Lagrangian fibrations by Prym varieties

Justin Sawon\textsuperscript{1}

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Overview

- Lagrangian fibrations
- fibrations by Prym varieties
- singularities and primitivity
- dual fibrations

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Holomorphic symplectic manifolds

Let $X$ be a compact Kähler manifold with $c_1 = 0$.

**Thm (Bogomolov):** ∃ finite étale cover $\tilde{X}$ of $X$ with

$$\tilde{X} = T \times \prod_i CY_i \times \prod_j IHS_j,$$

$T =$ torus, $CY_i =$ (strict) Calabi-Yau manifolds, and $IHS_j =$ ...

**Def:** A compact Kähler manifold $X$ is a *holomorphic symplectic manifold* if it admits a non-degenerate holomorphic two-form $\sigma$

In addition if $\pi_1(X) = 0$ and $H^0(\Omega^2)$ is generated by $\sigma$ then we say $X$ is an *irreducible holomorphic symplectic (IHS) manifold*. 
Examples of IHS manifolds

1. K3 surfaces $S$.

2. Hilbert schemes of points on K3 surfaces, $\text{Hilb}^n S \to \text{Sym}^n S$.

3. Generalized Kummer varieties, $\text{Hilb}^{n+1} A = A \times K_n(A)$.

4. Fano variety of lines in a cubic four-fold.


\[ \text{Ext}^1(\mathcal{E}, \mathcal{E}) \times \text{Ext}^1(\mathcal{E}, \mathcal{E}) \to \text{Ext}^2(\mathcal{E}, \mathcal{E}) \xrightarrow{\text{tr}} H^2(O) \cong \mathbb{C} \]


Up to deformation, just 2 or 3 examples known in each dimension.
Let $X$ be an IHS manifold of dimension $2n$.

**Thm (Matsushita):** If $X \to B$ is a proper fibration then
1. $\dim B = n = \dim F$,
2. $F$ is Lagrangian wrt the holomorphic symplectic form $\sigma$,
3. generic fibre is a complex torus.

**Rmk:** Lagrangian means $TF \subset TX$ is maximal isotropic wrt $\sigma$. Integrable means $T^*B \subset T^*X$ is maximal isotropic wrt $\sigma^{-1}$. Thus Lagrangian fibrations are equivalent to integrable systems.

**Rmk:** Hodge theory $\implies$ generic fibre is an abelian variety.

**Thm (Hwang):** $B$ is isomorphic to $\mathbb{P}^n$ if it is smooth.
Examples

1. Elliptic K3 surfaces $S \rightarrow \mathbb{P}^1$.

Lagrangian fibrations are like higher-dimensional elliptic K3s:

2. If $S$ is an elliptic K3 surface then the Hilbert scheme

$$\text{Hilb}^n S \rightarrow \text{Sym}^n S \rightarrow \text{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$$

is a Lagrangian fibration. Its fibres look like

$$E_1 \times E_2 \times \cdots \times E_n.$$
Examples

Lagrangian fibrations are also like compact Hitchin systems:

The $GL$-Hitchin system is an integrable system whose fibres are Jacobians of spectral curves $C \subset T^*\Sigma$.

3. **Beauville-Mukai system:** Let $C$ be a genus $g$ curve in a K3 $S$, with $|C| \cong \mathbb{P}^g$ and $C/\mathbb{P}^g$ the family of curves linearly equivalent to $C$.

$$X := \overline{\text{Jac}}^d (C/\mathbb{P}^g) \longrightarrow \mathbb{P}^g$$

is a Lagrangian fibration.

**Rmk:**

$$0 \longrightarrow TX_t \longrightarrow TX|_{X_t} \longrightarrow \pi^* T_t \mathbb{P}^g \longrightarrow 0$$

$TX_t = H^0(C, \Omega^1)^*$ is dual to $T_t \mathbb{P}^g = H^0(C, N_{C\subset S}) = H^0(C, \Omega^1)$.

Or $X \cong$ moduli space $M(0, [C], 1 - g + d)$ of stable sheaves on $S$. 
Fibrations by Jacobians

**Conj:** Let $C/\mathbb{P}^n$ be a family of genus $n$ curves. If $X = \text{Jac}^d(C/\mathbb{P}^n)$ is an IHSM then it must be a Beauville-Mukai integrable system.

**Thm (Markushevich):** True for genus $n = 2$.

**Thm (S-):** True in the following cases:
- genus $n = 3$,
- genus $n = 4, 5$ and non-hyperelliptic curves,
- arbitrary genus $n$ and degree of $\Delta \subset \mathbb{P}^n$ is $> 4n + 20$. 
(Generalized) Prym varieties

Let $\pi : C \to D$ be a double cover of curves with covering involution $\tau$. Then

$$\text{Fix}^0(\tau^*) = \pi^* \text{Jac}^0 D \subset \text{Jac}^0 C.$$ 

**Def:** The Prym variety of $C/D$ is

$$\text{Prym}(C/D) := \text{Fix}^0(-\tau^*),$$

an abelian variety of dimension $g_C - g_D$ and polarization type

$$\begin{pmatrix} 1, \ldots, 1, 2, \ldots, 2 \end{pmatrix}.$$

$g_C - 2g_D \quad g_D$

$\text{Prym}(C/D)$ is principally polarized iff $\pi : C \to D$ has zero or two branch points.
Families of Prym varieties

Let $\pi : S \rightarrow T$ be a K3 double cover of another surface with \textit{anti-symplectic} covering involution $\tau$. A curve $D \subset T$ has a double cover $C \subset S$,

$$
\begin{align*}
C & \subset S \\
2:1 & \downarrow \\
D & \subset T.
\end{align*}
$$

Let $D \rightarrow |D|$ be the complete linear system in $T$, $C = \pi^*D$.

\textbf{Thm (Markushevich-Tikhomirov, Arbarello-Saccà-Ferretti, Matteini):} We can construct a relative Prym variety

$$
Prym(C/D) := \text{Fix}^0(\mathcal{E} \mapsto \text{Ext}^1_S(\tau^*\mathcal{E}, \mathcal{O}(-C))) \subset \overline{\text{Jac}}^0(\tilde{C}/|C|).
$$

This is a symplectic variety and a Lagrangian fibration over $|D|$. 
Examples

1. **Markushevich-Tikhomirov system**: $S/T$ a K3 double cover of a degree two del Pezzo, $C/D$ a genus three cover of an elliptic curve, $\text{Prym}(C/D)$ an abelian surface of type $(1, 2)$.

Then $\text{Prym}(C/D) \to \mathbb{P}^2$ is an *irreducible* symplectic orbifold of dimension four, with 28 isolated singularities that look like $\mathbb{C}^4/\pm 1$.

2. **Arbarello-Saccà-Ferretti system**: $S/T$ a K3 double cover of an Enriques surface, $D$ genus $g$, $\text{Prym}(C/D)$ principally polarized.

Then $\text{Prym}(C/D) \to \mathbb{P}^{g-1}$ is a symplectic variety, which is

- birational to $\text{Hilb}^{g-1} K3$ if $D$ is hyperelliptic,
- simply connected, no symplectic resolution, otherwise,
- and irreducible if $g$ is odd.
Examples

3. **Matteini system**: $S/T$ a K3 double cover of a cubic del Pezzo, $C/D$ a genus four cover of an elliptic curve, $\text{Prym}(C/D)$ an abelian threefold of type $(1,1,2)$.

$\text{Prym}(C/D) \to \mathbb{P}^3$ is an *irreducible* symplectic orbifold of dimension six, with singularities that look like $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$ and $\mathbb{C}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. **Other systems (Matteini)**: K3 covers of other del Pezzo and Hirzebruch surfaces, give symplectic varieties with Lagrangian fibrations

$$\text{Prym}(C/D) \to |D|.$$  

**Questions:**

- What are the structure of the singularities?
- Are these varieties simply connected? Are they irreducible?
An example of dimension six

$S/T$ a K3 double cover of a degree one del Pezzo, $D \in |-2K_T|$, $C/D$ a genus five cover of a genus two curve. Then

$$\text{Prym}(C/D) := \text{Fix}^0(\pi) \subset \text{Jac}^0(\tilde{C}/|C|) \leftarrow \text{OG}10$$

is a symplectic variety of dim $^n$ six and a Lagrangian fibration with abelian fibres of type $(1, 2, 2)$ over $|D| \cong \mathbb{P}^3$.

**Lemma (Arbarello et al.):** If $C = C_1 \cup C_2$ with $C_1.C_2 = 2k$ then $\text{Prym}(C/D)$ looks locally like $\mathbb{C}^{N-2k} \times (\mathbb{C}^{2k}/\pm 1)$ at $[\mathcal{F}_1 \oplus \mathcal{F}_2]$.

**Thm (S-Shen):** $\text{Prym}(C/D)$ contains 120 isolated singularities that look like $\mathbb{C}^6/\pm 1$ (and thus there is no symplectic resolution).
The del Pezzo $T$ is a double cover of the quadric cone $Q$. The covering involution lifts to another anti-symplectic involution on $S$:

$$
\begin{array}{ccc}
S & \rightarrow & \mathbb{P}^2 \\
\downarrow & \swarrow & \searrow \\
T & \rightarrow & \bar{S} \\
\downarrow & & \downarrow \\
Q & & \bar{S}
\end{array}
$$

$\tilde{S}$ = resolution of $\bar{S}$

The anti-symplectic involutions commute and their composition gives a symplectic involution on $S$, with quotient a singular K3 surface $\bar{S}$ with 8 $A_1$-singularities.
A birational model

\[
\begin{align*}
C \subset S & \quad \longrightarrow \quad \mathbb{P}^2 & \quad \tilde{C} \subset \tilde{S} \\
\downarrow & \quad \downarrow & \quad \downarrow \\
D \subset T & \quad \longrightarrow \quad \overline{C} \subset \overline{S} \\
\downarrow & & \downarrow Q \\
\quad Q & \quad & \quad \end{align*}
\]

A generic $\tau$-invariant $C \subset S$ is an étale double cover of a genus three curve $\overline{C} \subset \overline{S}$, which is isomorphic to $\tilde{C} \subset \tilde{S}$.

Pull-back induces a map

\[\text{Jac}^0 \tilde{C} = \text{Jac}^0 \overline{C} \longrightarrow \text{Jac}^0 C\]

which is two-to-one onto its image $\text{Prym}(C/D)$.
A birational model

Let \( \widetilde{\mathcal{M}} := \text{Jac}^0(\tilde{\mathcal{C}}/\mathbb{P}^3) \) be the Beauville-Mukai system of \( \tilde{\mathcal{C}} \subset \tilde{\mathcal{S}} \). Then there is a rational dominant generically two-to-one map

\[
\widetilde{\mathcal{M}} \longrightarrow \text{Prym}(\mathcal{C}/\mathcal{D}).
\]

Moreover, \( \widetilde{\mathcal{M}} \) is deformation equivalent to \( \text{Hilb}^3\tilde{\mathcal{S}} \).

**Thm (S-Shen):** For \( \text{Prym}(\mathcal{C}/\mathcal{D}) \) we have

- the symplectic structure is unique up to a scalar, \( h^{2,0} = 1 \),
- \( \pi_1 \) is trivial or \( \mathbb{Z}/2\mathbb{Z} \), and thus \( h^{1,0} = 0 \).

**Rmk:** We say that \( \text{Prym}(\mathcal{C}/\mathcal{D}) \) is a *primitive* symplectic variety.
Pantazis’s bigonal construction

Given a tower $C \xrightarrow{2:1} D \xrightarrow{2:1} \mathbb{P}^1$ we can construct $C' \xrightarrow{2:1} D' \xrightarrow{2:1} \mathbb{P}^1$

$C' := \{\text{pairs of lifts } (c_1, c_3), (c_1, c_4), (c_2, c_3), (c_2, c_4)\}$.

This interchanges the branch points of the double covers.

Thm (Pantazis): Prym$(C'/D')$ is dual to Prym$(C/D)$. 
Dual of the Markushevich-Tikhomirov system

A K3 double cover $S/T$ of a degree two del Pezzo is given by two quartics $\Delta$ and $\Delta'$ in $\mathbb{P}^2$ that are tangent at eight points.

- $f : T \to \mathbb{P}^2$ is a double cover branched over $\Delta$
- $S \to T$ is branched over one component of $f^{-1}(\Delta')$

Applying the bigonal construction gives $S' \xrightarrow{2:1} T' \xrightarrow{2:1} \mathbb{P}^2$, with the roles of the quartics $\Delta$ and $\Delta'$ switched.

**Thm (Menet):** Prym($C'/D'$) over $\mathbb{P}^2$ is dual to Prym($C/D$).

Thus the dual of a Markushevich-Tikhomirov system is another Markushevich-Tikhomirov system.

**Question:** What is the dual of our fibration in dimension six?
Dual of our fibration

Fibres are $\text{Prym}(C/D)$ with $g_C = 5$ and $g_D = 2$. Pantazis gives:

Thus $\text{Prym}(C'/D')$ look like fibres of the Matteini system.

Questions: How to go from $S \xrightarrow{2:1} T \xrightarrow{2:1} Q$ to $S' \xrightarrow{2:1} T' \xrightarrow{??} Q$. 
Dual of our fibration

The dual of the abelian threefold $\text{Prym}(C/D)$ is

$$\text{Prym}(C'/D') \quad \longleftrightarrow \quad \text{Jac}^0 C' \subset \overline{\text{Jac}^0 C'}$$

2:1

$$\text{Jac}^0 \Sigma_3.$$
Start with $S \overset{2:1}{\rightarrow} T \overset{2:1}{\rightarrow} Q$, a K3 double cover of a degree one del Pezzo cover of a quadric cone. The bigonal construction gives

$$S' = \overline{S} \cup \mathbb{P}^2 \overset{2:1}{\rightarrow} T' = Q \cup Q \overset{2:1}{\rightarrow} Q.$$ 

**Thm (S-Shen):** Prym$(C'/D')$ over $\mathbb{P}^3$ is dual to Prym$(C/D)$.

**Rmk:** Prym$(C'/D')$ is a double cover of the same Beauville-Mukai system that Prym$(C/D)$ is a $\mathbb{Z}/2\mathbb{Z}$ quotient of.

**Question:** Is $S'/T'$ a degeneration of a K3 double cover of a cubic del Pezzo? Is Prym$(C'/D')$ a degeneration of the Matteini system?