Lagrangian fibrations by Prym varieties

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Overview

- Lagrangian fibrations
- fibrations by Prym varieties
- singularities and primitivity
- dual fibrations

Joint work with Chen Shen, PhD 2020 (on ProQuest).

Holomorphic symplectic manifolds

Let X be a compact Kähler manifold with $c_1 = 0$.

Thm (Bogomolov): $\exists$ finite étale cover $\tilde{X}$ of X with

$$\tilde{X} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

T = torus, CY_i = (strict) Calabi-Yau manifolds, and $IHS_j = \ldots$

Def: A compact Kähler manifold X is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form σ
In addition if $\pi_1(X) = 0$ and $\mathrm{H}^0(\Omega^2)$ is generated by σ then we say X is an *irreducible holomorphic symplectic (IHS) manifold*.

Examples of IHS manifolds

1. K3 surfaces S.
2. Hilbert schemes of points on K3 surfaces, $\mathrm{Hilb}^n S \to \mathrm{Sym}^n S$.
3. Generalized Kummer varieties, $\widetilde{\mathrm{Hilb}^{n+1}} A = A \times K_n(A)$.
4. Fano variety of lines in a cubic four-fold.
5. Mukai moduli spaces of stable sheaves on K3/abelian surfaces.

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathrm{H}^2(\mathcal{O}) \cong \mathbb{C}$$

6. O'Grady's spaces, OG6 and OG10.

Up to deformation, just 2 or 3 examples known in each dimension.

Fibrations

Let X be an IHS manifold of dimension $2n$.

Thm (Matsushita): If $X \to B$ is a proper fibration then

1. $\dim B = n = \dim F$,
2. F is Lagrangian wrt the holomorphic symplectic form σ,
3. generic fibre is a complex torus.

Rmk: Lagrangian means $TF \subset TX$ is maximal isotropic wrt σ. Integrable means $T^*B \subset T^*X$ is maximal isotropic wrt σ^{-1}. Thus Lagrangian fibrations are equivalent to integrable systems.

Rmk: Hodge theory $\implies$ generic fibre is an abelian variety.

Thm (Hwang): B is isomorphic to $\mathbb{P}^n$ if it is smooth.

Examples

1. Elliptic K3 surfaces $S \to \mathbb{P}^1$.

Lagrangian fibrations are like higher-dimensional elliptic K3s:

2. If S is an elliptic K3 surface then the Hilbert scheme

$$\mathrm{Hilb}^n S \to \mathrm{Sym}^n S \to \mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$$

is a Lagrangian fibration. Its fibres look like

$$E_1 \times E_2 \times \cdots \times E_n.$$

Examples

Lagrangian fibrations are also like compact Hitchin systems:

The GL-Hitchin system is an integrable system whose fibres are Jacobians of spectral curves $C \subset T^*\Sigma$.

3. Beauville-Mukai system: Let C be a genus g curve in a K3 S, with $|C| \cong \mathbb{P}^g$ and $\mathcal{C}/\mathbb{P}^g$ the family of curves linearly equivalent to C.

$$X := \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^g) \longrightarrow \mathbb{P}^g$$

is a Lagrangian fibration.

Rmk: $0 \longrightarrow TX_t \longrightarrow TX|_{X_t} \longrightarrow \pi^* T_t\mathbb{P}^g \longrightarrow 0$
$TX_t = \mathrm{H}^0(C, \Omega^1)^*$ is dual to $T_t\mathbb{P}^g = \mathrm{H}^0(C, N_{C\subset S}) = \mathrm{H}^0(C, \Omega^1)$.

Or $X \cong$ moduli space $M(0, [C], 1-g+d)$ of stable sheaves on S.

Fibrations by Jacobians

Conj: Let $\mathcal{C}/\mathbb{P}^n$ be a family of genus n curves. If $X = \overline{\mathrm{Jac}}^d(\mathcal{C}/\mathbb{P}^n)$ is an IHSM then it must be a Beauville-Mukai integrable system.

Thm (Markushevich): True for genus $n = 2$.

Thm (S-): True in the following cases:

- genus $n = 3$,
- genus $n = 4, 5$ and non-hyperelliptic curves,
- arbitrary genus n and degree of $\Delta \subset \mathbb{P}^n$ is $> 4n + 20$.

(Generalized) Prym varieties

Let $\pi : C \to D$ be a double cover of curves with covering involution τ. Then

$$\mathrm{Fix}^0(\tau^*) = \pi^*\mathrm{Jac}^0 D \subset \mathrm{Jac}^0 C.$$

Def: The Prym variety of C/D is

$$\mathrm{Prym}(C/D) := \mathrm{Fix}^0(-\tau^*),$$

an abelian variety of dimension $g_C - g_D$ and polarization type

$$(\underbrace{1, \ldots, 1}_{g_C - 2g_D}, \underbrace{2, \ldots, 2}_{g_D}).$$

$\mathrm{Prym}(C/D)$ is principally polarized iff $\pi : C \to D$ has zero or two branch points.

Families of Prym varieties

Let $\pi : S \to T$ be a K3 double cover of another surface with *anti-symplectic* covering involution τ. A curve $D \subset T$ has a double cover $C \subset S$,

$$\begin{array}{ccc} C & \subset & S \\ {\scriptstyle 2:1}\downarrow & & {\scriptstyle 2:1}\downarrow \\ D & \subset & T. \end{array}$$

Let $\mathcal{D} \to |D|$ be the complete linear system in T, $\mathcal{C} = \pi^*\mathcal{D}$.

Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini): We can construct a relative Prym variety

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(\mathcal{E} \mapsto \mathcal{E}xt^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|).$$

This is a symplectic variety and a Lagrangian fibration over $|D|$.

Examples

1. Markushevich-Tikhomirov system: S/T a K3 double cover of a degree two del Pezzo, C/D a genus three cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian surface of type $(1,2)$.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^2$ is an *irreducible* symplectic orbifold of $\dim^{\mathrm{n}}$ four, with 28 isolated singularities that look like $\mathbb{C}^4/\pm 1$.

2. Arbarello-Saccà-Ferretti system: S/T a K3 double cover of an Enriques surface, D genus g, $\mathrm{Prym}(C/D)$ principally polarized.

Then $\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^{g-1}$ is a symplectic variety, which is

- birational to $\mathrm{Hilb}^{g-1}K3$ if D is hyperelliptic,
- simply connected, no symplectic resolution, otherwise,
- and irreducible if g is odd.

Examples

3. Matteini system: S/T a K3 double cover of a cubic del Pezzo, C/D a genus four cover of an elliptic curve, $\mathrm{Prym}(C/D)$ an abelian threefold of type $(1,1,2)$.

$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to \mathbb{P}^3$ is an *irreducible* symplectic orbifold of $\dim^{\mathrm{n}}$ six, with singularities that look like $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$ and $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. Other systems (Matteini): K3 covers of other del Pezzo and Hirzebruch surfaces, give symplectic varieties with Lagrangian fibrations

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) \to |D|.$$

Questions:

- What are the structure of the singularities?
- Are these varieties simply connected? Are they irreducible?

An example of dimension six

S/T a K3 double cover of a degree one del Pezzo, $D \in |-2K_T|$, C/D a genus five cover of a genus two curve. Then

$$\mathrm{Prym}(\mathcal{C}/\mathcal{D}) := \mathrm{Fix}^0(-) \subset \overline{\mathrm{Jac}}^0(\widetilde{\mathcal{C}}/|C|) \leftarrow \mathrm{OG10}$$

is a symplectic variety of dim$^{\mathrm{n}}$ six and a Lagrangian fibration with abelian fibres of type $(1, 2, 2)$ over $|D| \cong \mathbb{P}^3$.

Lemma (Arbarello et al.): If $C = C_1 \cup C_2$ with $C_1.C_2 = 2k$ then $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ looks locally like $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k}/\pm 1)$ at $[\mathcal{F}_1 \oplus \mathcal{F}_2]$.

Thm (S-Shen): $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ contains 120 isolated singularities that look like $\mathbb{C}^6/\pm 1$ (and thus there is no symplectic resolution).

A birational model

The del Pezzo T is a double cover of the quadric cone Q. The covering involution lifts to another anti-symplectic involution on S:

$$\begin{array}{ccccc}
S & \longrightarrow & \mathbb{P}^2 & & \widetilde{S} = \text{resolution of } \overline{S} \\
\downarrow & \searrow & & \swarrow & \\
T & & \overline{S} & & \\
\downarrow & & & & \\
Q & & & &
\end{array}$$

The anti-symplectic involutions commute and their composition gives a symplectic involution on S, with quotient a singular K3 surface $\overline{S}$ with 8 A_1-singularities.

A birational model

$$\begin{array}{ccccc} C \subset S & \longrightarrow & \mathbb{P}^2 & & \tilde{C} \subset \tilde{S} \\ \downarrow & \searrow & & \swarrow & \\ D \subset T & & \overline{C} \subset \overline{S} & & \\ \downarrow & & & & \\ Q & & & & \end{array}$$

A generic τ-invariant $C \subset S$ is an étale double cover of a genus three curve $\overline{C} \subset \overline{S}$, which is isomorphic to $\tilde{C} \subset \tilde{S}$.

Pull-back induces a map

$$\mathrm{Jac}^0 \tilde{C} = \mathrm{Jac}^0 \overline{C} \longrightarrow \mathrm{Jac}^0 C$$

which is two-to-one onto its image $\mathrm{Prym}(C/D)$.

A birational model

Let $\widetilde{\mathcal{M}} := \overline{\mathrm{Jac}}^0(\tilde{\mathcal{C}}/\mathbb{P}^3)$ be the Beauville-Mukai system of $\tilde{C} \subset \tilde{S}$. Then there is a rational dominant generically two-to-one map

$$\widetilde{\mathcal{M}} \dashrightarrow \mathrm{Prym}(\mathcal{C}/\mathcal{D}).$$

Moreover, $\widetilde{\mathcal{M}}$ is deformation equivalent to $\mathrm{Hilb}^3 \tilde{S}$.

Thm (S-Shen): For $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ we have

- the symplectic structure is unique up to a scalar, $h^{2,0} = 1$,
- π_1 is trivial or $\mathbb{Z}/2\mathbb{Z}$, and thus $h^{1,0} = 0$.

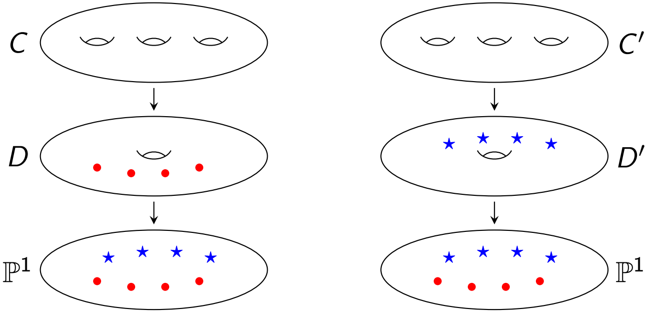
Rmk: We say that $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is a *primitive* symplectic variety.

Pantazis's bigonal construction

Given a tower $C \xrightarrow{2:1} D \xrightarrow{2:1} \mathbb{P}^1$ we can construct $C' \xrightarrow{2:1} D' \xrightarrow{2:1} \mathbb{P}^1$

$$C' := \{\text{pairs of lifts } (c_1, c_3), (c_1, c_4), (c_2, c_3), (c_2, c_4)\}.$$

This interchanges the branch points of the double covers.



Thm (Pantazis): $\mathrm{Prym}(C'/D')$ is dual to $\mathrm{Prym}(C/D)$.

Dual of the Markushevich-Tikhomirov system

A K3 double cover S/T of a degree two del Pezzo is given by two quartics Δ and Δ' in $\mathbb{P}^2$ that are tangent at eight points.

- $f : T \to \mathbb{P}^2$ is a double cover branched over Δ
- $S \to T$ is branched over one component of $f^{-1}(\Delta')$

Applying the bigonal construction gives $S' \xrightarrow{2:1} T' \xrightarrow{2:1} \mathbb{P}^2$, with the roles of the quartics Δ and Δ' switched.

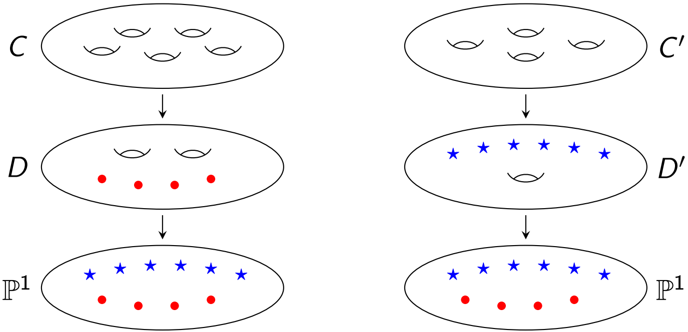
Thm (Menet): $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ over $\mathbb{P}^2$ is dual to $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$.

Thus the dual of a Markushevich-Tikhomirov system is another Markushevich-Tikhomirov system.

Question: What is the dual of our fibration in dimension six?

Dual of our fibration

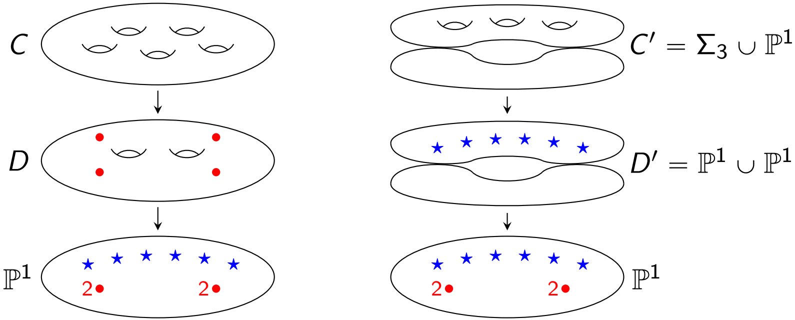
Fibres are $\mathrm{Prym}(C/D)$ with $g_C = 5$ and $g_D = 2$. Pantazis gives:



Thus $\mathrm{Prym}(C'/D')$ look like fibres of the Matteini system.

Questions: How to go from $S \xrightarrow{2:1} T \xrightarrow{2:1} Q$ to $S' \xrightarrow{2:1} T' \xrightarrow{???} Q$.

Dual of our fibration



The dual of the abelian threefold $\mathrm{Prym}(C/D)$ is

$$\begin{array}{ccccc} \mathrm{Prym}(C'/D') & \hookrightarrow & \mathrm{Jac}^0 C' & \subset & \overline{\mathrm{Jac}}^0 C' \\ & {\scriptstyle 2:1} \searrow & \downarrow & & \\ & & \mathrm{Jac}^0 \Sigma_3. & & \end{array}$$

Dual of our fibration

Start with $S \xrightarrow{2:1} T \xrightarrow{2:1} Q$, a K3 double cover of a degree one del Pezzo cover of a quadric cone. The bigonal construction gives

$$S' = \overline{S} \cup \mathbb{P}^2 \xrightarrow{2:1} T' = Q \cup Q \xrightarrow{2:1} Q.$$

Thm (S-Shen): $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ over $\mathbb{P}^3$ is dual to $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$.

Rmk: $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ is a double cover of the same Beauville-Mukai system that $\mathrm{Prym}(\mathcal{C}/\mathcal{D})$ is a $\mathbb{Z}/2\mathbb{Z}$ quotient of.

Question: Is S'/T' a degeneration of a K3 double cover of a cubic del Pezzo? Is $\mathrm{Prym}(\mathcal{C}'/\mathcal{D}')$ a degeneration of the Matteini system?