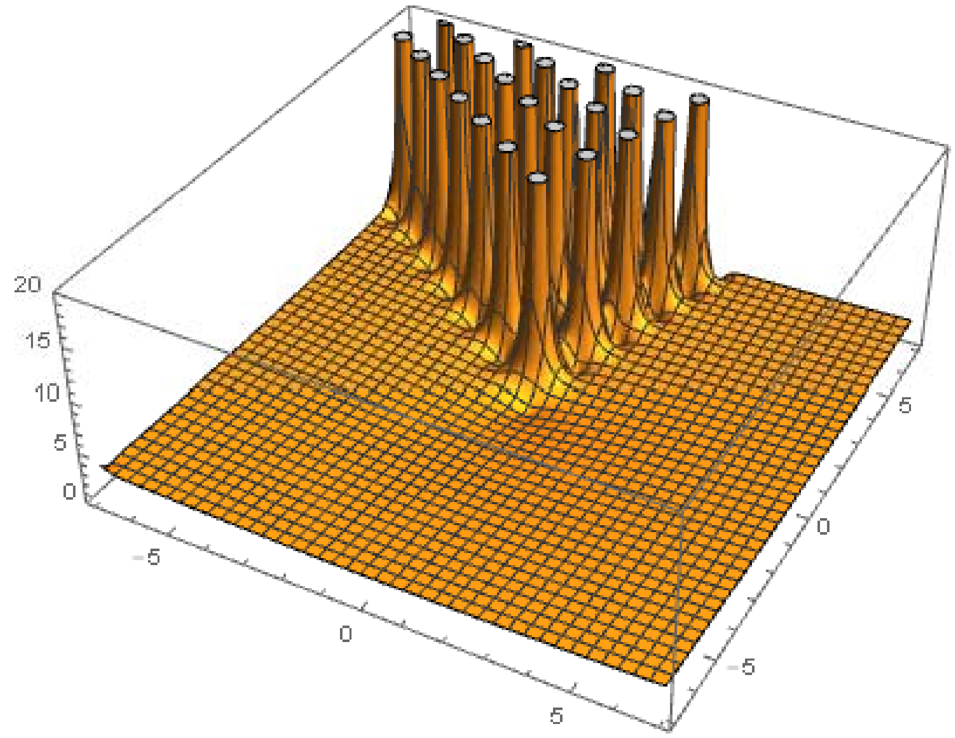


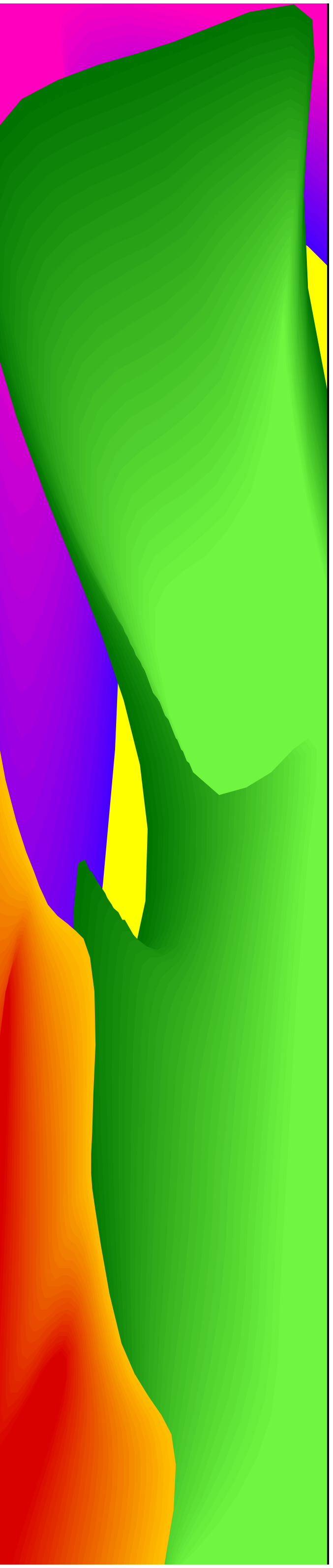


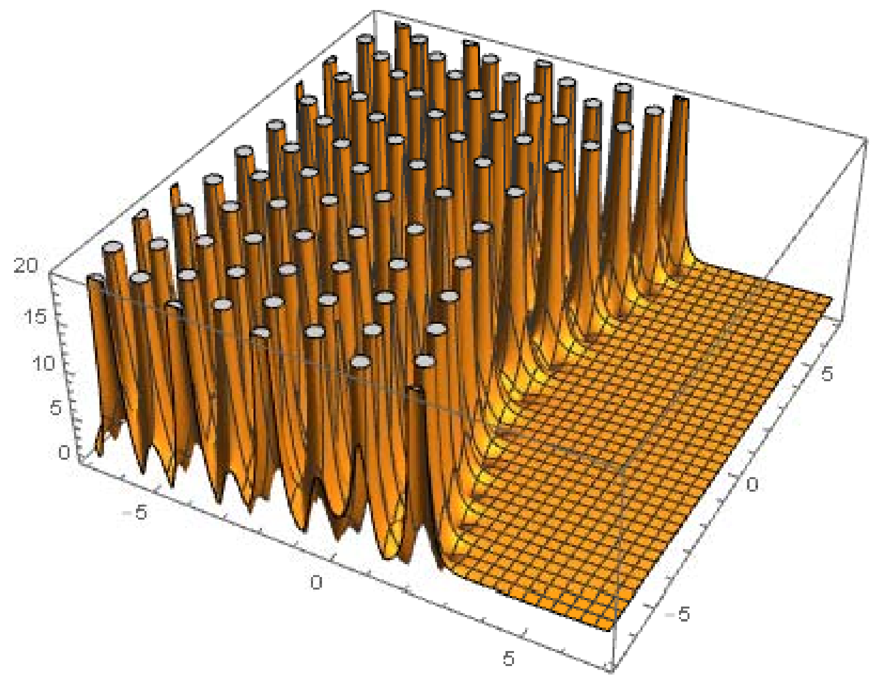
Numerical solutions

Results (3): tritronquée solution

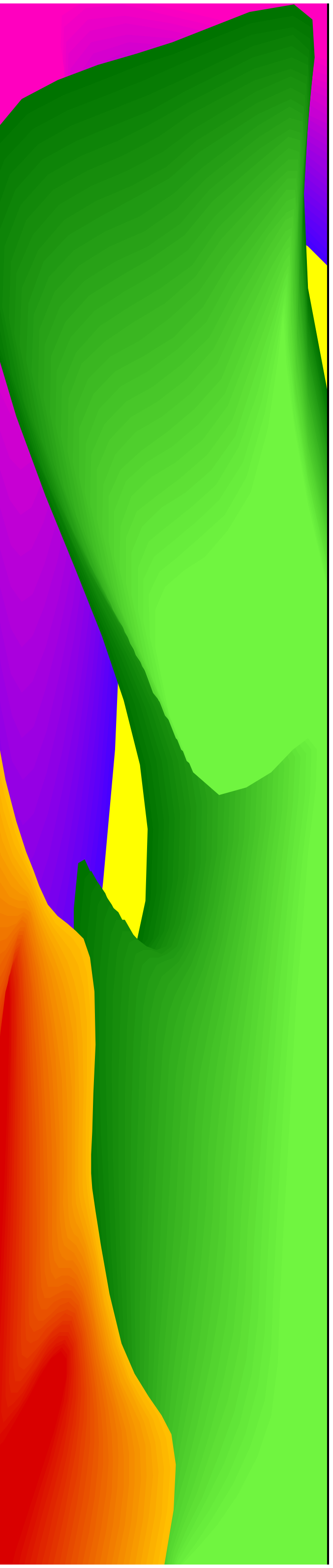








3. Transseries solution



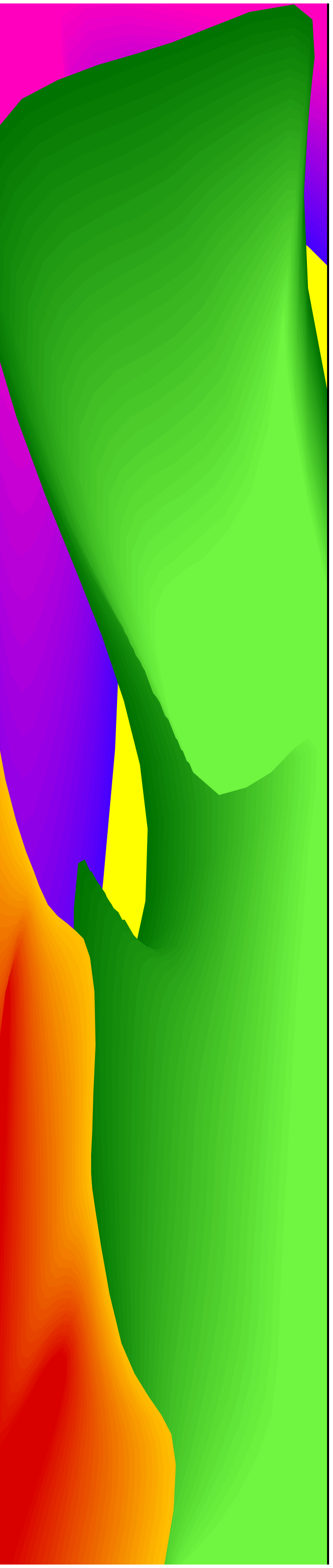
Transseries solution

In the pole-free regions, Painlevé I solutions behave asymptotically as $u \sim \sqrt{z}$.

Perturbative asymptotic expansion:

$$u_{\text{pert}}(z) \simeq \sqrt{z}\left(1 - \frac{1}{48}\,z^{-\frac{5}{2}} - \frac{49}{4608}\,z^{-5} - \frac{1225}{55296}\,z^{-\frac{15}{2}} - \ldots\right)$$

Coefficients grow as *(2g)!* Physical interpretation: $z^{-5/4}$ is the string coupling g_s.



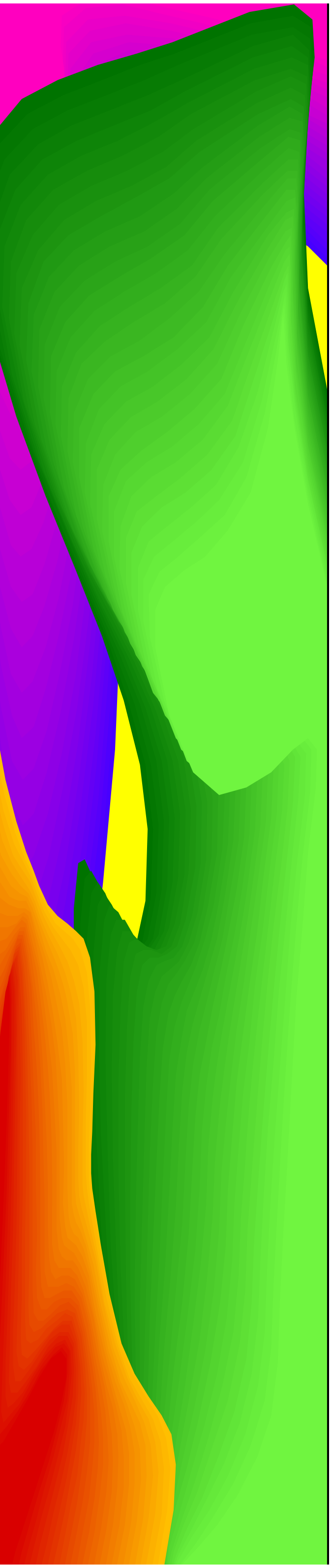
Transseries solution

As usual, the asymptotic growth indicates that we should extend the perturbative series to a resurgent transseries.

Naïve way (use $x=z^{-5/4}$):

$$u(x;\sigma) = x^{-\frac{2}{5}} \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-\frac{nA}{x}} \, x^{n\beta} \sum_{g=0}^{+\infty} u_g^{(n)} \, x^g$$

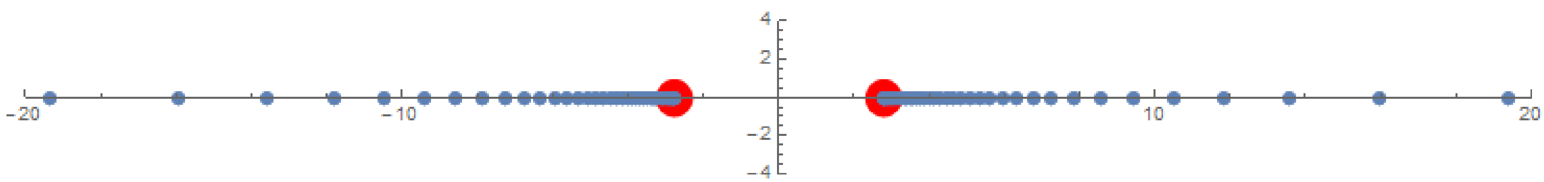
This does provide a 1-parameter family of formal solutions, but not all!



Transserries solution

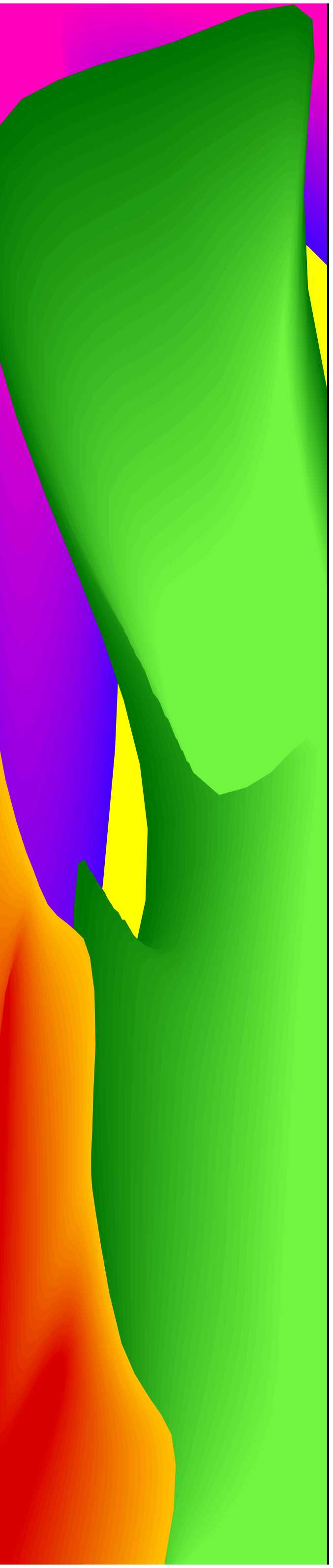
Indications that there should be more:

- Instanton action can be $A = \pm 8\sqrt{3}/5$
- Borel plane has positive and negative branch points at these values



- Painlevé I is a 2nd order ODE, so we expect two constants of integration

So at least formally, we expect to have a 2-parameter transseries solution.

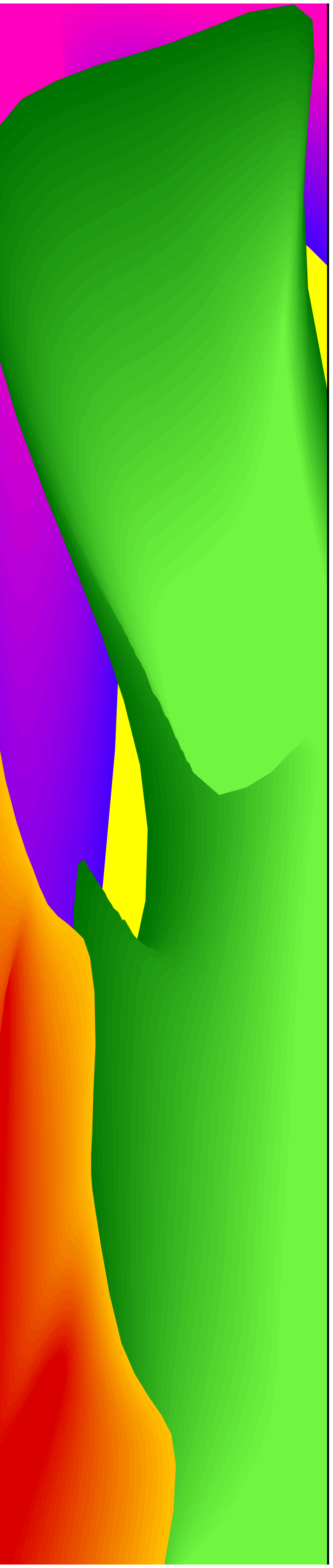


Transseries solution

Since the two instanton actions are opposite, there will be resonance, as first shown by Garoufalidis, Its, Kapaev, Mariño [2010].

The equation for the transseries cannot be solved unless one includes log(x) as an additional transmonomial.

In 1106.5922, we constructed the full resonant 2-parameter transseries solution.



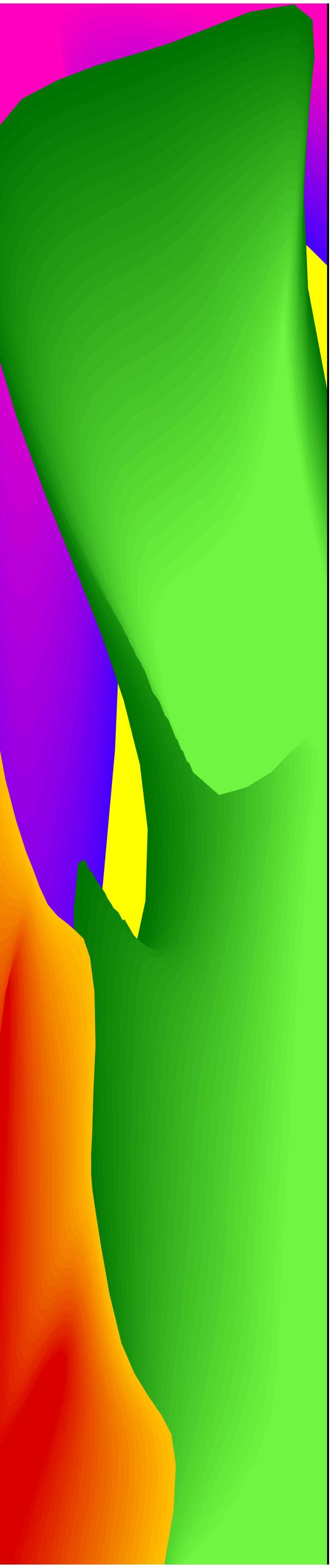
Transseries solution

$$u(x;\sigma_1,\sigma_2) = x^{-\frac{2}{5}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_{nm}(x;\sigma_1,\sigma_2)\, \sigma_1^n \sigma_2^m \, e^{\frac{(m-n)A}{x}} \, x^{\beta_{nm}} \, \Phi^{(n|m)}(x)$$

$$L_{nm}(x;\sigma_1,\sigma_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2}{\sqrt{3}} (m-n)\sigma_1\sigma_2 \log x \right)^k$$

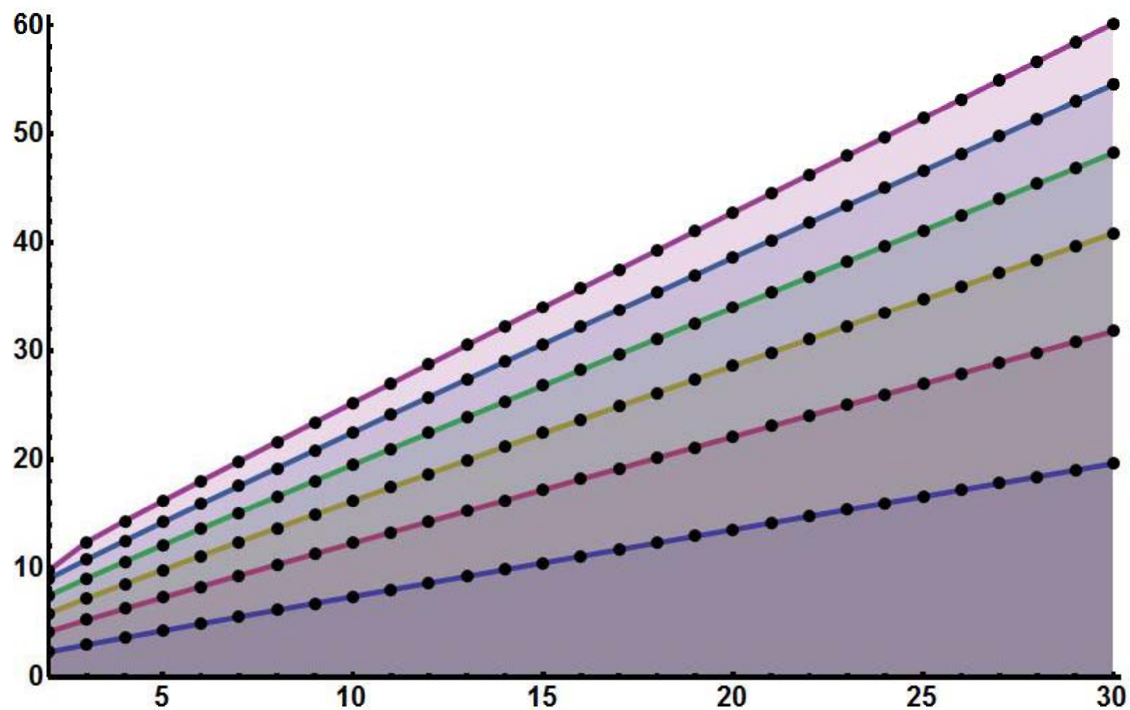
- Coefficients of log terms are multiples of the coefficients of non-log terms
- At given order in σ_1, σ_2 there is only a finite number of logs.
- Logs can be formally summed:

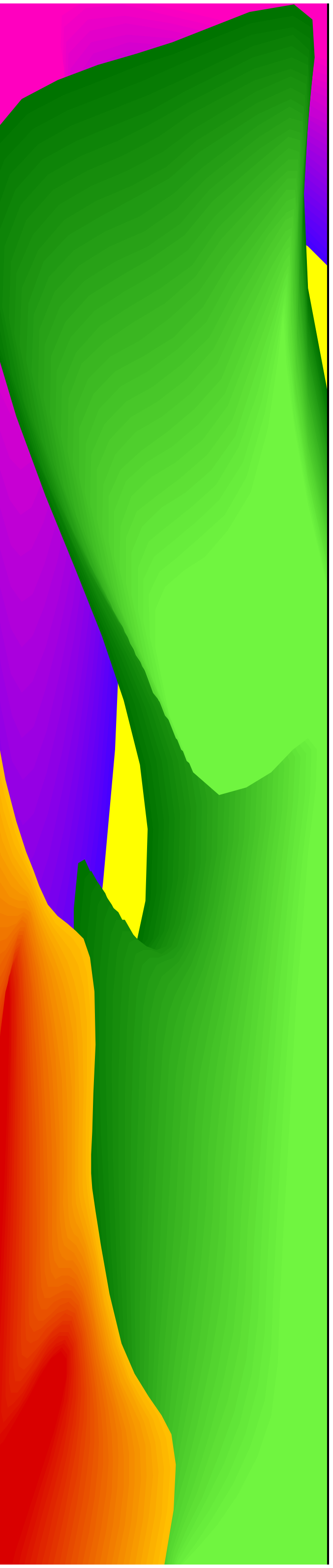
$$L_{nm}(x;\sigma_1,\sigma_2) \sim x^{\frac{2}{\sqrt{3}}(m-n)\sigma_1\sigma_2}$$

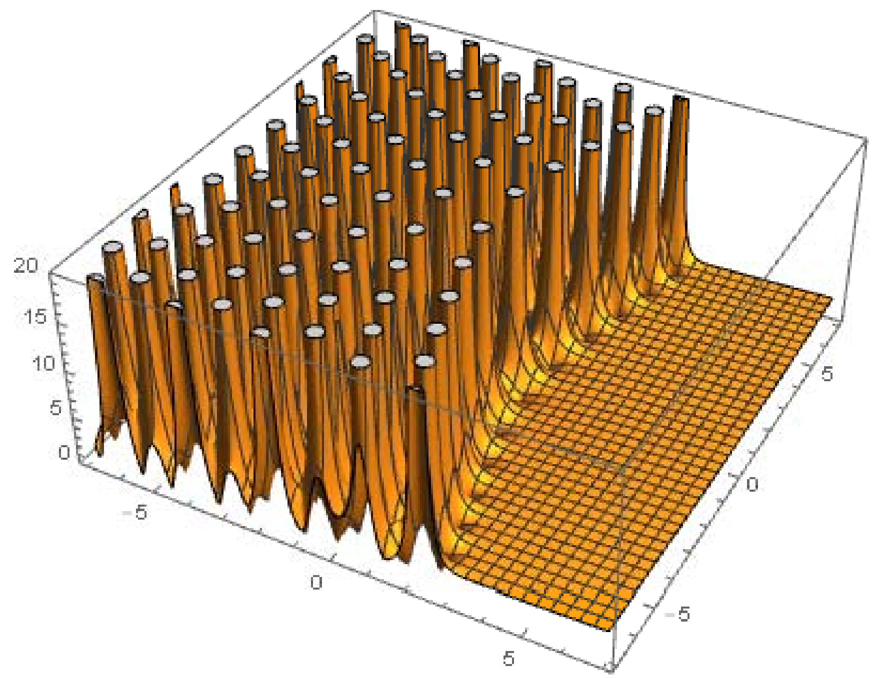


Transseries solution

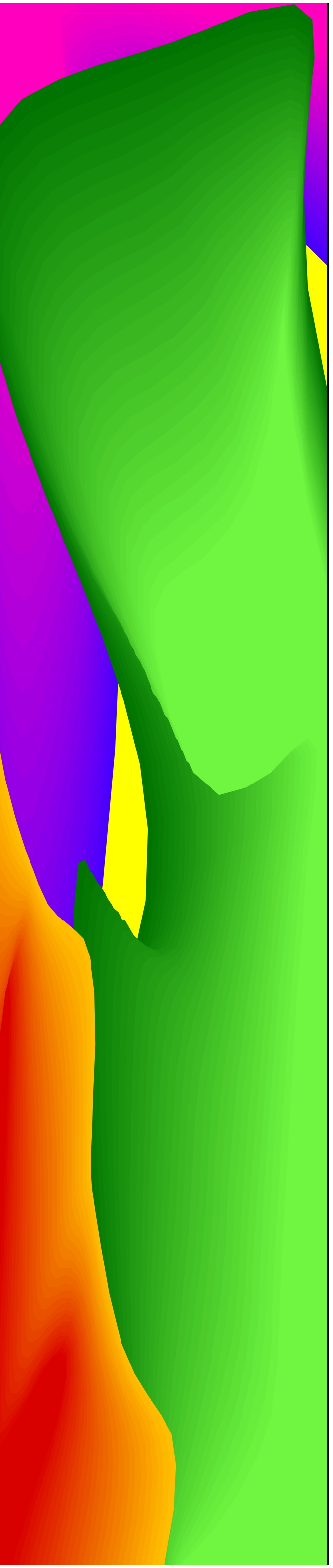
In 1106.5922, we computed many coefficients and checked the resurgent large order relations between sectors.







4. Transasymptotics: beyond first poles



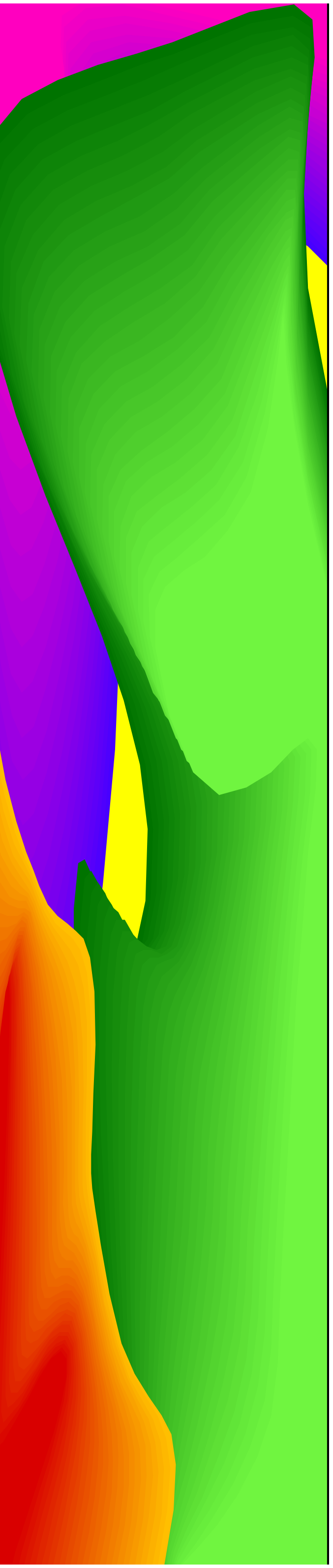
Transasymptotics

For now, let's go back to the 1-parameter transseries:

$$u(x;\sigma) = x^{-\frac{2}{5}} \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-\frac{nA}{x}} \, x^{n\beta} \sum_{g=0}^{+\infty} u_g^{(n)} \, x^g$$

The *g*-sums are asymptotic, but if we exchange summation order, the *n*-sums have finite radius of convergence.

What if we do the *n*-sums first?



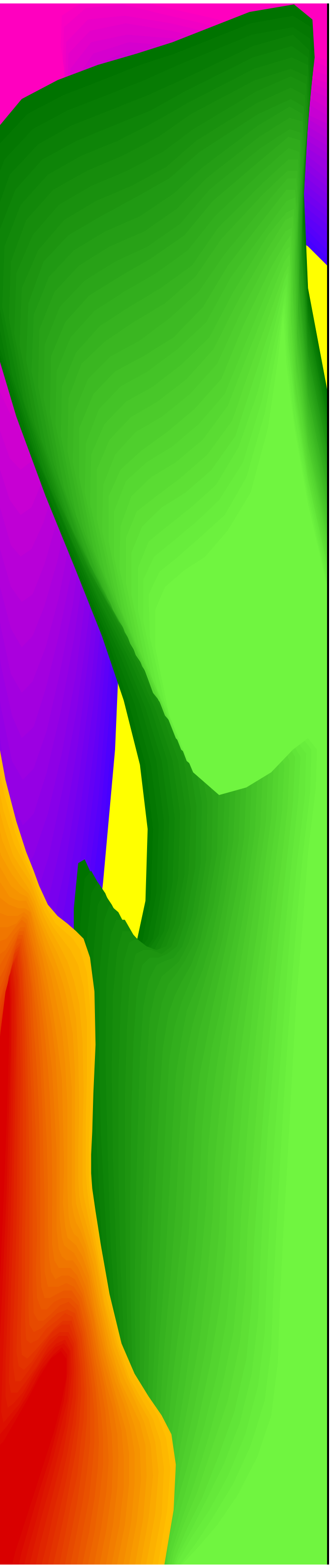
Transasymptotics

It was pointed out by Costin/Costin (2002) and worked out in detail for Painlevé I in Costin/Costin/Huang (2013) that the *n*-sums can often be done exactly. This goes under the name of transasymptotics.

For example:

$$u_0^{(n)} = \frac{n}{12^{n-1}} \qquad\qquad (n \geq 1)$$

with the exceptional case $u_0^{(0)}=1$.



Transasymptotics

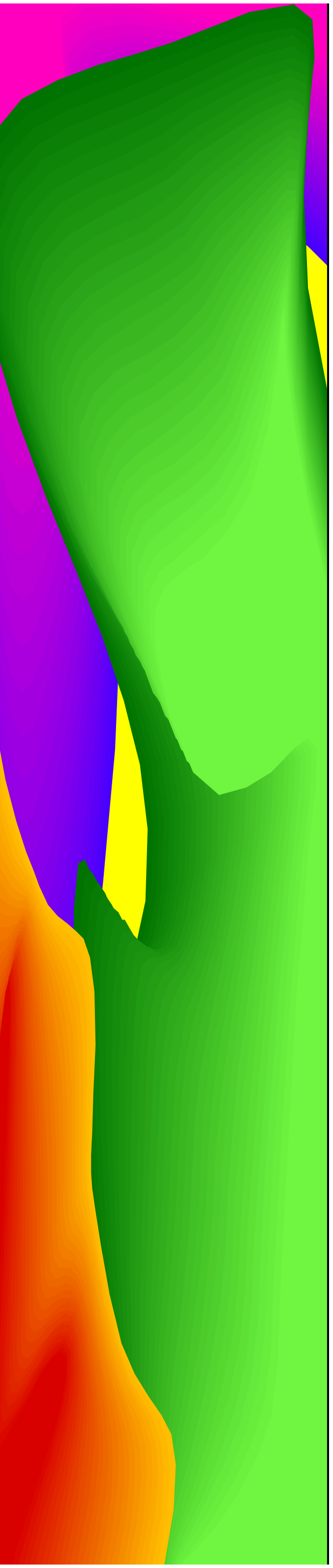
$$u_0^{(n)} = \frac{n}{12^{n-1}} \qquad\qquad (n \geq 1)$$

Using this, we can sum over all *n*:

$$\begin{aligned} u_0(x;\sigma) &:= \sum_{n=0}^{+\infty} \sigma^n \, \mathrm{e}^{-\frac{nA}{x}} \, x^{n\beta} u_0^{(n)} \\ &= 1 + 12 \sum_{n=1}^{+\infty} n\tau^n \\ &= \frac{1 + 10\tau + \tau^2}{(1-\tau)^2} \end{aligned}$$

where

$$\tau := \frac{x^\beta \sigma}{12} e^{-A/x}$$



Transasymptotics

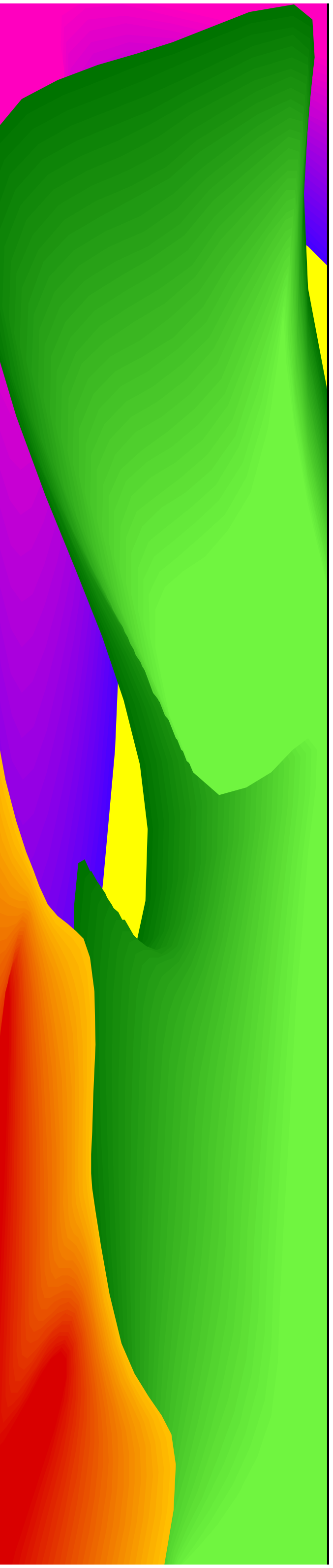
$$u_0(x;\sigma) = \frac{1+10\tau+\tau^2}{(1-\tau)^2}$$

Crucial observation: doing the infinite sum, poles appear in the expression!

$$\tau := \frac{x^\beta \sigma}{12} e^{-A/x}$$

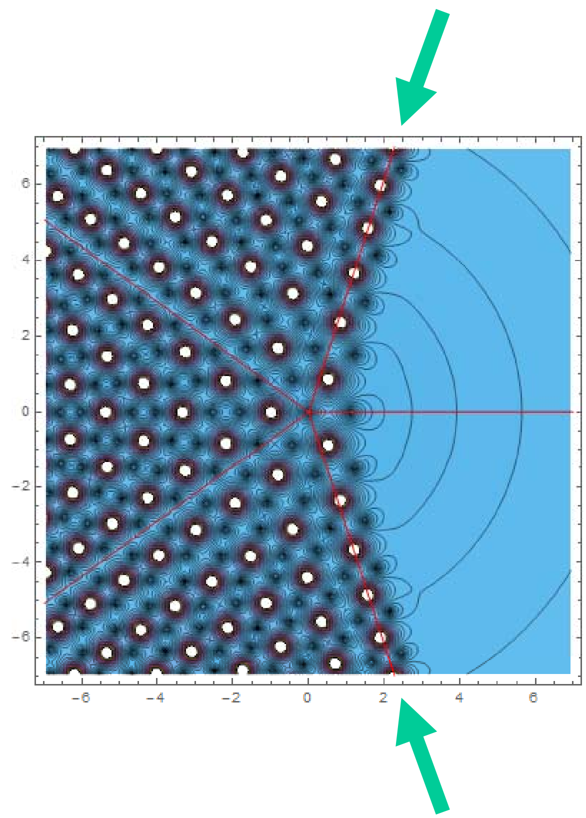
The equation *τ=1* has an array of solutions in terms of the Lambert *W*-function:

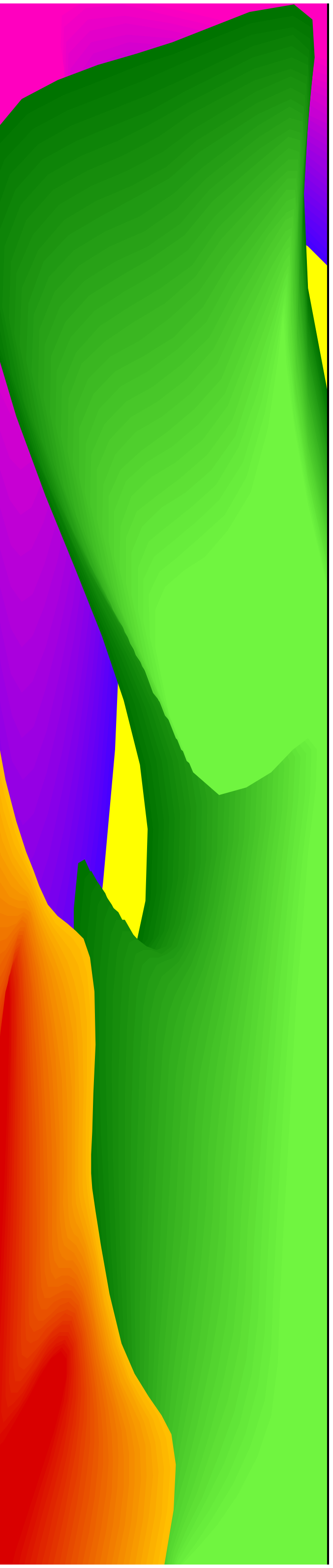
$$x_i = \frac{2A}{W_i\left(\frac{\sigma^2 A}{72}\right)}$$



Transasymptotics

Costin et al. prove (and one can check numerically) that these approximate the first array of poles in the region where $u \sim \sqrt{z}$ no longer holds.





Transasymptotics

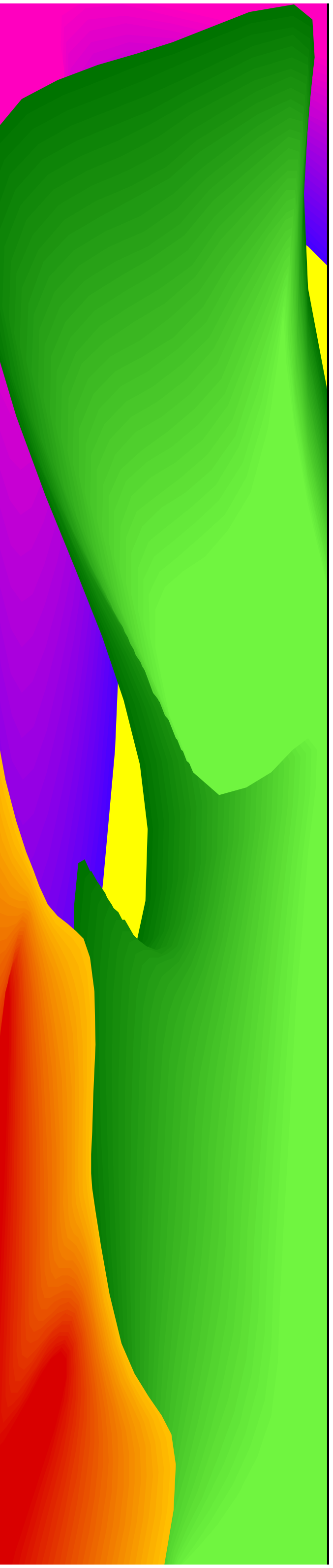
Including subleading orders in *g* (again summed over all *n*) gives corrections in *x* to these positions.

In practice, it turns out to be easier to go to the partition function Z. Recall:

pole of *u* ↔ zero of *Z*

One finds:

$$
\begin{aligned}
Z_0 &= \tau - 1 \\
Z_1 &= \frac{2\tau^2 - 111\tau}{192\sqrt{3}} \cdot x \\
Z_2 &= \ldots \cdot x^2 \\
&\vdots
\end{aligned}
$$



Transasymptotics

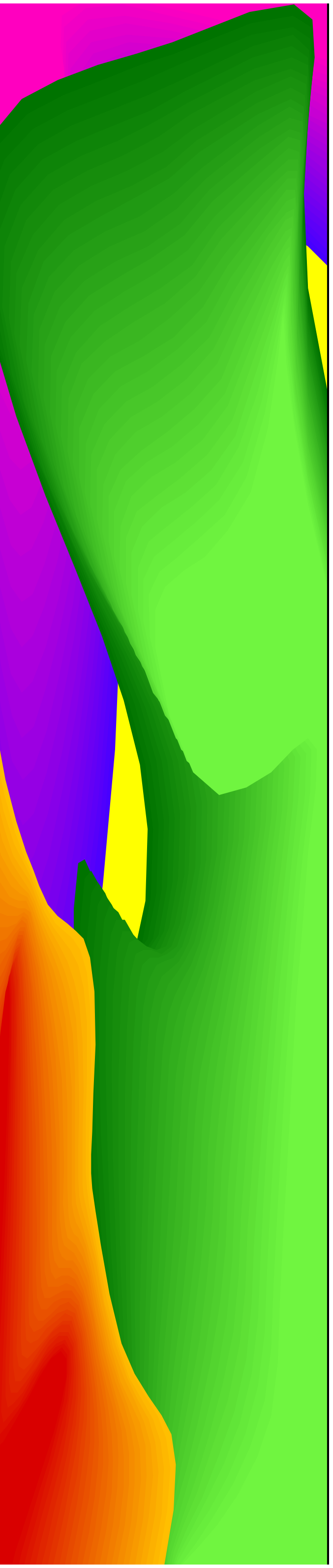
Consider the sum of the first two terms:

$$Z_0 + Z_1 = \tau - 1 + \frac{2\tau^2 - 111\tau}{192\sqrt{3}} \cdot x$$

Making the Ansatz *τ=1+cx* and setting the leading two orders two zero we find the corrected solution

$$\tau = 1 + \frac{109}{192\sqrt{3}} x + \cdots$$

Next order obtained by including Z_2, etc.



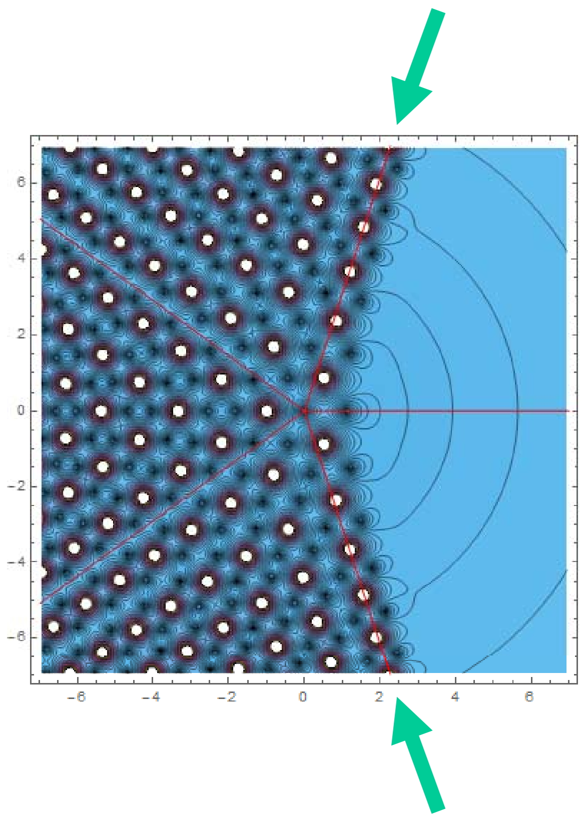
Transasymptotics

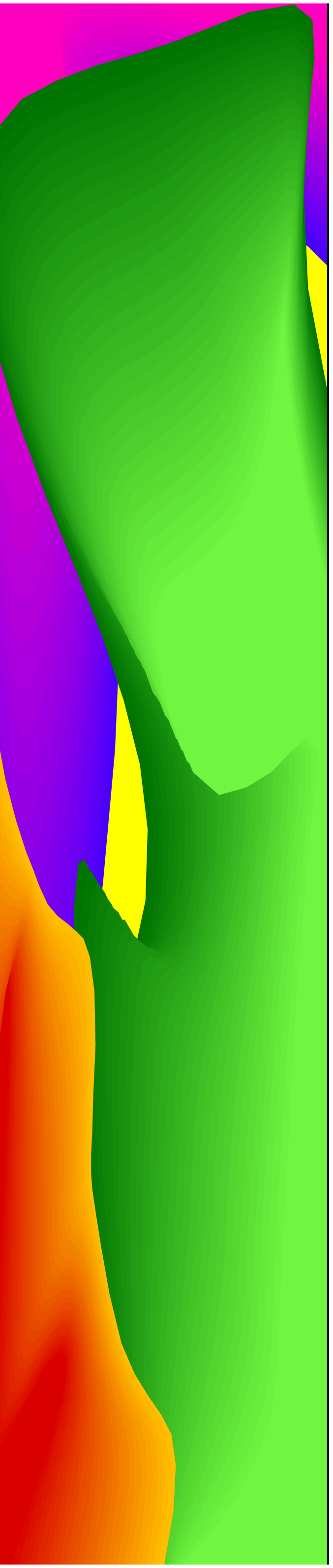
Note: this gives corrections for the first array of poles.

However,

$$Z_0 + Z_1 = \tau - 1 + \frac{2\tau^2 - 111\tau}{192\sqrt{3}} \cdot x$$

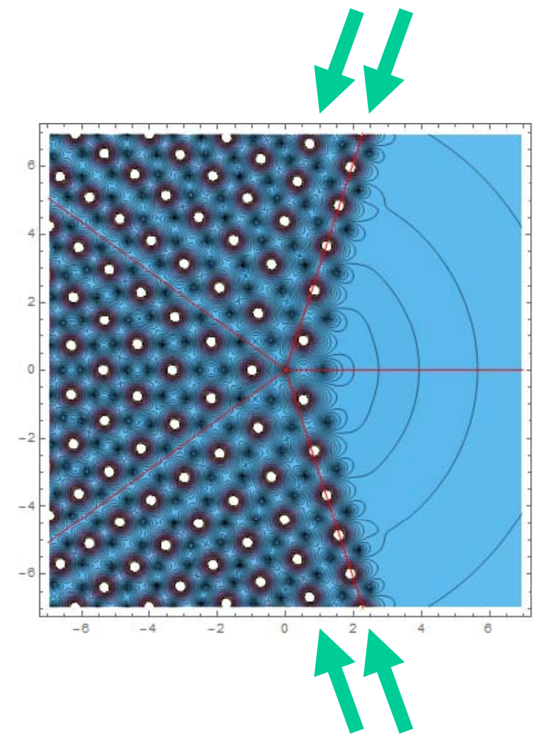
has two zeroes!





Transasymptotics

The second zero, which is absent when *x=0*, is



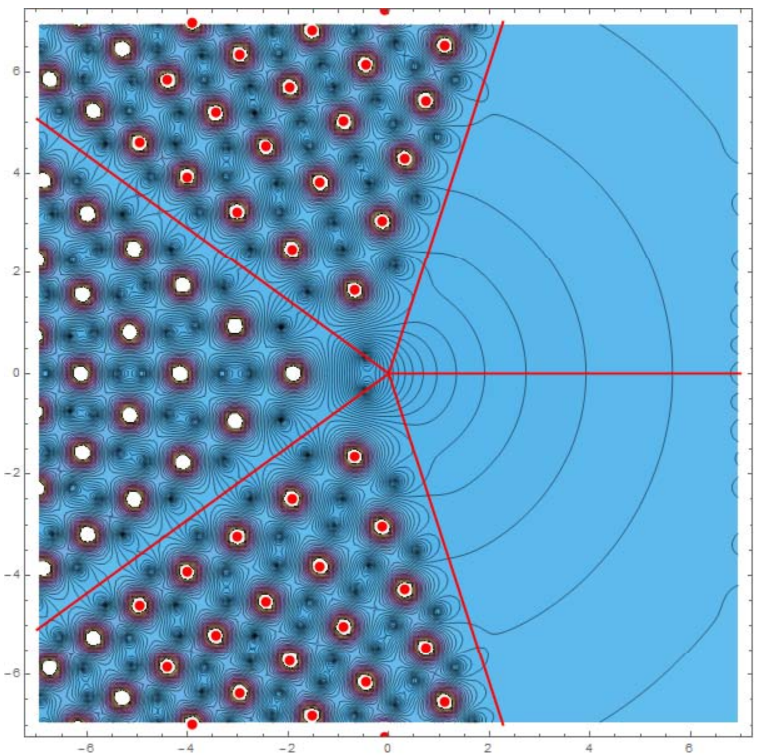
$$\tau = \frac{-96\sqrt{3}}{x} - \frac{673}{2} + \ldots$$

Corrections are again obtained by including Z_2, etc.

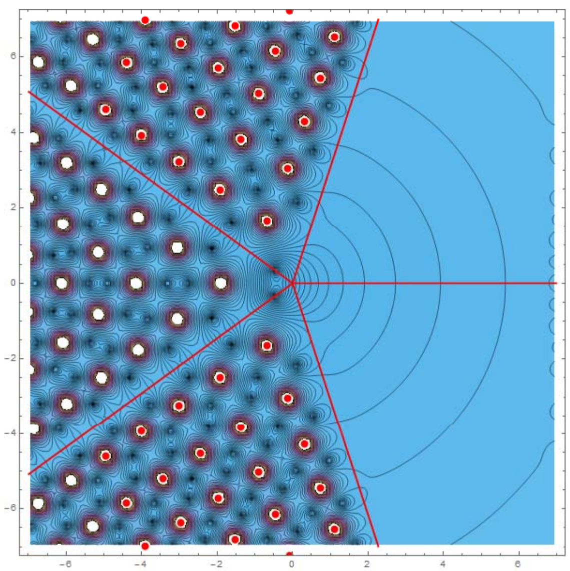
Using all solutions for the cubic polynomial $Z_1+Z_2+Z_3$ gives a third solution scaling as x^{-2}, and so on.

Transasymptotics

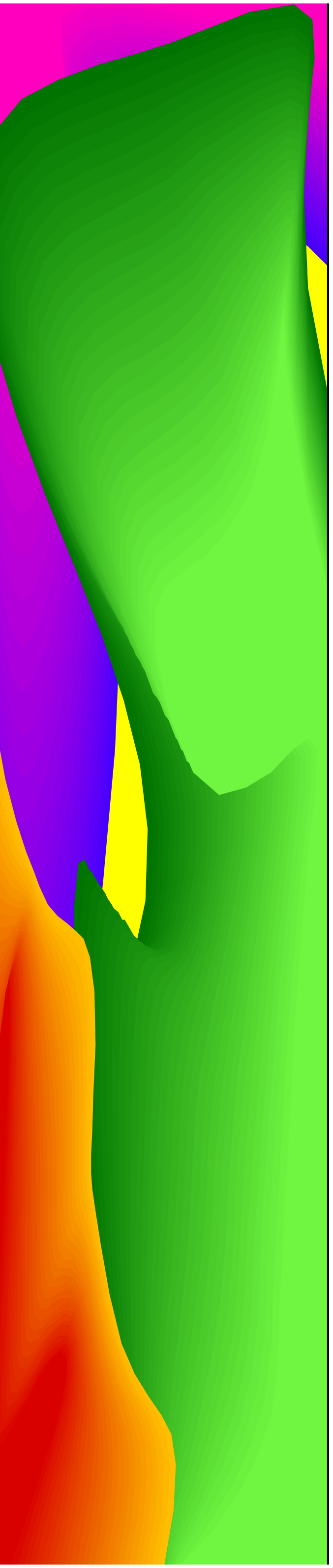
So… does this work? We leave a complete proof to the mathematicians, but the numerics is very convincing:

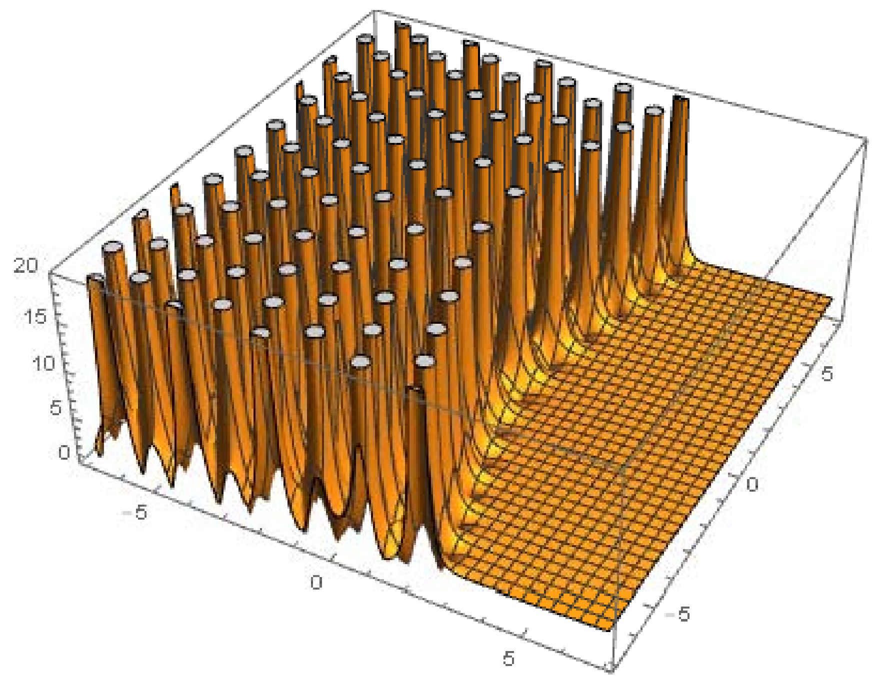


Transasymptotics

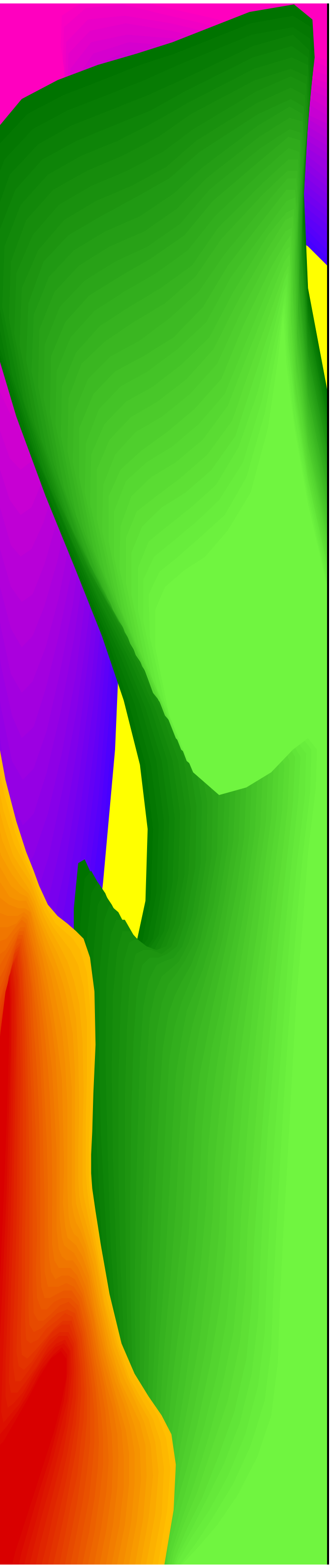


Unfortunately, the fifth sector seems hard to reach using this method. (But see the remarks that follow.)





5. The second parameter



The second parameter

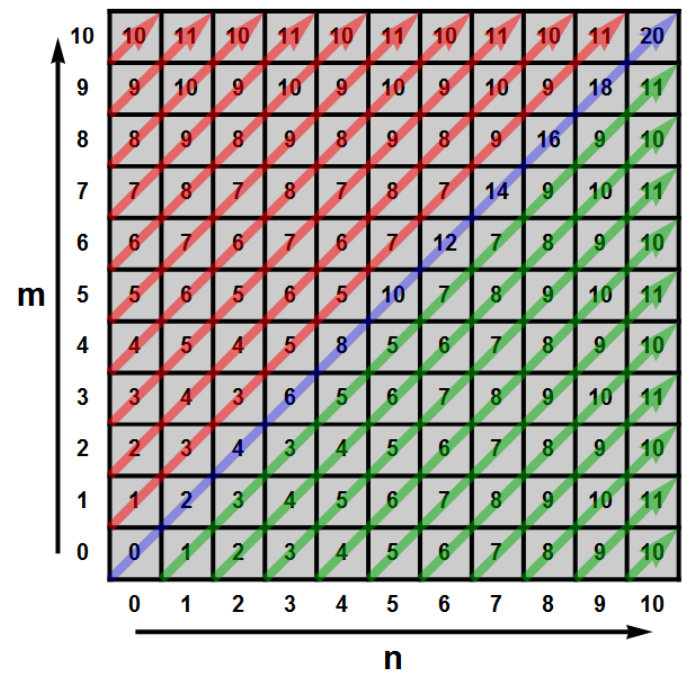
As mentioned, the two-parameter transseries solution was completely determined in 1106.5922. (Cf. question after Takei's talk.)

Side remark: the Painlevé II case was analogously treated by Schiappa and Vaz in 1302.5138.

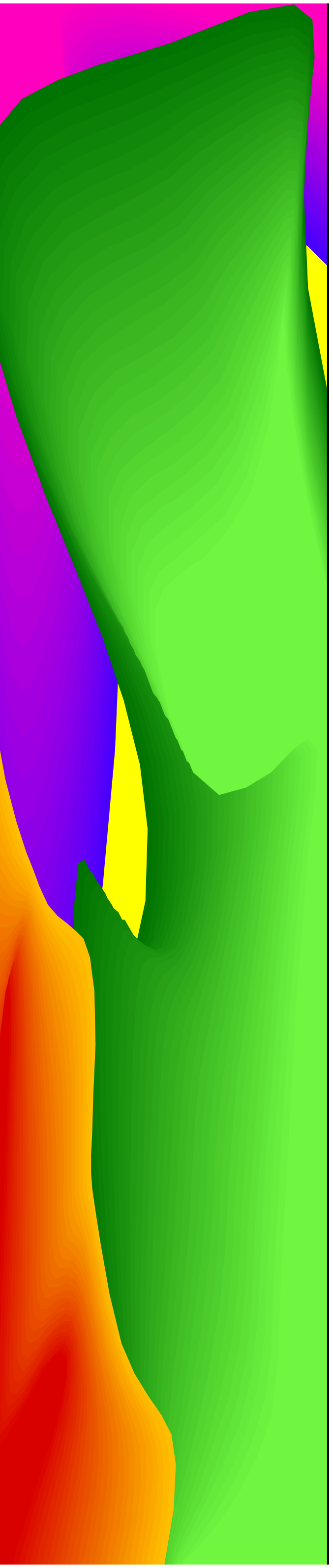
Can we do a similar transasymptotic analysis? How to deal with the fact that $e^{+A/x}$ is large?

The second parameter

Schematic depiction of how we want to sum:



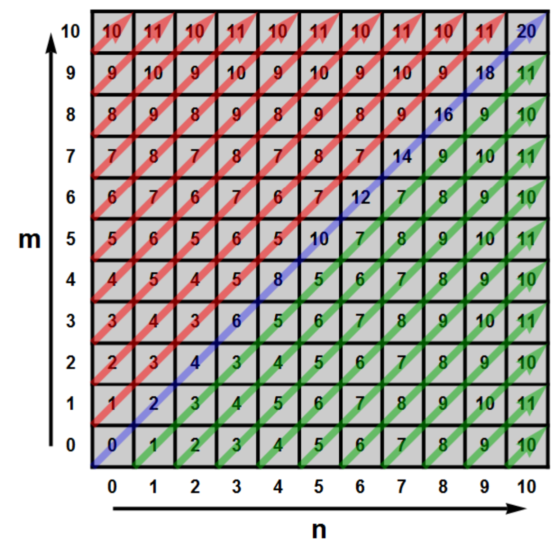
Integers give the order in x^{β} at which a certain sector starts.



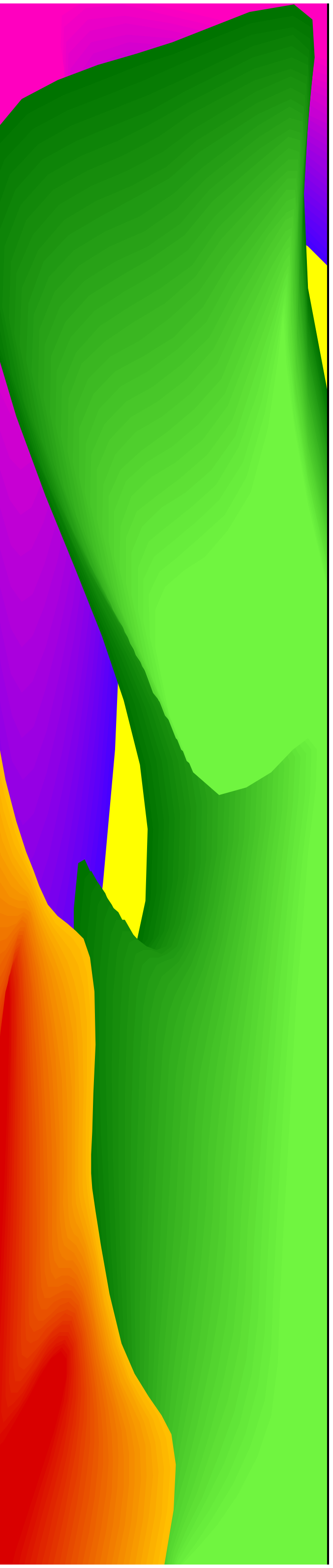
The second parameter

Recall that τ contained a factor of x^β, which allowed us to sum ‘horizontally’.

$$\tau := \frac{x^\beta \sigma}{12} e^{-A/x}$$



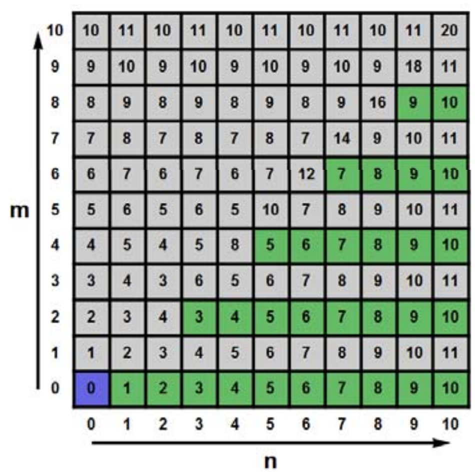
To be able to sum leading terms, we need to define a second parameter ρ without such a factor.

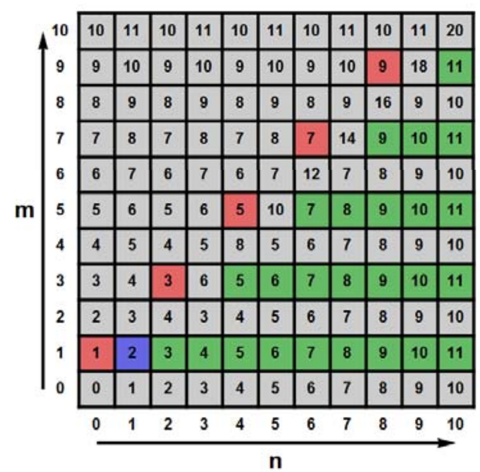


The second parameter

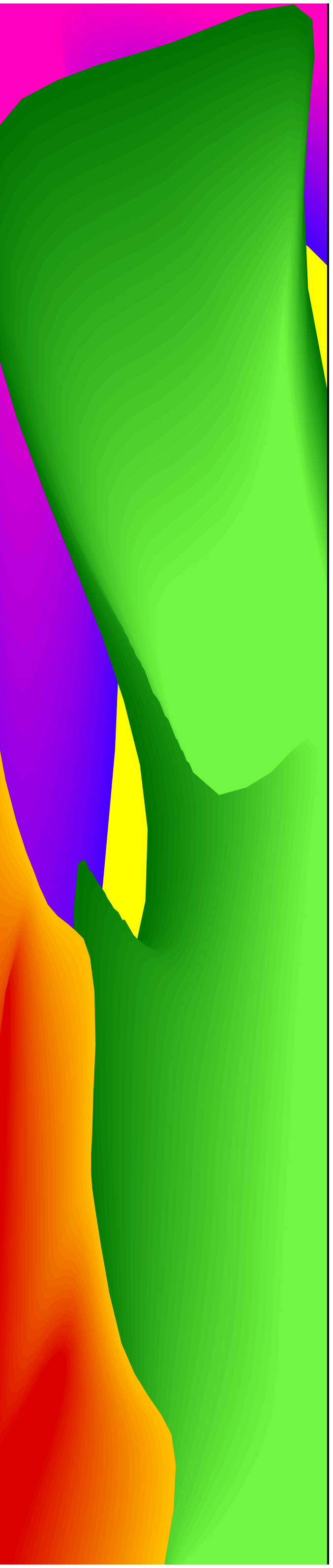
$$\rho_2 \equiv \sigma_2 \, e^{+A/x}, \qquad \tau_1 \equiv \frac{x^{1/2}\sigma_1}{12} \, e^{-A/x}$$

Now, the leading and subleading terms become:





Finite number of $e^{+A/x}$-lines!



The second parameter

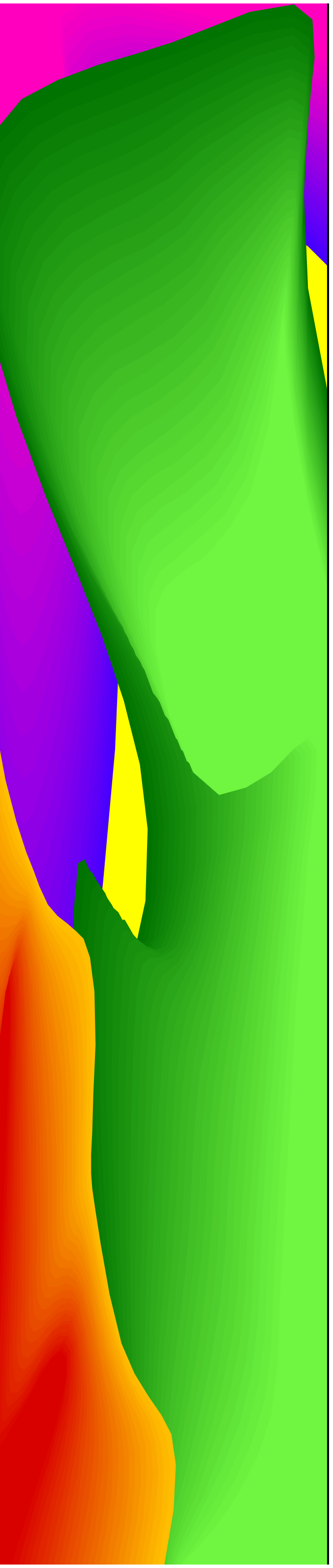
Amazingly, the summation can be done:

- Closed expressions for the coefficient at each order in *x* can be found,
- The log-terms can be summed over,
- The remaining sum over *n* and *m* can be done.

Leading term:

$$u_0 = 1 + \frac{12T}{(1-T)^2}$$

$$T \equiv \frac{x^{1/2}\sigma_1}{12}\,\exp\left(-\frac{47\sqrt{3}}{72}x\sigma_1^2\sigma_2^2 - \frac{2\sqrt{3}}{3}\sigma_1\sigma_2\log x - \frac{A}{x}\right)$$



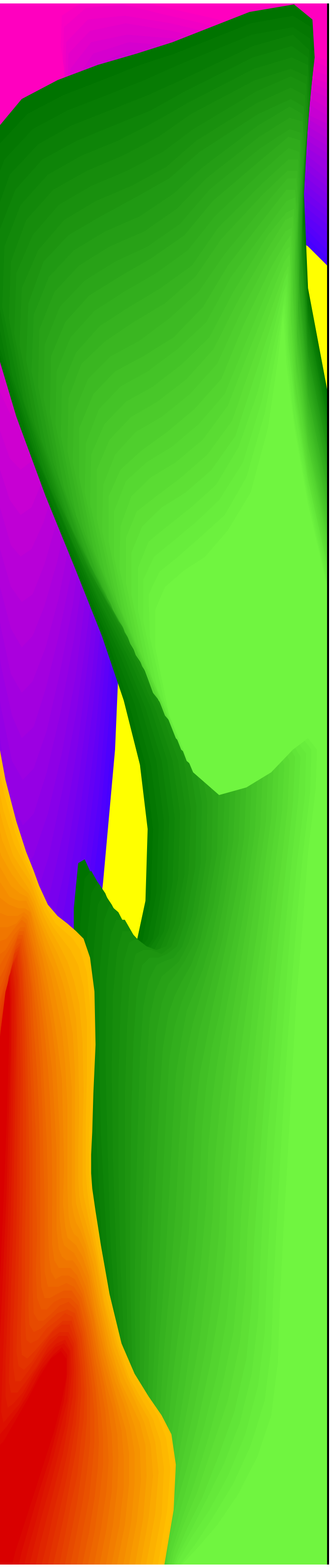
The second parameter

Subleading term:

$$u_1 = \tau_1\rho_2 \left(\frac{1}{T} + \frac{T^4 - 147T^3 + 174T^2 - 454T + 144}{12(T-1)^3}\right) - \tau_1^2\rho_2^2 \cdot \frac{7717}{432\sqrt{3}}\frac{T(T+1)}{(T-1)^3}$$

The expressions get complicated fast, but of course the computer doesn’t care.

Can we use this to match every numerical solution? Optimistic, but work in progress…



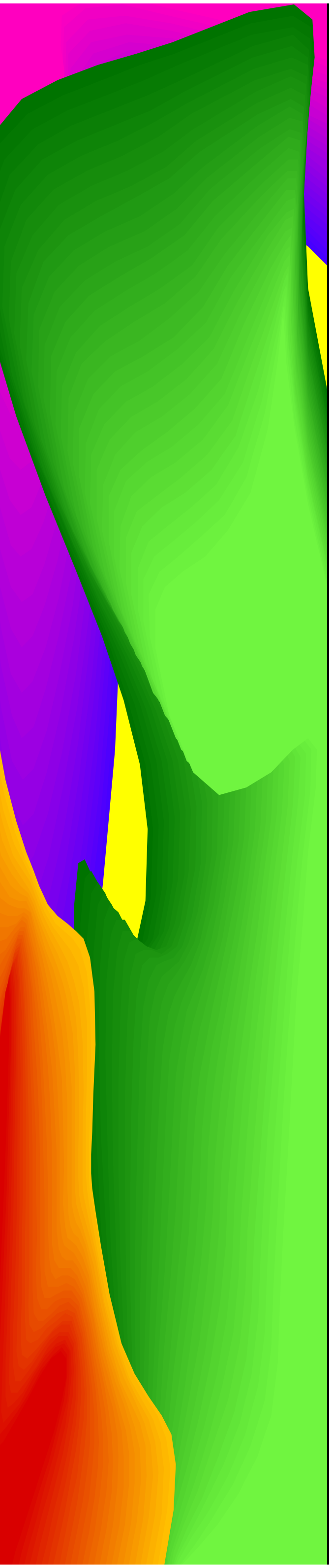
The second parameter

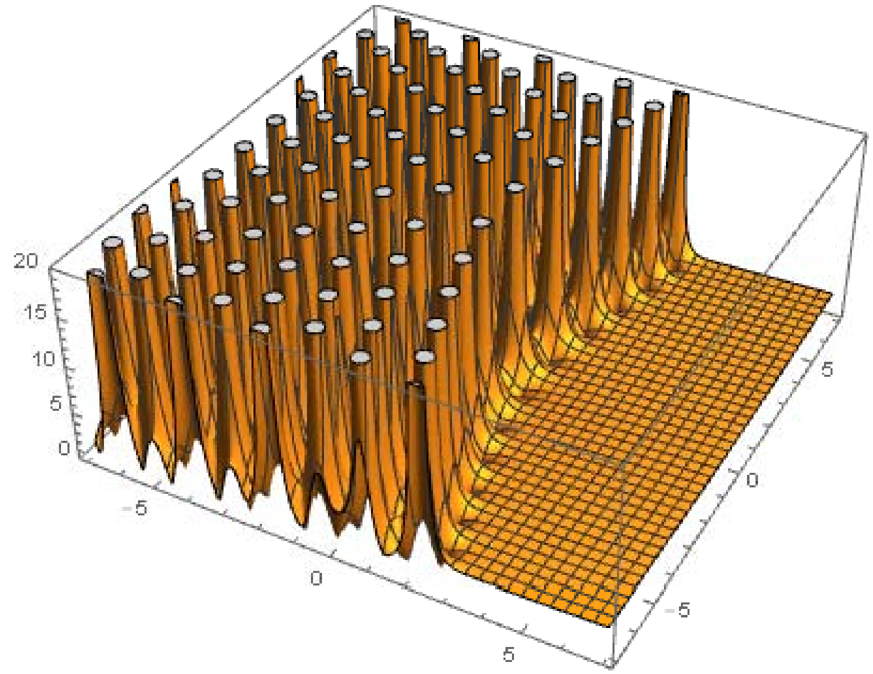
The hope is (and we have indications that)

• One can now match pole fields also for non-tronquées solutions (1-parameter is always tronquée), at least in some near-tronquée region of parameter space.

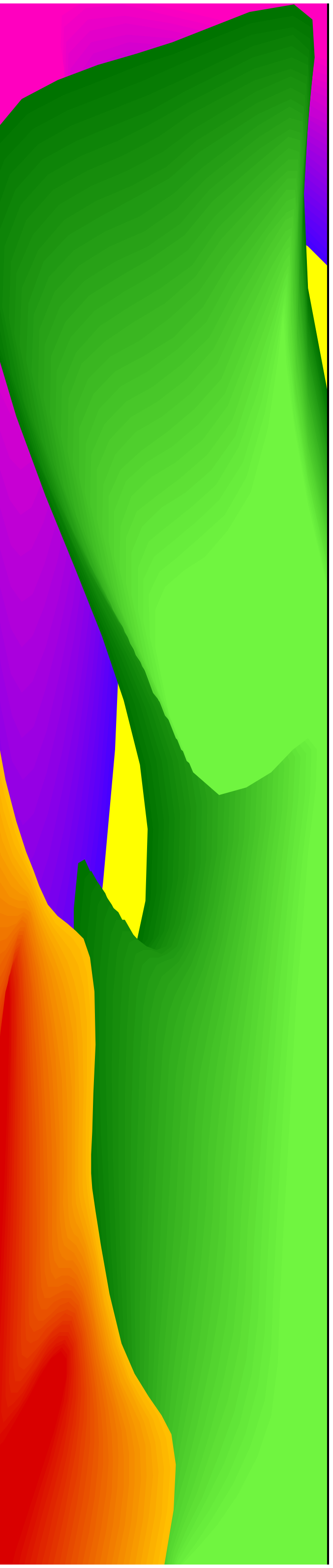
• One can see the poles 'coming in from infinity'.

• This allows one to see the 'fifth sector' using the $\mathbf{Z}_5$-symmetry.





6. Modularity



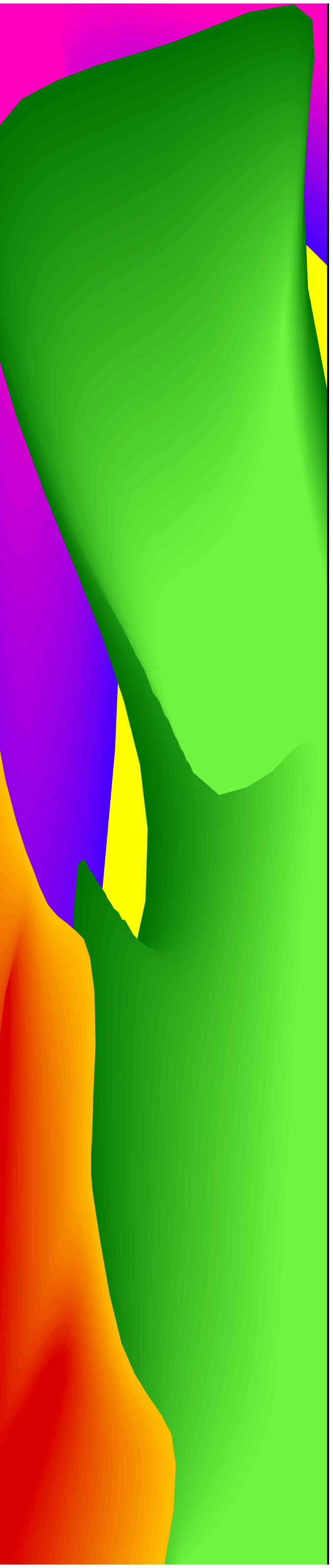
Modularity

Several indications that modularity plays an important role:

• Far from x=0, Painlevé I is well approximated by a Weierstrass equation.

• Transasymptotic terms satisfy degenerate Weierstrass equations (Costin et al.)

• Exact sums over instanton number in matrix models lead to Jacobi theta functions. (David/Eynard/Marino/…)



Modularity

Very much work in progress, but as a teaser: when we organize the 1-parameter transasymptotic expansion by powers of τ, we get expressions of the form

$$Z_{\text{inst}} = \sum_{n=0}^{\infty} \left\{ (-1)^n G_2(n+1)(q\tau)^{\frac{1}{2}(n^2-n)} \times \sum_{g=0}^{\infty} f_g(n) x^g \right\}$$

where $G_2(n)$ is the Barnes function, and

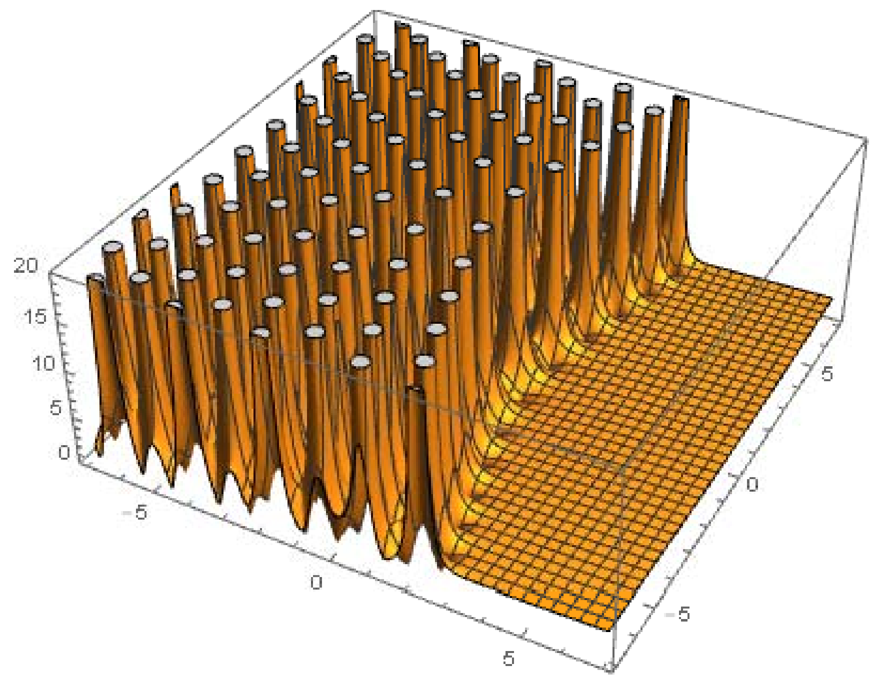
$$q = -\frac{1}{96\sqrt{3}}$$

Modularity

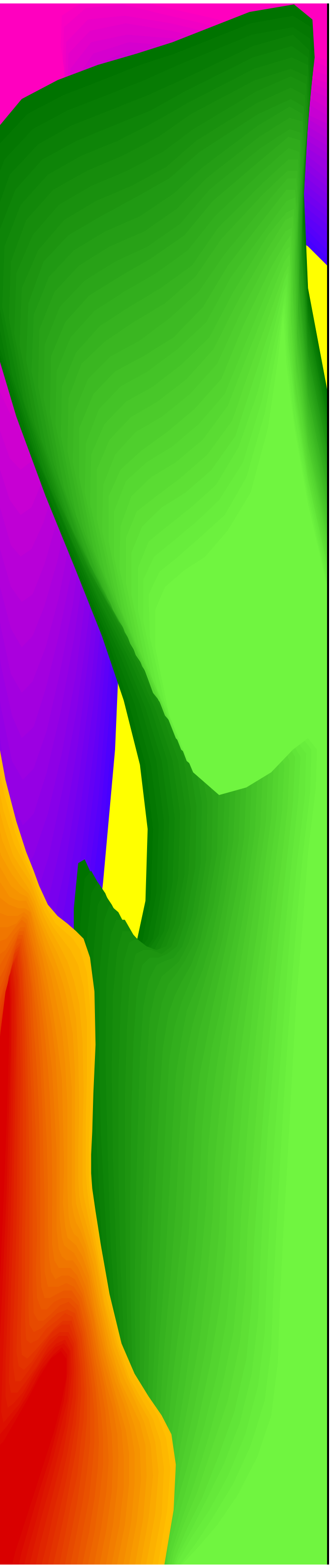
$$Z_{\text{inst}} = \sum_{n=0}^{\infty} \left\{ (-1)^n G_2(n+1)(q\tau)^{\frac{1}{2}(n^2-n)} \times \sum_{g=0}^{\infty} f_g(n) x^g \right\}$$

Smells like a theta function, but what about (very divergent!) Barnes-insertions?

In a much simpler example (‘grand canonical Gaussian’) we were able to show that the Barnes factors can be rewritten as a well-behaved sum of derivative operators acting on a ‘true’ Jacobi theta function.

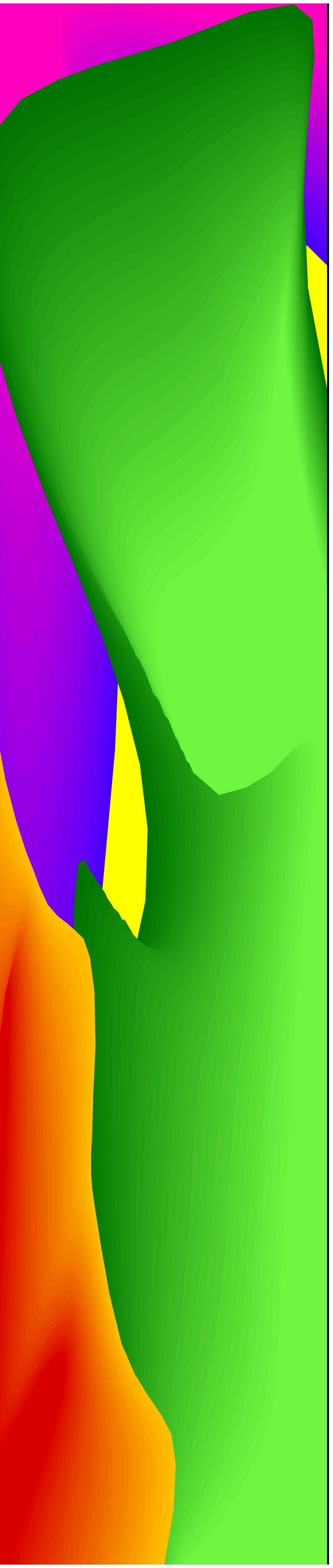


Conclusion and outlook



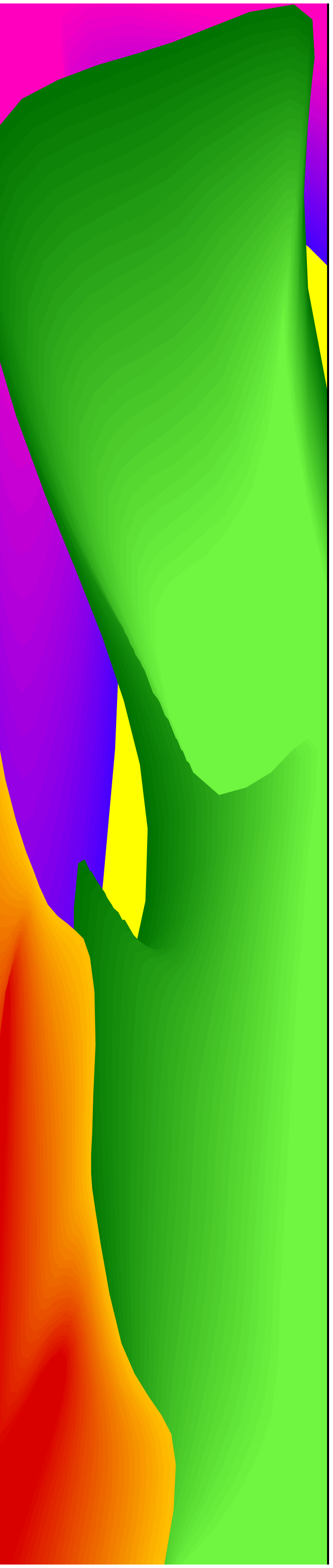
Conclusion

- Improved transasymptotics allows us to match formal transseries expressions to honest solutions even deep inside the pole fields.
- All this seems to work also in the two-parameter case, and therefore away from tronquées solutions.
- Interesting links to modularity exist, which can hopefully made precise in terms of theta function (re)summations of the transseries.



Outlook

- **To do:** show that all of this works as nicely as expected in the 2-parameter case, and make theta function (re)summations precise.
- Original motivation: matrix models = string equations = discrete Painlevé. (Cf. Takei's talk)
- For these, the full two-parameter transseries was also constructed in 1106.5922.



Outlook

- Here, the locations of the poles tell us about the different phases of the model. (Compare to Lee-Yang zeroes.) These techniques allow us to get a much deeper understanding of the phases of matrix models. (Long-term work in progress with Ines Aniceto and Ricardo Schiappa.)