COMPLEX SADDLES IN SEMI-CLASSICS OF PATH INTEGRALS

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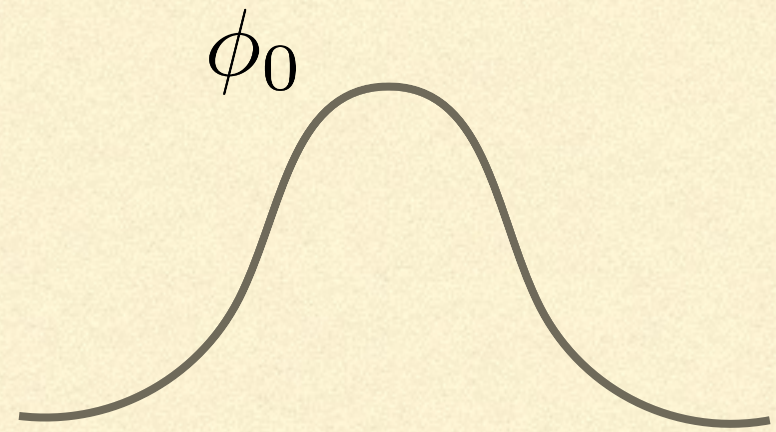
PRL 116 (2016) no.1, 011601/arXiv:1510.00978
arXiv:1510.03435

SEMI-CLASSICS AND GROUND STATE ENERGY

$$Z = e^{-\beta H} = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

path-integral over real paths



Lump of "Φ"

suppressed by $e^{-S[\phi_0]}$

$$Z \approx 1 + V\beta c e^{-S(\phi_0)} + \frac{1}{2}(V\beta c e^{-S(\phi_0)})^2 + \dots$$

$$Z = e^{-\beta E_0^{NP}} + \cdots \approx e^{-\left(-\beta V c e^{-S(\phi_0)}\right)}$$

$$E_0^{NP} \approx -V c e^{-S(\phi_0)}$$

INCONSISTENCY WITH SUSY

$$H = \sum_\alpha \{Q_\alpha^\dagger, Q_\alpha\} \geq 0$$

$$E_n \geq 0$$

Perturbative corrections are vanishing, implying that if the classical semi-classics is true, no non-perturbative semi-classical contributions can exist.

But we know non-perturbative contributions can and do exist in SUSY theories (e.g. spontaneous SUSY breaking). So what is going on?

RESOLUTION: PICARD-LEFSCHETZ THEORY

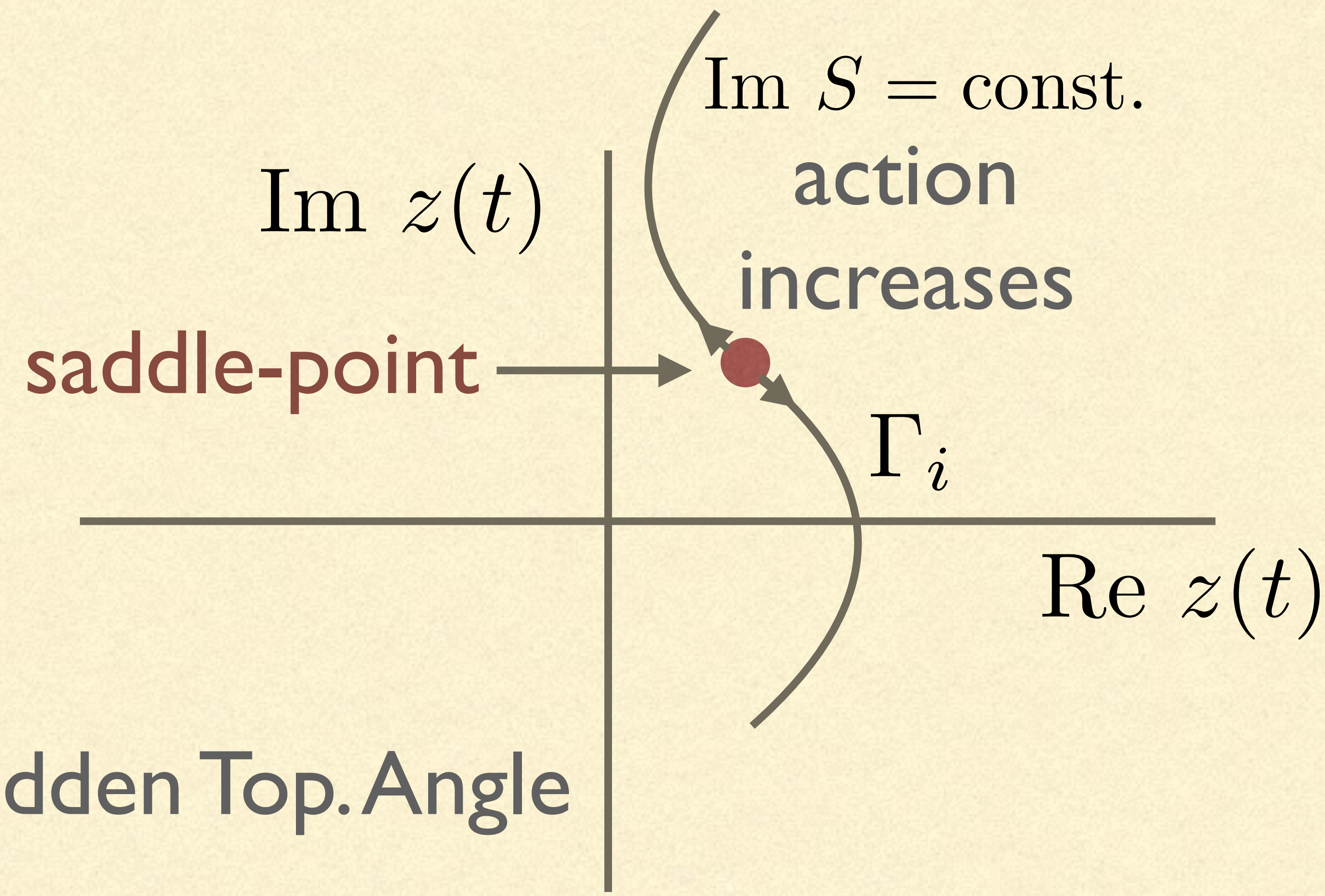
$$\int \mathcal{D}x \; e^{-S[x]} \to \sum_i n_i \int_{\Gamma_i} \mathcal{D}z \; e^{-S[z]}$$

$$\Gamma_i : \quad \frac{dz}{ds} = \frac{\delta \bar{S}[\bar{z}]}{\delta \bar{z}(t)}$$

$$\frac{d}{ds}\mathrm{Im}\ S[z] = 0$$

$$\frac{d}{ds}\mathrm{Re}\ S[z] > 0$$

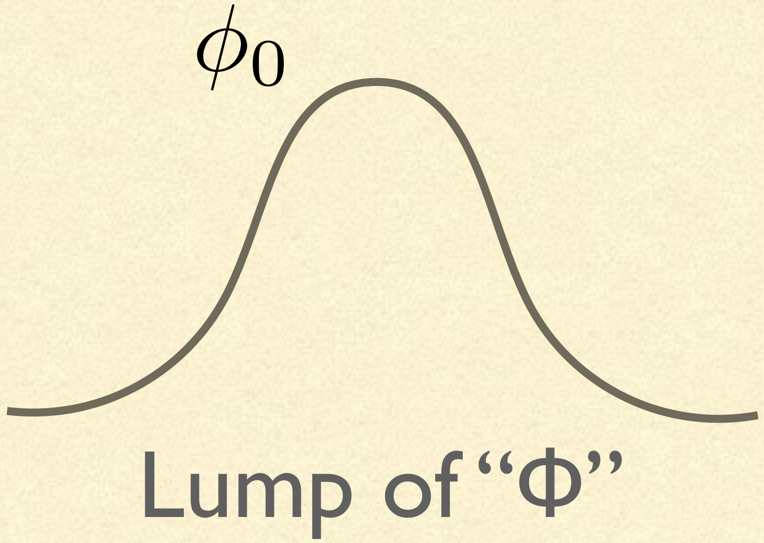
$$\mathrm{Im}\ S[z] = \theta_{HTA}$$ —Hidden Top. Angle



SEMI-CLASSICS AND GROUND STATE ENERGY

$$Z = e^{-\beta H} = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

↑ path-integral over real paths



suppressed by $e^{-\mathrm{Re}\ S(\phi_0) - i\mathrm{Im}\ S(\phi_0)}$

$$Z \approx 1 + V\beta c e^{-S(\phi_0)} + \frac{1}{2}(V\beta c e^{-S(\phi_0)})^2 + \dots$$

$$Z = e^{-\beta E_0^{NP}} + \cdots \approx e^{-\left(-\beta V c e^{-S(\phi_0)}\right)}$$

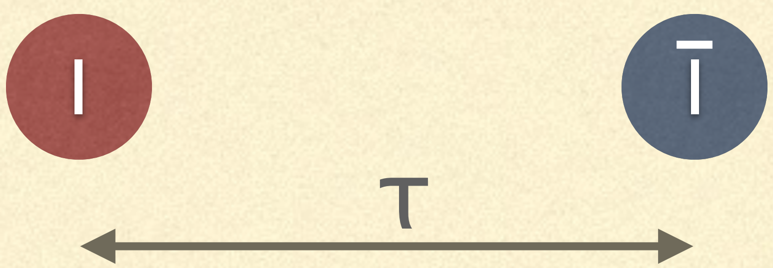
$$E_0^{NP} \approx -V c e^{-\mathrm{Re}\ S(\phi_0) - i\mathrm{Im}\ S(\phi_0)}$$

"QUASI-ZEROMODE THIMBLES"

$$\mathcal{D}x(t) \to \mathcal{D}x'(t) \times \text{(quasi-)zeromodes}$$

small fluctuations

Treated by PL theory

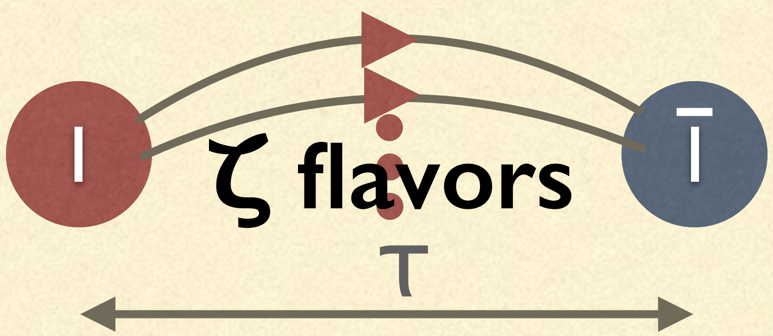


$$E_{NP} \approx -Ce^{-2S_{inst}} \int d(\omega\tau) e^{\frac{A}{g}} e^{-\omega\tau}$$

“QUASI-ZEROMODE THIMBLES”

$$\mathcal{D}x(t) \to \mathcal{D}x'(t) \times \text{(quasi-)zeromodes}$$

small fluctuations → $\mathcal{D}x'(t)$; Treated by PL theory → (quasi-)zeromodes



$$E_{NP} \approx -Ce^{-2S_{inst}} \int d(\omega\tau) e^{\frac{A}{g} e^{-\omega\tau}} e^{-\zeta\omega\tau}$$

$$S_{int} = -e^{\frac{A}{g} e^{-\omega\tau}}$$

$$S_{eff}(\tau) = \frac{A}{g} e^{-\omega\tau} - N_f \omega\tau$$

$$S'_{eff}(\tau_0) = 0$$

$$E_{NP} \approx -Ce^{-2S_{inst}} \Gamma(\zeta) \left(\frac{g}{A}\right)^{\zeta} e^{-i\zeta\pi}$$

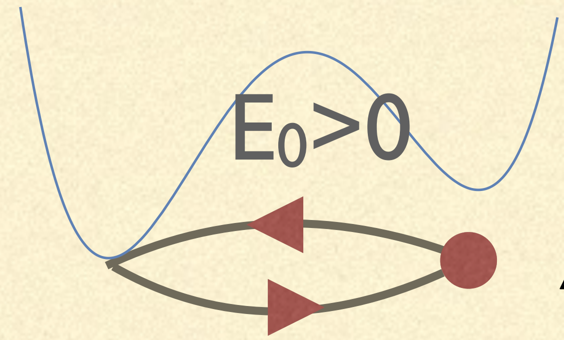
COMPLEX SADDLES IN SUSY

See also talk by Tatsuhiro Misumi

$$V_\pm(x) = \frac{W'^2}{2} \pm g\frac{W''(x)}{2}$$

N=1 quantum mechanics
(comes from fermions (i.e. spin))

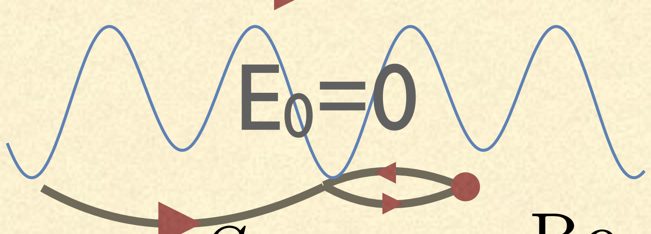
Balitsky-Yung
Nucl.Phys. B274 (1986) 475



$$\Delta E = -e^{-\text{Re } S_{cl}} e^{+i\pi} > 0$$

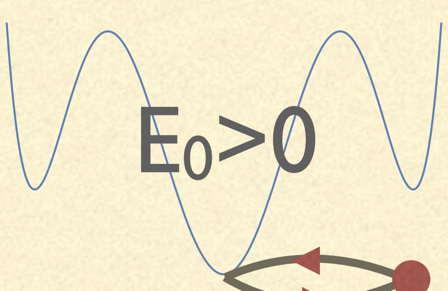
$$W'(x) = x^2 - 1$$

Behtash. Dunne, Schafer, TS, Unsal
arXiv:1510.03435 (2015)
Phys.Rev.Lett. 116 (2016) no.1, 011601
arXiv:1510.00978

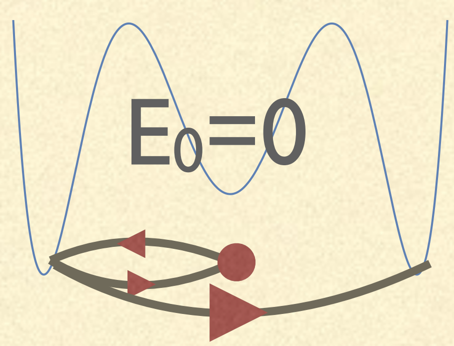


$$-e^{-S_{cl}} \quad -e^{-\text{Re } S_{cl}} e^{+i\pi}$$

$$W'(x) = \sin(x)$$



$$-e^{-\text{Re } S_{cl}} e^{+i\pi}$$



$$-e^{-S_{cl}} - e^{-\text{Re } S_{cl}} e^{+i\pi}$$

$$W'(x) = x^2(x-1)^2$$

G. Dunne, TS, M. Unsal
(in progress)

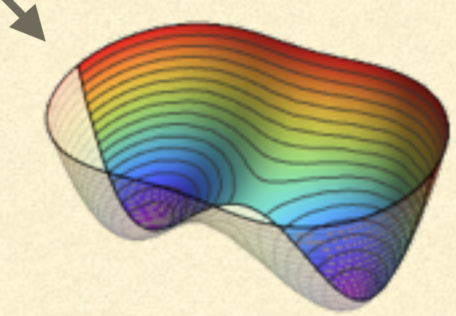
COMPLEX SADDLES IN SUSY CONTINUED

E. Poppitz, TS, M.Unsal

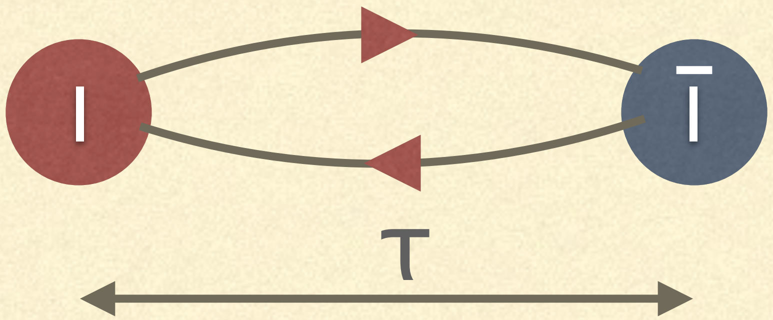
JHEP 1511 (2015) 175/arXiv:1507.04063

N=2 SUSY QM

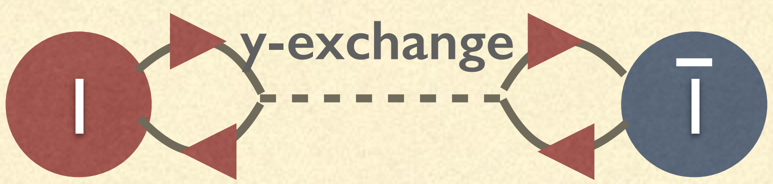
$$g\mathcal{L}_E = |\dot{z}(t)|^2 + |W'(z)|^2$$
$$+ \begin{pmatrix}\bar{\chi}_1 & \chi_2\end{pmatrix} \left(-\partial_t + \begin{pmatrix} 0 & \overline{W''(z)} \\ W''(z) & 0 \end{pmatrix}\right) \begin{pmatrix}\chi_1 \\ \bar{\chi}_2\end{pmatrix} .$$



$$z(t) = x + iy(t) \qquad W'(z) = z^2 - 1$$



$$E^1_{NP} \approx -C_1 e^{-2S_{inst}} \int d(\omega\tau) e^{\frac{A}{g} e^{-\omega\tau} - 2\omega\tau}$$



(naively higher order in g)

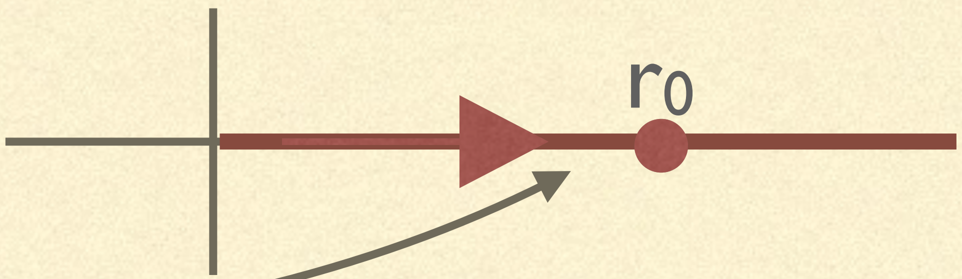
$$E^2_{NP} \approx -C_2 e^{-2S_{inst}} \int d(\omega\tau) e^{\frac{A}{g} e^{-\omega\tau} - \omega\tau}$$

$$E^1_{NP} + E^2_{NP} = 0\,!$$

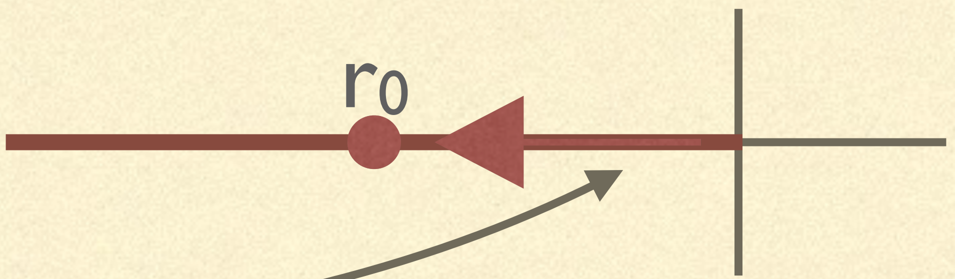
THE SUPER YANG-MILLS THEORY

The gluon condensate is zero due to SUSY

$$[RB] = C \int dr \; e^{-\frac{1}{\lambda r} - 2\log(r)}$$

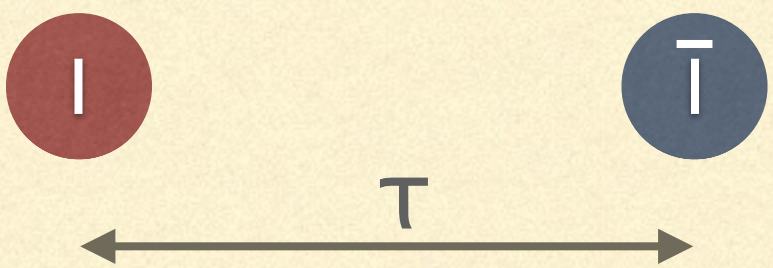


$$[CB] = C \int dr \; e^{\frac{1}{\lambda r} - 2\log(r)}$$



$$\langle \mathrm{tr}\; F_{\mu\nu}^2 \rangle \propto [RB] + [CB] = 0$$

COMMENDT ON THE BOGOMOLNY-ZINN-JUSTIN PERSCRIPTION



$$E_{NP} \approx -Ce^{-2S_{inst}} \int_0^\infty d(\omega\tau) e^{\frac{A}{g} e^{-\omega\tau}}$$

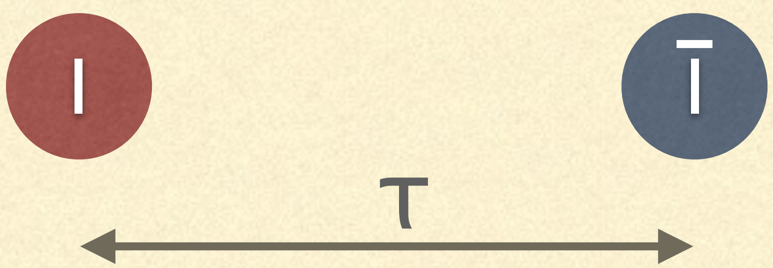
BZJ: treat g as a number with the negative real part (after subtracting the uncorrelated-instantons term):

$$E_{NP}^{I\bar{I}} \approx +\log(g/A) \pm i\pi - \gamma + \frac{g e^{A/g}}{A}$$

Exp. small for Re(g)<0, so dropped?

But for Re(g)>0, it is exponentially large!

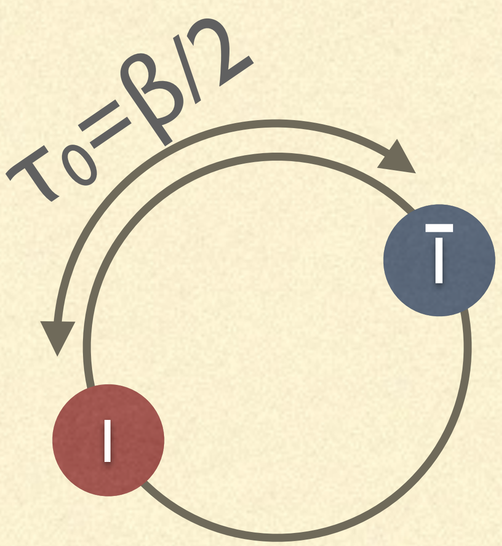
COMMENDT ON THE BOGOMOLNY-ZINN-JUSTIN PERSCRIPTION

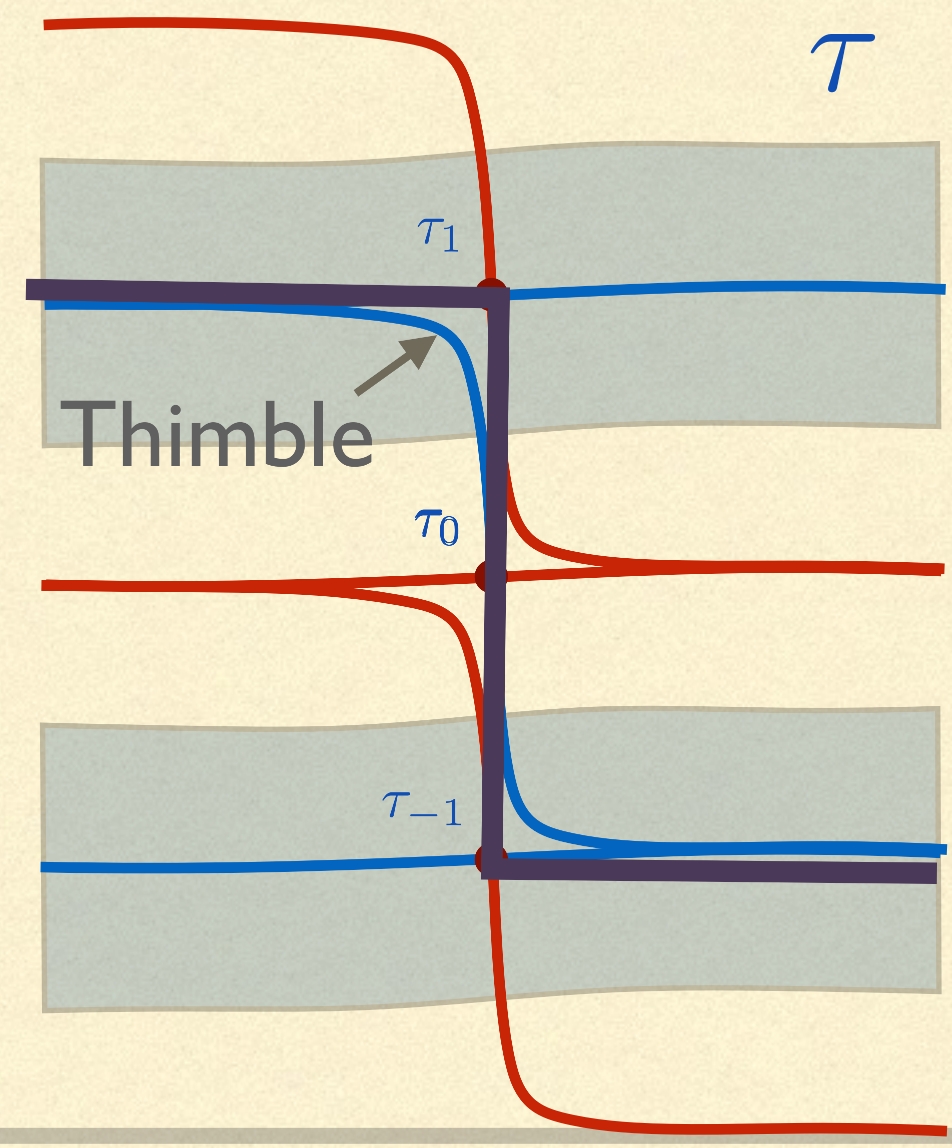


$$E_{NP} \approx -Ce^{-2S_{inst}} \int d(\omega\tau) e^{\frac{A}{g}\left[e^{-\omega\tau}+e^{-\omega(\beta-\tau)}\right]}$$

Periodic boundary condition:

$$\tau = \tau_0 + \tilde{\tau}$$





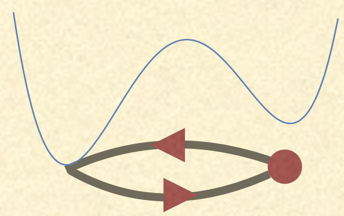
After subtracting the uncorrelated-instantons term:

$$E_{NP}^{I\bar{I}} \approx +\log(g/A) \pm i\pi - \gamma$$

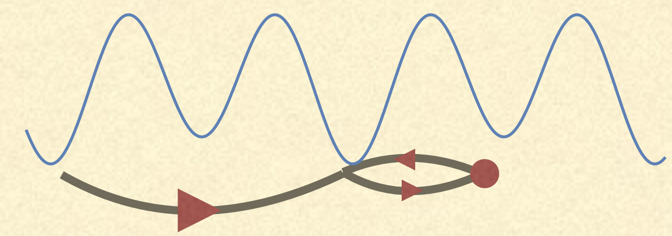
THE BEHAVIOR OF PERTURBATION THEORY AND QUASI-EXACT SOLVABILITY

with Can Kozcas, Yuya Tanizaki, Mithat Unsal

$$V_{\pm}(x) = \frac{W'^2}{2} \pm \zeta g \frac{W''(x)}{2}$$



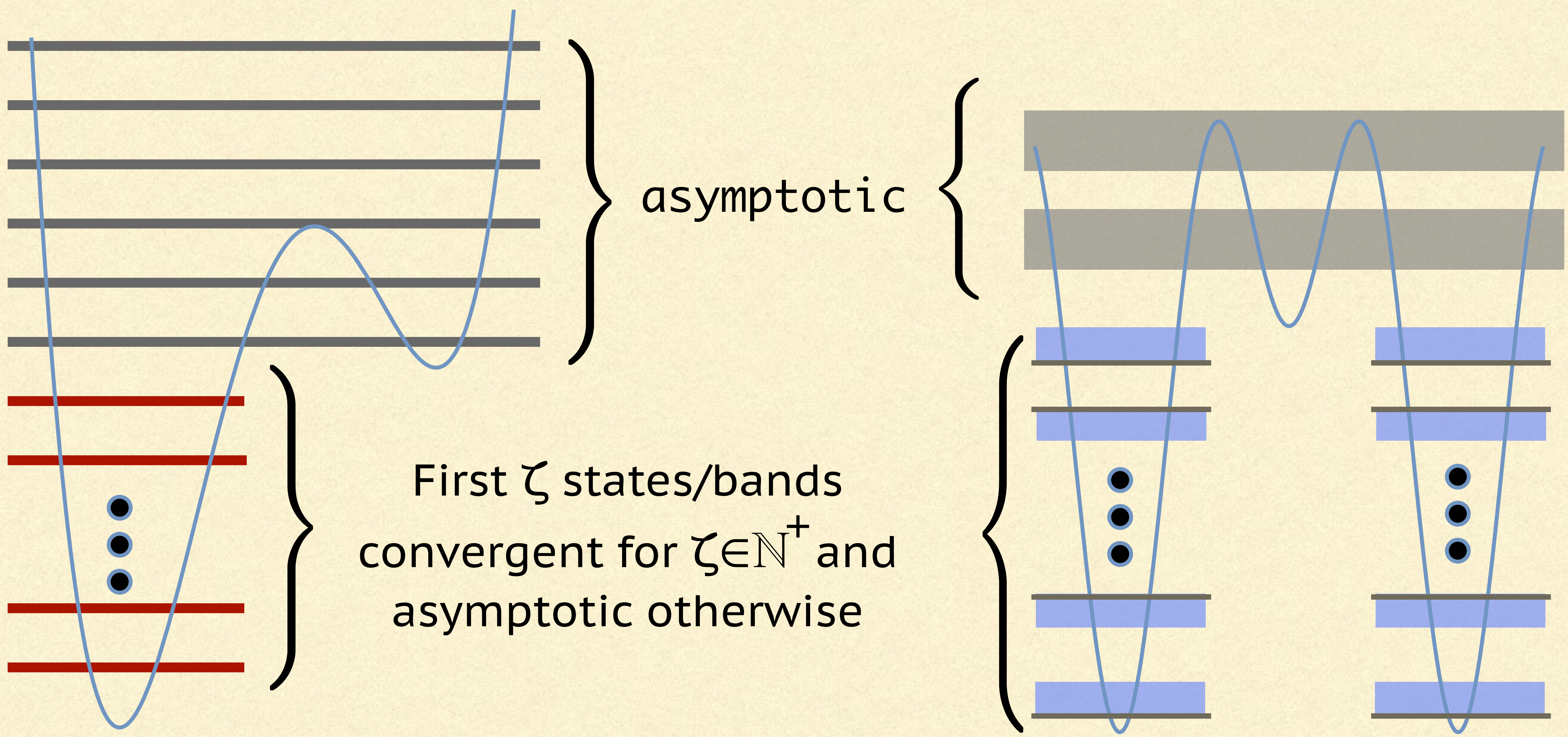
$$E^{NP} \approx -Ce^{-2S_{inst}} e^{\pm i\zeta\pi}$$



$$E^{NP} \approx -Ce^{-2S_{inst}} (\cos\theta + e^{\pm i\zeta\pi})$$

- The integer ζ theories are special
- The perturbation theory is CONVERGENT for the first ζ states
- In the case of Double Sine Gordon, a part of a spectrum is exactly solvable (Turbiner 1988), and the exact solution is reproduced by the perturbation theory
- In the case of the Tilted Double Well potential, the perturbation theory, although convergent for lowest ζ states, does not give the correct answer, i.e. it is missing the non-perturbative contribution which is unambiguous.

THE INTEGER ζ THEORY



THE BenderWu PACKAGE: STUDYING LARGE ORDERS

Bender-Wu (1973): anharmonic oscillator

- J. J. M. Verbaarschot, P. C. West, and T. T. Wu, *Large order behavior of the supersymmetric anharmonic oscillator*, *Phys. Rev.* D42 (1990) 1276–1284.
- C. M. Bender and G. V. Dunne, *Large order perturbation theory for a nonHermitian PT symmetric Hamiltonian*, *J. Math. Phys.* 40 (1999) 4616–4621, [quant-ph/9812039].
- C. M. Bender, G. V. Dunne, P. N. Meisinger, and M. Simsek, *Quantum complex Henon-Heiles potentials*, *Phys. Lett.* A281 (2001) 311–316, [quant-ph/0101095].
- M. Stone and J. Reeve, *Late Terms in the Asymptotic Expansion for the Energy Levels of a Periodic Potential*, *Phys. Rev.* D18 (1978) 4746.

CONSTRUCTIVE RESURGENCE

$$E_{NP} = -\frac{1}{\pi}\left(\frac{g}{8}\right)^{\zeta-1}\Gamma(\zeta)(\cos\theta + e^{\pm i\pi\zeta})e^{-2S_0/g}\mathcal{P}_{\text{fluc}}(g,\zeta)$$

Due to real saddle
"The Real Bion"

Due to complex saddle
"The Complex Bion"

$$\mathcal{P}_{fluct}(\nu, g; \zeta) = (1 + b_1 g + b_2 g + \dots)$$

Asymptotic growth of PT required by resurgence:

$$a_n \approx \frac{1}{2\pi}\frac{1}{8^{\zeta-1}}\frac{(n-\zeta)!}{\Gamma(1-\zeta)(2S_0)^{n-\zeta+1}}$$

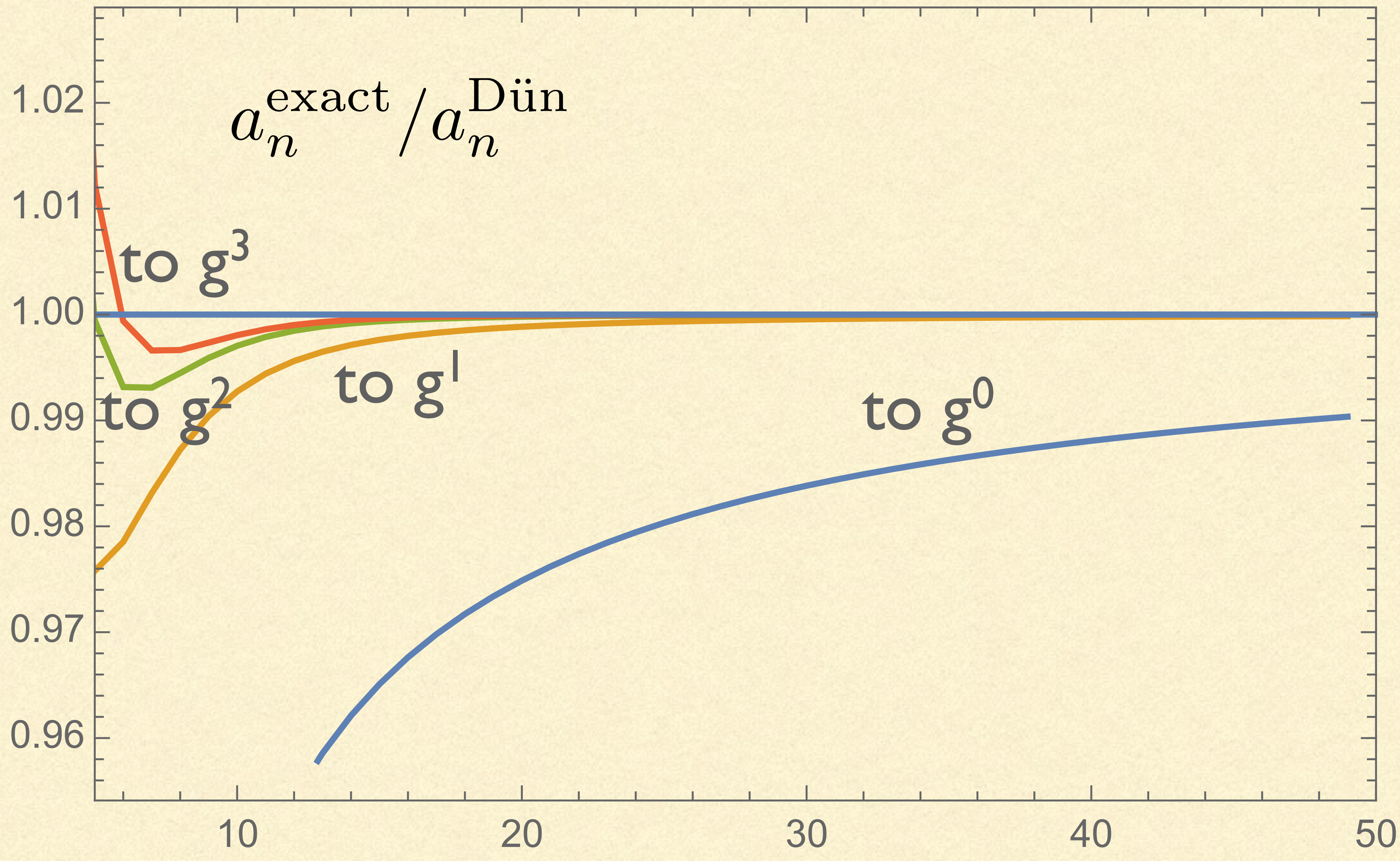
$$\times\left(1 + b_1\frac{2S_0}{n-\zeta} + b_2\frac{(2S_0)^2}{(n-\zeta)(n-\zeta-1)} + \dots\right)$$

CONSTRUCTIVE RESURGENCE

$$\mathcal{P}_{fluc}(\nu, g; \zeta) = \frac{\partial E_{PT}(\nu, g)}{\partial \nu} \exp\left[2S_0 \int_0^g dg\, g^{-2} \left[\frac{\partial E_{PT}(\nu, g)}{\partial \nu} - \left(1 - \frac{g(2\nu + 1 - \zeta))}{2S_0}\right)\right]\right]$$

- a version of Dünne-Unsal (or Dün relation) 2014— (see Chris Howls' and Gokçe Basar's talk.)
- relates petrubation theory around trivial vacuum to the "complex bion" (instanton—anti-instanton) fluctuation
- But a complex bion dictates late orders of perturbation theory
- Hence we have a relation between early terms of PT and late terms of PT

CONSTRUCTIVE RESURGENCE



CONCLUSIONS

- The nature of semi-classics is inextricably linked to the complexification of the path-integrals
- The machinery of resurgence guarantees the reality of all real physical observables
- Fascinating early-terms—late-terms relation in the same saddle-sector
- SUSY and integer ζ-deformed theories are special, with resurgent cancellation not needed for certain observables (i.e. energy-levels)
- The resurgence mechanism is not lost, and is restored with slight deformation of such theories
- Potential connection with emergent symmetries in QCD(adj) (Ask Aleksey Cherman!)