The first law of heterotic stringy black hole mechanics at zeroth order in α'

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Outline

Motivation

- Wald showed that for a gravitational theory invariant under diffeomorphisms, entropy is the Noether charge of the invariance
- Difficult to determine entropy when matter terms are added
- Iyer-Wald formalism assumes all terms are tensors. What about non-tensor cases?
- Gauge-covariant Lie derivatives are required

Motivation

- We study the use of momentum maps, which arise when symmetries have to be related to gauge transformations
- Interesting relationships between zeroth law and momentum map can be found
- Momentum maps will allow us to construct forms which are closed on the bifurcation surface

Heterotic string Framework

$$
\begin{aligned}
S[e^a, B, \phi, \mathcal{A}^I, \phi^x] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int e^{-2\phi} \Big[& (-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} \\
& -4 d\phi \wedge \star d\phi - \tfrac{1}{8} dM_{IJ} \wedge \star dM^{IJ} \\
& +(-1)^d \tfrac{1}{2} M_{IJ} \mathcal{F}^I \wedge \star \mathcal{F}^J + \tfrac{1}{2} H \wedge \star H \Big] \\
& \equiv \int \mathbf{L}\,.
\end{aligned} \tag{1}
$$

Levi-Civita curvature $\qquad R^{ab} \equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}$ (2)

KK/winding vector $\mathcal{F}^I \equiv \begin{pmatrix} F^m \\ G_m \end{pmatrix}, F^m = dA^m\,, G_m = dB_m\,,$

$$
H \equiv dB - \tfrac{1}{2}\mathcal{A}_I \wedge d\mathcal{A}^I\,, \qquad \mathcal{A}^I \equiv \begin{pmatrix} A^m \\ B_m \end{pmatrix}. \tag{3}
$$

Equations of motion

$$\delta S = \int \Big\{ \mathrm{E}_a \wedge \delta e^a + \mathrm{E}_B \wedge \delta B + \mathrm{E}_\phi \delta\phi + (\tilde{\mathrm{E}}_I + \frac{1}{2}\mathrm{E}_B \wedge \mathcal{A}_I) \wedge \delta \mathcal{A}^I + \mathrm{E}_x \delta\phi^x + d\Theta(\varphi, \delta\varphi) \Big\}. \quad (4)$$

Variation under diffeomorphism

Only the dilaton ϕ and the $O(n,n)/(O(n)\times O(n))$ scalars ϕ^x transform as a tensor under diffeomorphisms. The rest require compensating gauge transformations. These can be determined by

1. Requiring gauge-covariance of the complete transformation law (which can then be interpreted as a gauge-covariant Lie derivative) and
2. Imposing that, for diffeomorphisms which are symmetries of the field configuration that we are considering (in particular, for isometries), the complete transformation (covariant Lie derivative) vanishes.

We will use k to denote the Killing vector of the metric.

Lie-Maxwell derivative

The transformation of the Abelian vector fields $\mathcal{A}^I$ under diffeomorphisms can be defined as

$$\delta_\xi \mathcal{A}^I = -\mathbb{L}_\xi \mathcal{A}^I \,, \tag{5}$$

where

$$\mathbb{L}_\xi \mathcal{A}^I \equiv \imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I \,. \tag{6}$$

Here $\mathcal{P}_\xi{}^I$ is a gauge-invariant $\mathrm{O}(n,n)$ vector of functions that depends on $\mathcal{A}^I$.

$$d\mathcal{P}_k{}^I = -\imath_k \mathcal{F}^I \,. \tag{7}$$

Lorentz derivatives

- For the Vierbein,

$$\mathbb{L}_\xi e^a = \mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\,, \tag{8}$$

where

$$P_\xi{}^{ab} \equiv \nabla^{[a}\xi^{b]}\,, \tag{9}$$

leading us once more to a Momentum map

$$\imath_k R^{ab} = -\mathcal{D}P_k{}^{ab}. \tag{10}$$

- The curvature and spin connection likewise have a Lie-Lorentz derivative.
- In asymptotically-flat stationary black-hole spacetimes with bifurcate horizon

$$P_k{}^{ab} = \nabla^{[a}k^{b]} \overset{\mathcal{BH}}{=} \kappa n^{ab}\,. \tag{11}$$

Kalb-Ramond

Using the Bianchi identity, one finds

$$\delta_\xi H = -d\left(\imath_k H + \mathcal{P}_{k\,I}\mathcal{F}^I\right) = 0\,, \tag{12}$$

which can be defined as a momentum map

$$-\imath_k H - \mathcal{P}_{k\,I}\mathcal{F}^I = dP_k\,. \tag{13}$$

Using this, it is possible to determine the variation of B

$$\delta_\xi B = -\left(\imath_\xi H + \mathcal{P}_{\xi\,I}\mathcal{F}^I + dP_\xi\right) - \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I\,. \tag{14}$$

Covariant derivative summary

$$\delta_\xi \phi = -\imath_\xi d\phi\,, \tag{15a}$$

$$\delta_\xi \phi^x = -\imath_\xi d\phi^x\,, \tag{15b}$$

$$\delta_\xi \mathcal{A}^I = -\left(\imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\right)\,, \tag{15c}$$

$$\delta_\xi e^a = -\left(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\right)\,, \tag{15d}$$

$$\delta_\xi \omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab}\right)\,, \tag{15e}$$

$$\delta_\xi B + \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I = -\left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right)\,. \tag{15f}$$

Wald-Noether charge

Substituting in our transformations, the new variation takes the form

$$\delta_\xi S = -\int \Big\{ \mathrm{E}_a \wedge \mathcal{D}\imath_\xi e^a + \mathrm{E}_B \wedge \imath_\xi H + \tilde{\mathrm{E}}_I \wedge \imath_\xi \mathcal{F}^I + \mathrm{E}_\phi \imath_\xi d\phi + \mathrm{E}_x \imath_\xi d\phi^x - d\left[\Theta(\varphi, \delta_\xi\varphi) - P_\xi \wedge \mathrm{E}_B + (-1)^d \mathcal{P}_\xi{}^I \tilde{\mathrm{E}}_I\right]\Big\} . \tag{16}$$

Integrating by parts and applying the Noether identity leaves us with

$$\delta_\xi S = \int d\Theta'(\varphi, \delta_\xi\varphi) , \tag{17}$$

where

$$\Theta'(\varphi, \delta_\xi\varphi) = \Theta(\varphi, \delta_\xi\varphi) + (-1)^d \mathrm{E}_a \xi^a - P_\xi \wedge \mathrm{E}_B + (-1)^d \mathcal{P}_\xi{}^I \tilde{\mathrm{E}}_I . \tag{18}$$

Wald-Noether Charge

Under the combined transformations,

$$\delta_\xi S = -\int d\imath_\xi \mathsf{L}\,, \tag{19}$$

which, combined with Eq. (17), leads to the identity

$$d\mathsf{J} = 0, \tag{20}$$

where J is given by

$$\mathsf{J} \equiv \Theta'(\varphi, \delta_\xi \varphi) + \imath_\xi \mathsf{L}\,. \tag{21}$$

As J is closed, there exists a $(d-2)$ form such that.

$$\mathsf{J} = d\mathsf{Q}[\xi]\,. \tag{22}$$

Up to total derivatives and an overall factor, the Noether Charge can be found to be

$$\mathsf{Q}[\xi] = (-1)^d \star (e^a \wedge e^b) \left[e^{-2\phi} P_{\xi\, ab} - 2\imath_a d e^{-2\phi} \xi_b \right]$$

Zeroth law

- States that surface gravity is constant over the horizon
- Restricting ourselves to only the bifurcation surface, we can arrive at "restricted generalized zeroth laws"
- Generalized zeroth laws states the closedness of the electostatic potential and similar higher-rank forms
- If all field strengths are regular on the horizon,

$$\imath_k \mathcal{F}^I \stackrel{\mathcal{BH}}{=} 0\,, \tag{24a}$$

$$\imath_k H \stackrel{\mathcal{BH}}{=} 0\,. \tag{24b}$$

- The first equation implies closedness of the components of the momentum map $\mathcal{P}_k^I$ on $\mathcal{BH}$
- The second equation implies

$$0 \stackrel{\mathcal{BH}}{=} -\imath_k H = dP_k + \mathcal{P}_{k\,I}\mathcal{F}^I \stackrel{\mathcal{H}}{=} d\left(P_k + \mathcal{P}_{k\,I}\mathcal{A}^I\right)\,. \tag{25}$$

First Law

- We start by defining the *pre-symplectic* $(d-1)$-*form*

$$\omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1\Theta(\varphi, \delta_2\varphi) - \delta_2\Theta(\varphi, \delta_1\varphi)\,, \tag{26}$$

and the *symplectic form* relative to the Cauchy surface Σ

$$\Omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \int_\Sigma \omega(\varphi, \delta_1\varphi, \delta_2\varphi)\,. \tag{27}$$

- If $\delta_1\varphi = \delta\varphi$ is an arbitrary variation of the fields and $\delta_2\varphi = \delta_\xi\varphi$ is variation under diffeomorphism,

$$\omega(\varphi, \delta\varphi, \delta_\xi\varphi) = \delta \mathrm{J} + d\imath_\xi\Theta' = \delta d\mathrm{Q}[\xi] + d\imath_\xi\Theta'\,, \tag{28}$$

- For $\xi = k$, integrating and using Stokes' theorem yields

$$\int_{\delta\Sigma}\left(\delta\mathrm{Q}[k] + \imath_k\Theta'\right) = 0\,. \tag{29}$$

- For asymptotically flat, stationary, black-hole spacetimes

$$k^\mu = t^\mu + \Omega^n\phi_n^\mu\,. \tag{30}$$

First Law

- If we choose Σ to be the space between infinity and the bifurcation sphere ($\mathcal{BH}$) on which $k = 0$, then

$$\delta \int_{\mathcal{BH}} \mathrm{Q}[k] = \int_{\mathrm{S}^{d-2}_{\infty}} \left(\delta \mathrm{Q}[k] + \imath_k \Theta'\right) . \tag{31}$$

- Right hand side is simply $\delta M - \Omega^m \delta J_n$
- Restoring the overall factor $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$, and adding and subtracting $\mathcal{P}_{k\,I}\mathcal{A}^I \wedge \left(e^{-2\phi} \star H\right)$ yields

$$\begin{aligned}\delta \int_{\mathcal{BH}} \mathrm{Q}[k] =& \frac{(-1)^{d-1} g_s^{(d)\,2}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} \mathcal{P}_k{}^I e^{-2\phi} \left[M_{IJ} \star \mathcal{F}^J + \star H \wedge \mathcal{A}_I\right] \\ & - \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} \left(P_k + \mathcal{P}_{k\,I}\mathcal{A}^I\right) \wedge \left(e^{-2\phi} \star H\right) \\ & + \frac{(-1)^d g_s^{(d)\,2}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} e^{-2\phi} \star (e^a \wedge e^b) P_{k\,ab} .\end{aligned} \tag{32}$$

First Law

- The first two integrals are simply $\Phi^I \delta \mathcal{Q}_I$ and $\Phi^i \delta \mathcal{Q}_i$ respectively, where $Q_i \equiv Q[\Lambda_{h\,i}]$, and $\Lambda_{h,i}$ are the harmonic 1-forms on $\mathcal{BH}$, and Φ^i can be interpreted as the potential for a given charge of the KR field
- The final integral is given as

$$\begin{aligned}
&\frac{(-1)^d \kappa}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} e^{-2(\phi-\phi_\infty)} \star (e^a \wedge e^b) n_{ab} \\
&\qquad = -\frac{\kappa}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} e^{-2(\phi-\phi_\infty)} n^{ab} n_{ab} \\
&\qquad = T\delta \frac{\mathcal{A}_{\mathcal{H}}}{4 G_N^{(d)}}
\end{aligned}$$

$$\mathcal{A}_{\mathcal{H}} \equiv \int_{\mathcal{B}} d^{d-2} S e^{-2(\phi-\phi_\infty)}$$

First Law

Combined, we finally achieve

$$\delta M = T\delta \frac{\mathcal{A}_{\mathcal{H}}}{4G_N^{(d)}} + \Omega^m \delta J_m + \Phi^i \delta Q_i + \Phi^I \delta \mathcal{Q}_I \, , \qquad (33)$$

Conclusion

- We have introduced the notion of Momentum maps, which allow for easier computation of the first law and producing a generalized zeroth law
- We have derived the first law of black hole mechanics in the context of the effective action of the Heterotic Superstring compactified on a torus at leading order in α'
- Applying momentum map formalism to other more complex cases should be interesting (has been applied to higher order α' corrections and Black Ring example)
- How can the definition of the potential Φ^i be applied to cases not on the bifurfaction surface?