Arithmetic of decay walls through continued fractions

A new exact dyon counting solution in $\mathcal{N} = 4$ CHL models

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with Gabriel Cardoso and Suresh Nampuri, arXiv:2007.10302

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Introduction

Understanding the microscopic origin of Black Hole entropy remains a central question in Quantum Gravity.

$$S_{stat}(Q) = \ln d(Q) \qquad \leftrightarrow \qquad S_{BH}(Q)$$

Microscopic — Macroscopic

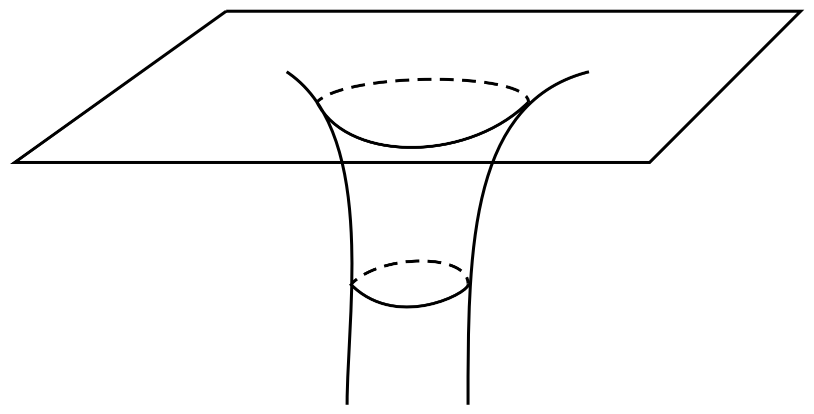
Address it in $\mathcal{N} = 4$ supersymmetric String Theory. Concretely: study the microscopic degeneracies of a special type ($\Delta < 0$) of $1/4-$BPS dyons in CHL models.

Rich interplay between Physics and Number Theory.

Inspired by [Chowdhury, Kidambi, Murthy, Reys, Wrase '19]. Here we propose a new systematic way to tackle these issues.

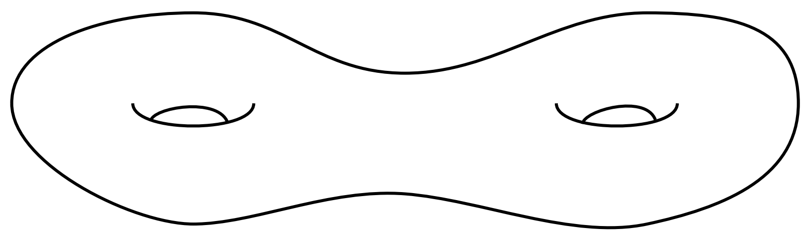
Introduction

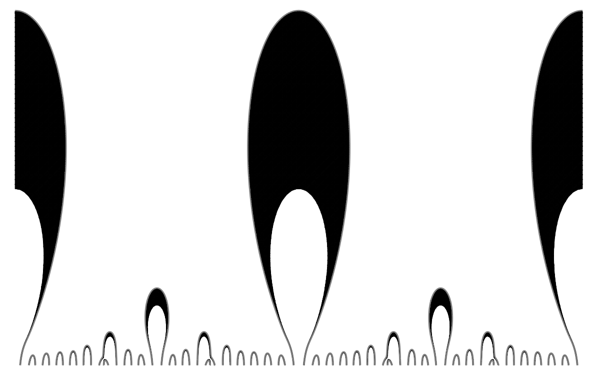
Dyonic degeneracies



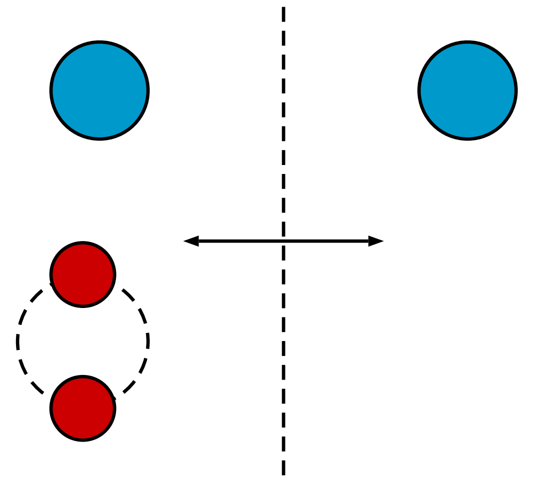
Siegel modular forms

Mock Jacobi forms





Wall-crossing



Continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Setup

Heterotic string theory on T^6.

$S-$duality group is $SL(2,\mathbb{Z})$

$T-$duality group, $SO(22,6;\mathbb{Z})$

$U-$duality group is $SL(2,\mathbb{Z}) \times SO(22,6;\mathbb{Z})$

$\mathcal{N} = 4$ supersymmetry and 28 $U(1)$ gauge groups

$1/4$-BPS states carry electric $\vec{Q}$ and magnetic $\vec{P}$ charges: Dyons

$T-$duality invariants $\quad m = P^2/2 \in \mathbb{Z}, \quad n = Q^2/2 \in \mathbb{Z}, \quad \ell = P \cdot Q \in \mathbb{Z}$

$$d(\vec{P}, \vec{Q}) = d(m, n, \ell)$$

Relevant $U-$duality invariant:

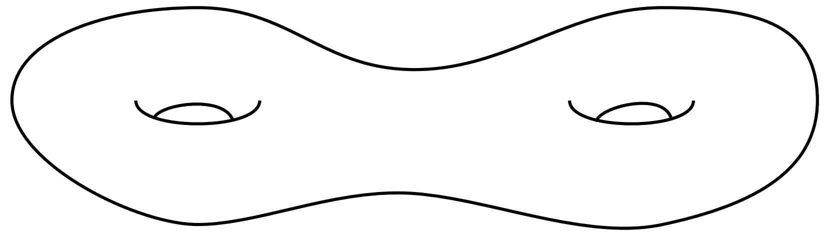
$$\Delta = Q^2P^2 - (Q \cdot P)^2 = 4mn - \ell^2 \qquad \text{Area} \sim \sqrt{\Delta}$$

Siegel modular forms

The generating function for $1/4-$BPS dyonic degeneracies is a modular form of the genus-2 modular group $Sp(2,\mathbb{Z})$

[Dijgkraaf, Verlinde, Verlinde '96]

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{\substack{m,n\geq -1\\ m,n,\ell\in\mathbb{Z}}} (-1)^{\ell+1} d(m,n,\ell) e^{2\pi i(m\rho+n\sigma+\ell v)}$$



Φ_{10} is the Igusa cusp form, invariant under $SL(2,\mathbb{Z})$.

$$p = e^{2\pi i\rho}$$
$$q = e^{2\pi i\sigma}$$
$$y = e^{2\pi i v}$$

$$d(m,n,\ell) = (-1)^{\ell+1}\int_C d\rho d\sigma dv\, p^{-m}q^{-n}y^{-\ell}\frac{1}{\Phi_{10}(\rho,\sigma,v)}$$

$C : 0 \leq \rho_1, \sigma_1, v_1 \leq 1$

ρ_2, σ_2, v_2 fixed, $\rho_2\sigma_2 - v_2^2 \gg 0$

Problem: Meromorphic

Dyon spectrum

Two types of $1/4-$BPS dyons:

Immortal

Single centre dyonic black holes with finite or zero horizon area in two-derivative gravity

Two-centred bound states of $1/2$-BPS constituents **Can decay**

[Cheng, Verlinde '07]

Single centre $1/4$-BPS black holes with finite horizon area have $\Delta > 0$.

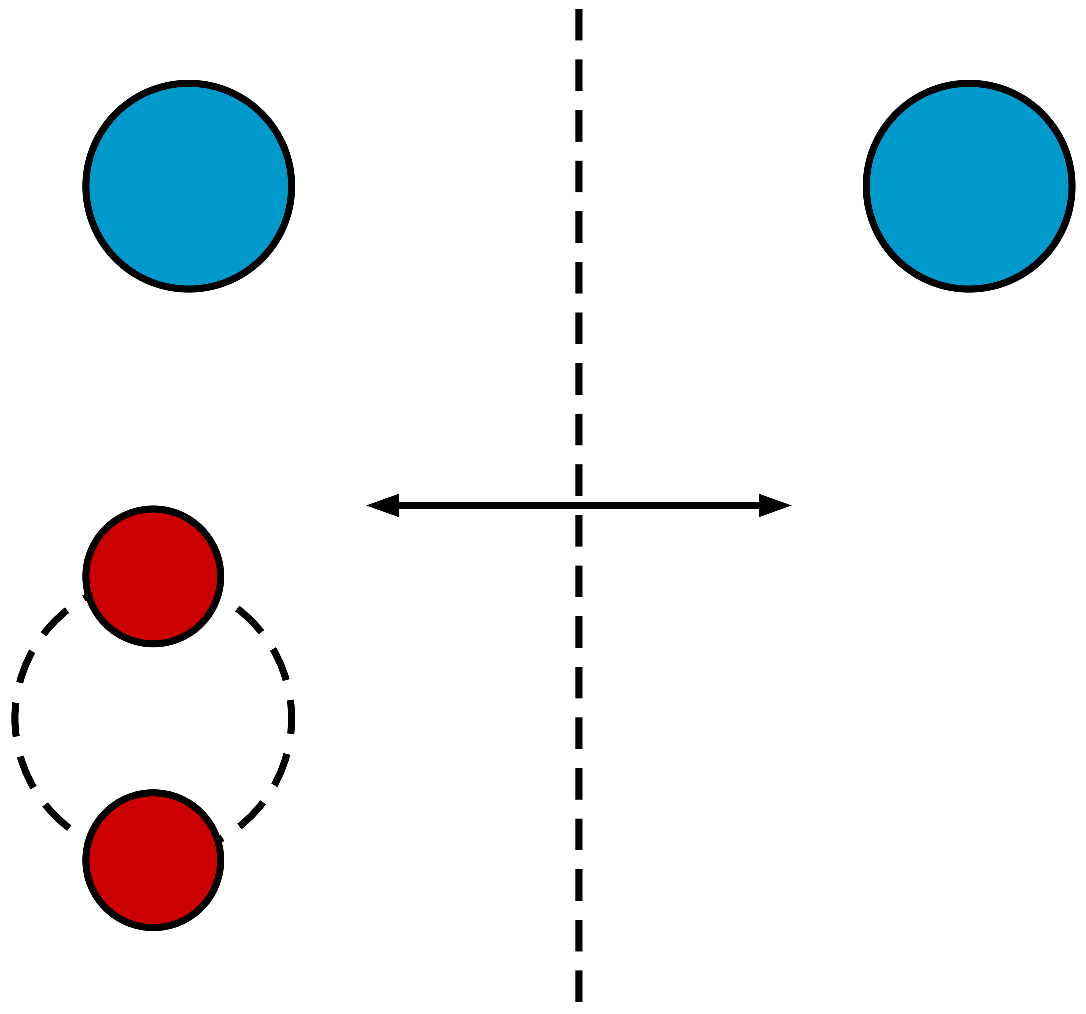
We will focus on

$$\Delta = 4mn - \ell^2 < 0$$

$\Delta < 0$ are always two-centred states

Wall-crossing

Pole in the Siegel modular form ⟷



$$d(m,n,\ell) = (-1)^{\ell+1}\int_C d\rho d\sigma dv\, p^{-m}q^{-n}y^{-\ell}\Phi_{10}^{-1}$$

$$C: 0 \leq \rho_1, \sigma_1, v_1 \leq 1$$

ρ_2, σ_2, v_2 fixed, $\rho_2\sigma_2 - v_2^2 \gg 0$

Changing ρ_2, σ_2, v_2 in contour C

$d(m,n,\ell)$ can jump

Ex: $\dfrac{1}{1-x} = \sum_{n\geq 0} x^n$ or $-\sum_{n\geq 1} x^{-n}$

$|x| < 1$ $\qquad$ $|x| > 1$

[Sen, '07]

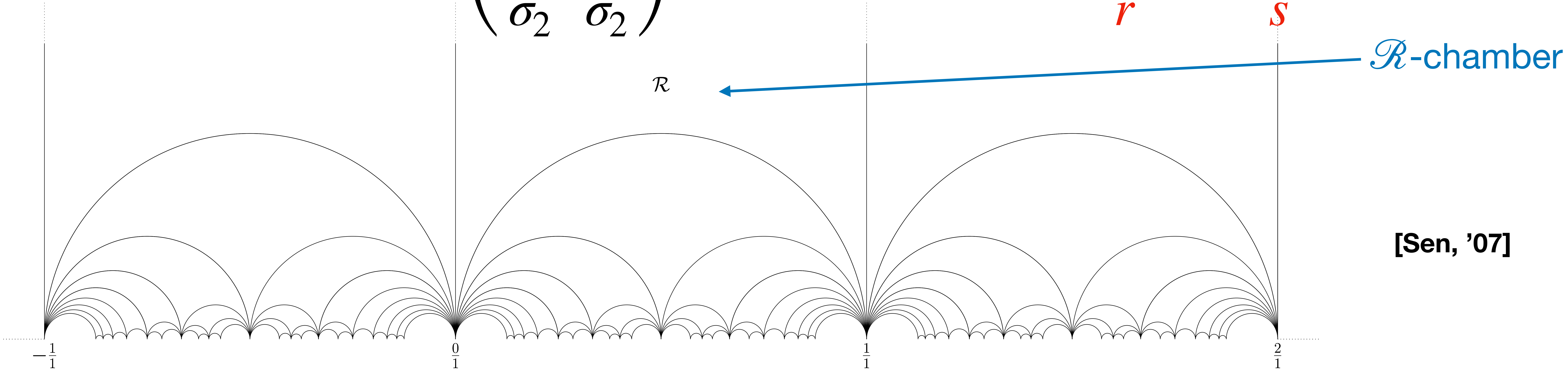
[Dabholkar, Gaiotto Nampuri '07]

Poles and walls

$\dfrac{1}{\Phi_{10}}$ has an infinite family of second order poles in the (ρ, σ, v) space

$$pq\sigma_2 + rs\rho_2 + (ps + qr)v_2 = 0, \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in PSL(2,\mathbb{Z})$$

Represent walls in the $\left(\dfrac{v_2}{\sigma_2}, \dfrac{\rho_2}{\sigma_2}\right)$ plane by lines joining $\dfrac{p}{r}$ and $\dfrac{q}{s}$



[Sen, '07]

Dyonic decay

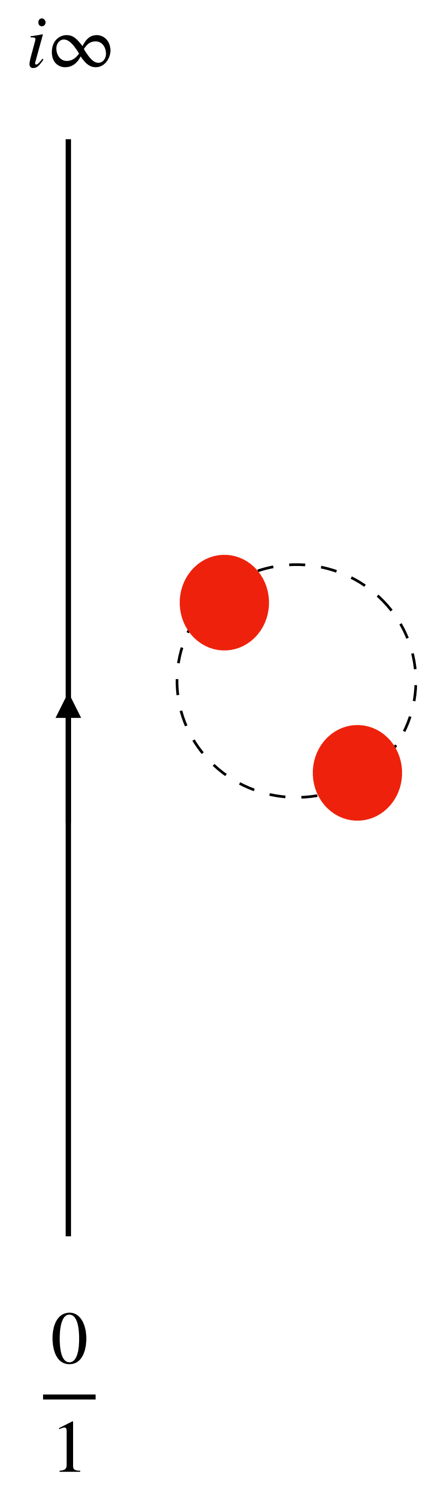
Decay mode at the wall of marginal stability corresponding to the identity matrix

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} Q \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ P \end{pmatrix}, \ v_2 = 0$$

Wall crossing contribution given by the residue at the pole

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} \xrightarrow{v\to 0} \frac{1}{v^2}\,\frac{1}{\eta^{24}(\rho)}\,\frac{1}{\eta^{24}(\sigma)} : \quad (-1)^{\ell+1}\ell\, d(m)d(n)$$

where $\dfrac{1}{\eta^{24}(\rho)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n\rho}$ counts $1/2-$BPS states



[Sen, '07]

Wall-crossing formula

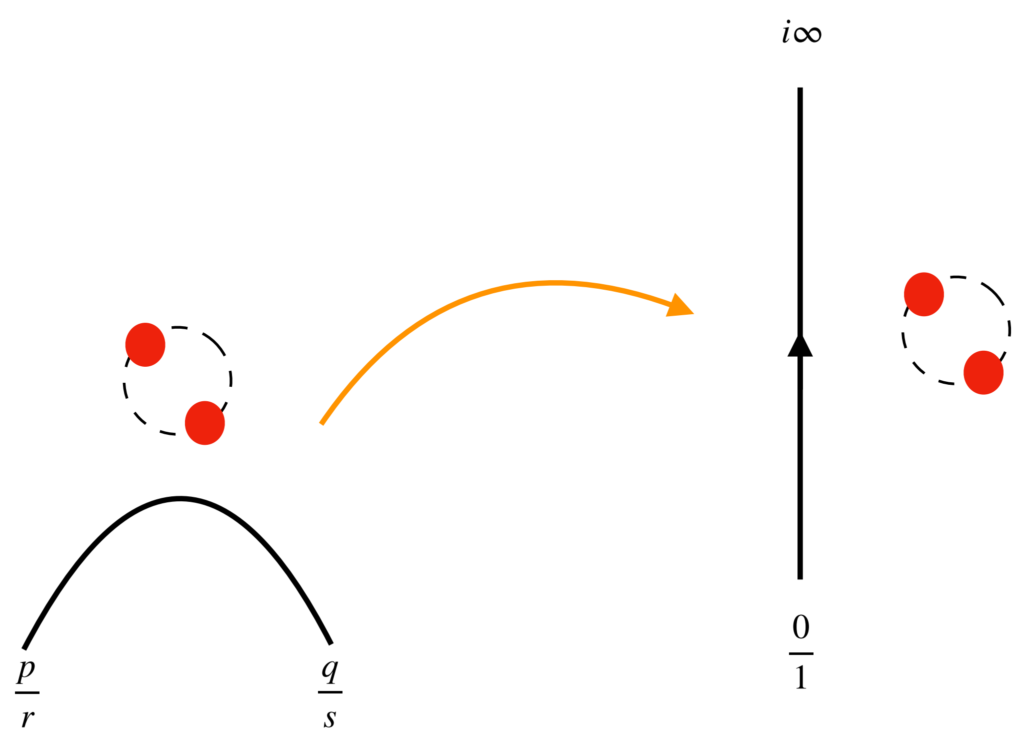
Using $SL(2,\mathbb{Z})$ invariance, the generic contribution at $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$ will be given in terms of the transformed charge bilinears $(m_\gamma, n_\gamma, \ell_\gamma)$

$$\Delta_\gamma\, d(m,n,\ell) = (-1)^{\ell_\gamma + 1}\, |\ell_\gamma|\, d(m_\gamma)\, d(n_\gamma)\ .$$

$$m_\gamma = r^2 n + p^2 m - pr\,\ell$$

$$n_\gamma = s^2 n + q^2 m - qs\,\ell\,,$$

$$\ell_\gamma = -\,2rs\,n\ -2pq\,m\ + (ps + qr)\,\ell$$



[Sen, ’07]

[Cheng, Verlinde ’07]

[Sen, ’11]

Dyon counting problem

Compute $d(m, n, \ell)$ with $\Delta = 4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$ in $\mathscr{R}$-chamber

$\Delta < 0 \Longrightarrow$ Two centred-states only

[Sen, '11]

[Chowdhury, Kidambi, Murthy, Reys, Wrase '19]

The solution must have the form

$$d(m, n, \ell) = \sum_{i=1}^{k} \Delta_i = (-1)^{\ell+1} \sum_{\substack{i=1 \\ \gamma_i \in W(m,n,\ell)}}^{k} |\ell_{\gamma_i}| \, d(m_{\gamma_i}) \, d(n_{\gamma_i})$$

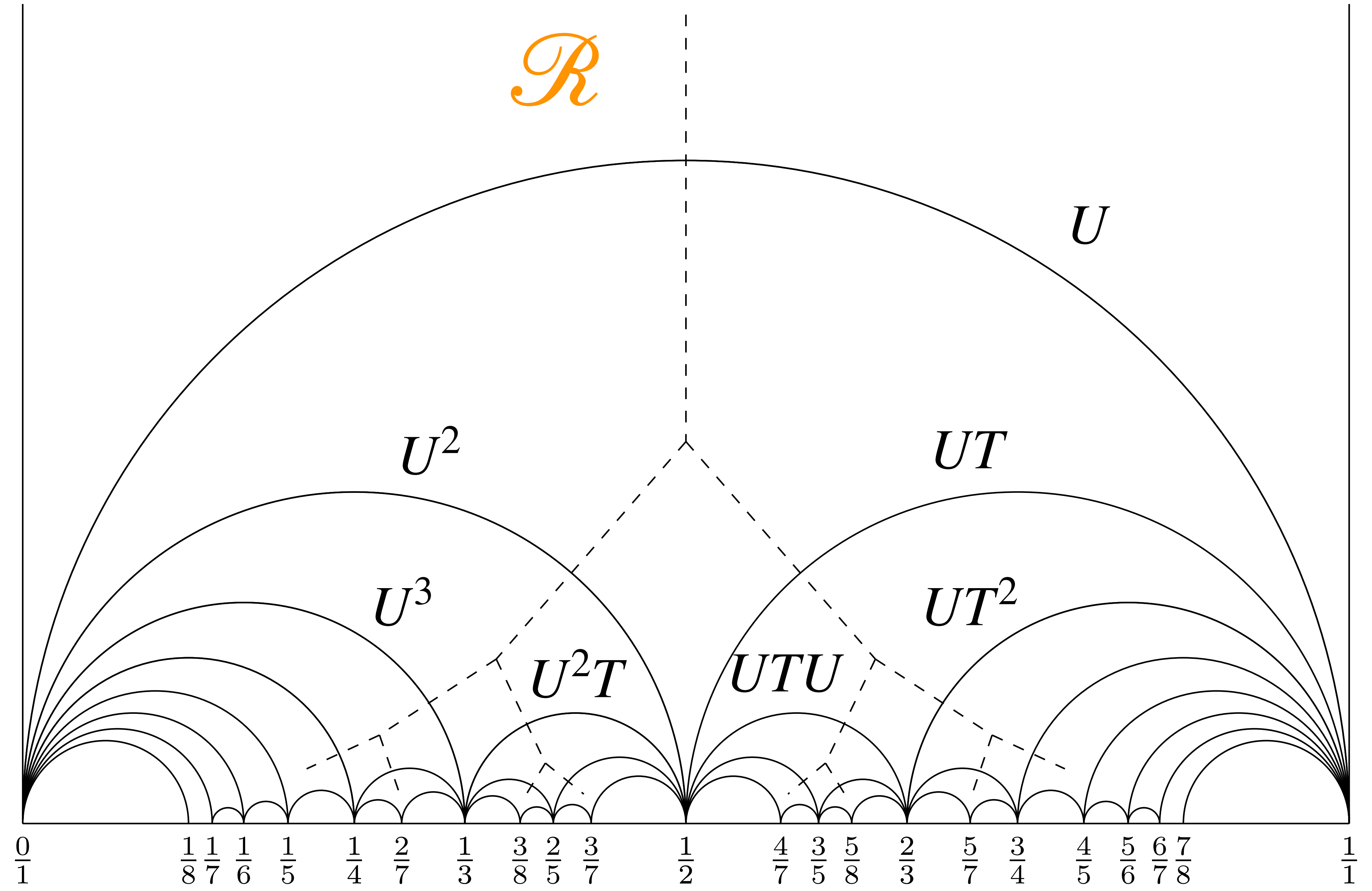
Q: How can we characterize $W(m, n, \ell)$?

Solution

Downward:

left-right choice associated to

$$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



$$W(m,n,\ell) = \{U, U^2, \ldots, U^{s_1}, U^{s_1}T, \ldots, U^{s_1}T^{s_2}, U^{s_1}T^{s_2}U, \ldots, U^{s_1}T^{s_2}U^{s_3}, \ldots, \gamma_*\}$$

γ_* determines all s_i: Only need to determine γ_*

Look for $\gamma_* = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ such that $m_{\gamma_*} < 0 \implies$ $\frac{\ell}{2m} - \frac{\sqrt{-\Delta}}{2m} < \frac{p}{r} < \frac{\ell}{2m} + \frac{\sqrt{-\Delta}}{2m}$

Solved by $0 \le \frac{\ell}{2m} - \frac{q}{s} \le \frac{1}{rs}$

$$\begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} \ell/g \\ 2m/g \end{pmatrix}, \quad \gamma_* = \begin{pmatrix} \ell/g & q \\ 2m/g & s \end{pmatrix} \text{ with } g = \gcd(\ell, 2m)$$

The continued fraction of $\frac{\ell}{2m} = [a_0; a_1, \ldots, a_r] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_r}}}}$ yields

$$\gamma_* = \begin{pmatrix} \ell/g & q \\ 2m/g & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_r & 1 \end{pmatrix}$$

Result

Given m, n, ℓ with $\Delta = 4mn - \ell^2 < 0$ and $0 \leq \ell \leq m$,

$$\ell/2m = [a_0, a_1, \ldots, a_r] \quad \text{defines} \quad W(m, n, \ell)$$

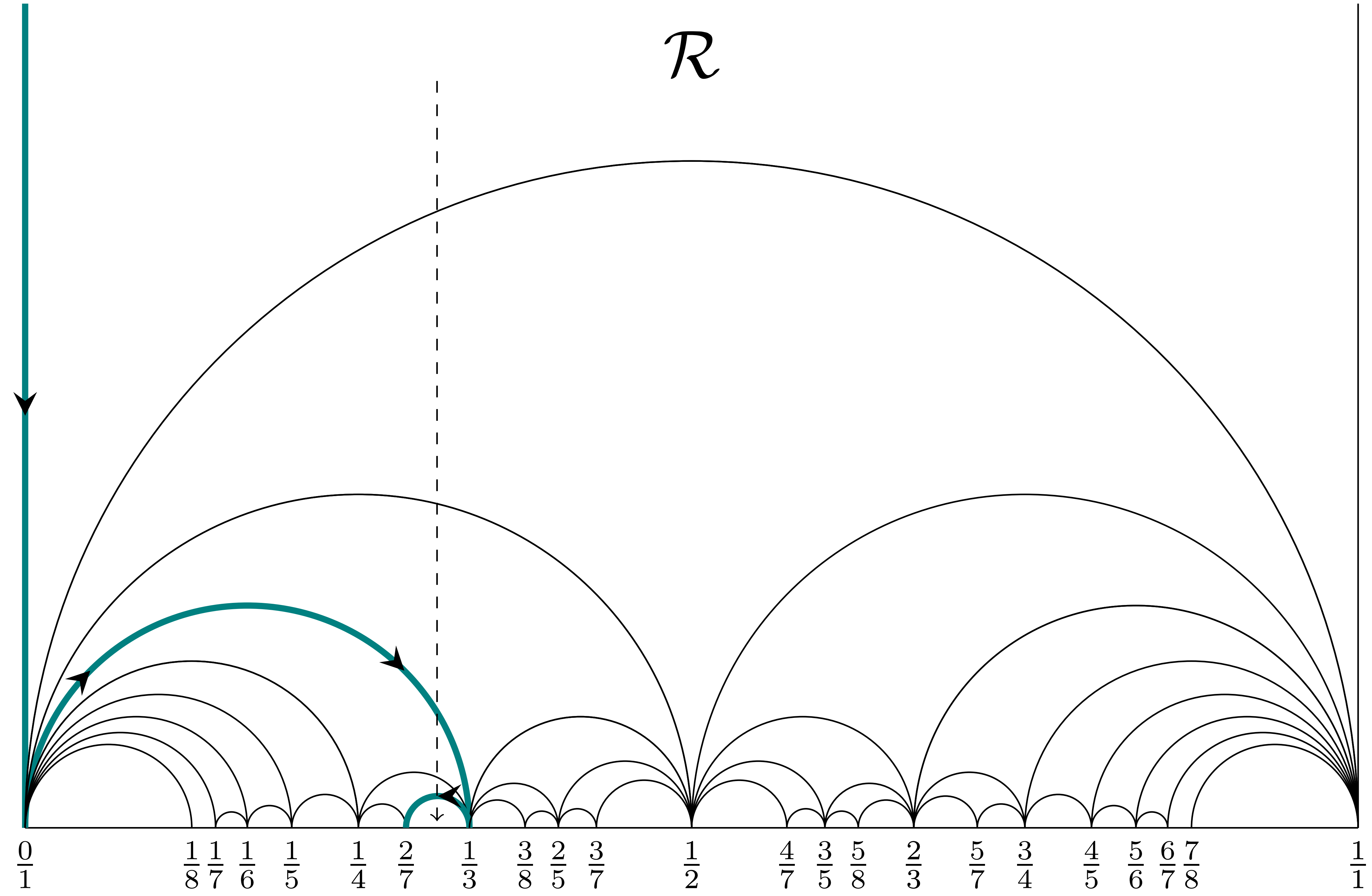
in the $\mathscr{R}$-chamber,

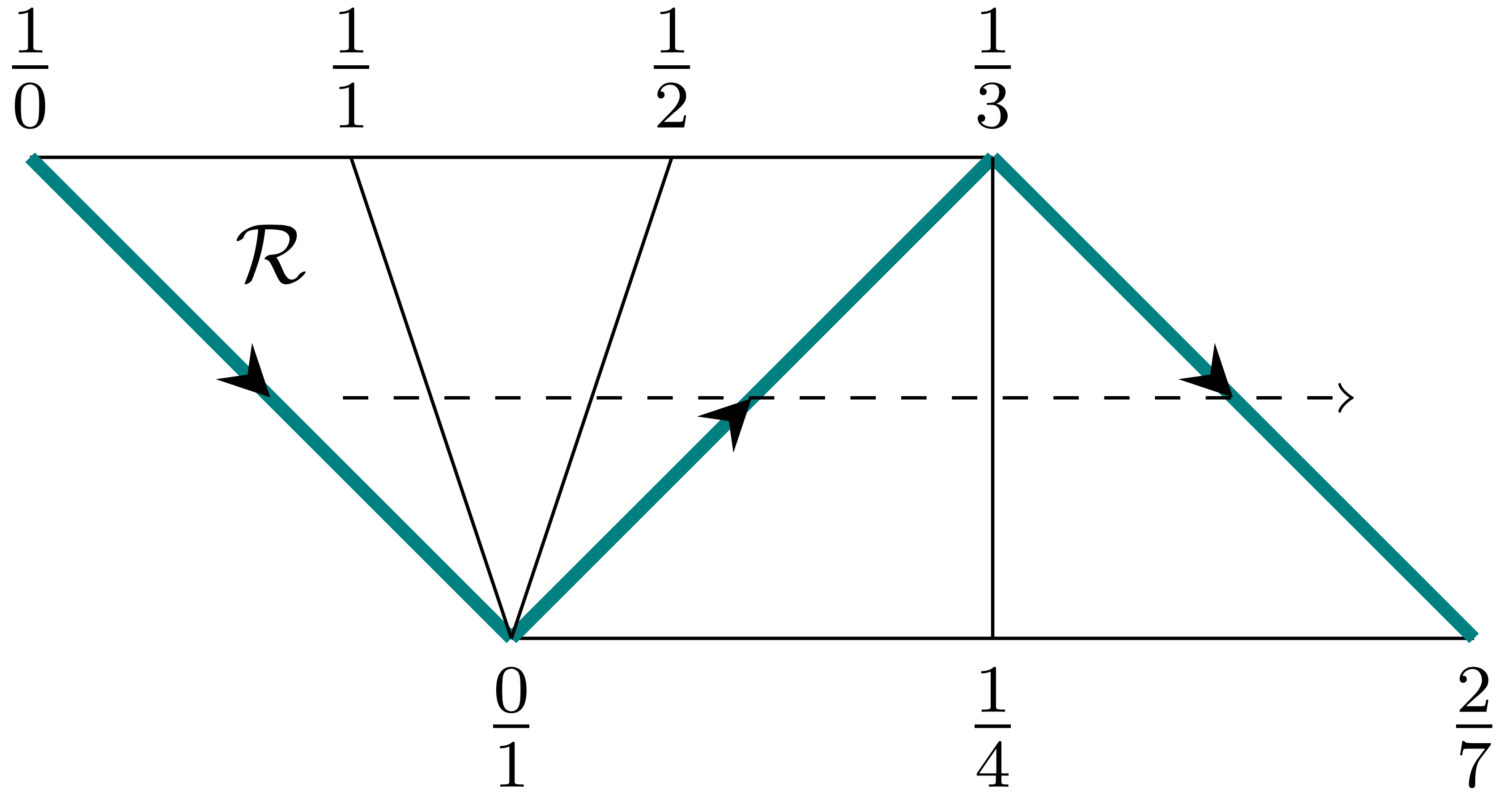
$$d(m, n, \ell) = d_* + (-1)^{\ell+1} \sum_{\substack{i=1 \\ \gamma_i \in W(m,n,\ell)}}^{k} |\ell_{\gamma_i}|\, d(m_{\gamma_i})\, d(n_{\gamma_i})$$

Diagrammatic representation

Take $\ell/2m = 2/7 = [0; 3,2] = \cfrac{1}{3+\cfrac{1}{2}}$

$$\gamma_* = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$





$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

Jacobi forms

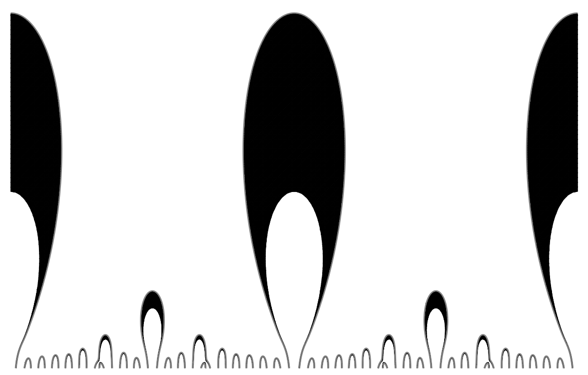
Φ_{10}^{-1} has a Fourier-Jacobi expansion

$$\frac{1}{\Phi_{10}(\rho,\sigma,v)} = \sum_{m\geq -1} \psi_m(\sigma,v)e^{2\pi i m\rho}$$

where $\psi_m(\sigma,v)$ are Jacobi forms of weight -10 and index m

$$\psi_m\left(\frac{a\sigma+b}{c\sigma+d},\frac{v}{c\sigma+d}\right) = (c\sigma+d)^{-10}\, e^{\frac{2\pi i m c v^2}{c\sigma+d}}\,\psi_m(\sigma,v)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})$$

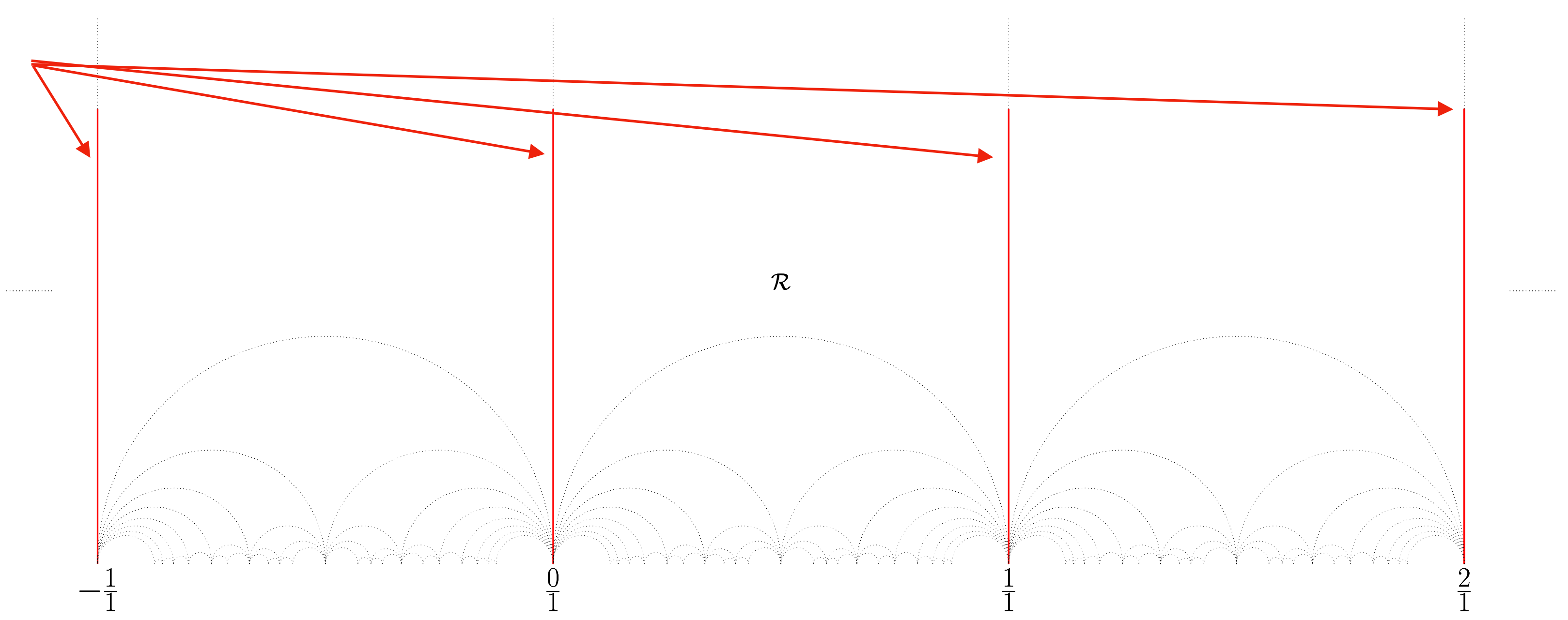
$$\psi_m(\sigma, v+\lambda\sigma+\mu) = e^{-2\pi i m(\lambda^2\sigma+2\lambda v)}\,\psi_m(\sigma,v)\,,\ \lambda,\mu\in\mathbb{Z}$$



Mock Jacobi forms

[Dabholkar, Murthy, Zagier '12]

$$\psi_m(\sigma, v) = \psi_m^F(\sigma, v) + \psi_m^P(\sigma, v)$$



split into **mock** Jacobi forms: a finite and a polar part.

[Ramanujan '1920]
[Zwegers '2001]

$\psi_m^F(\sigma, v) = \sum_{n,\ell} c_m^F(n,\ell) q^n y^\ell$ has no poles in (σ, v) **Immortal**

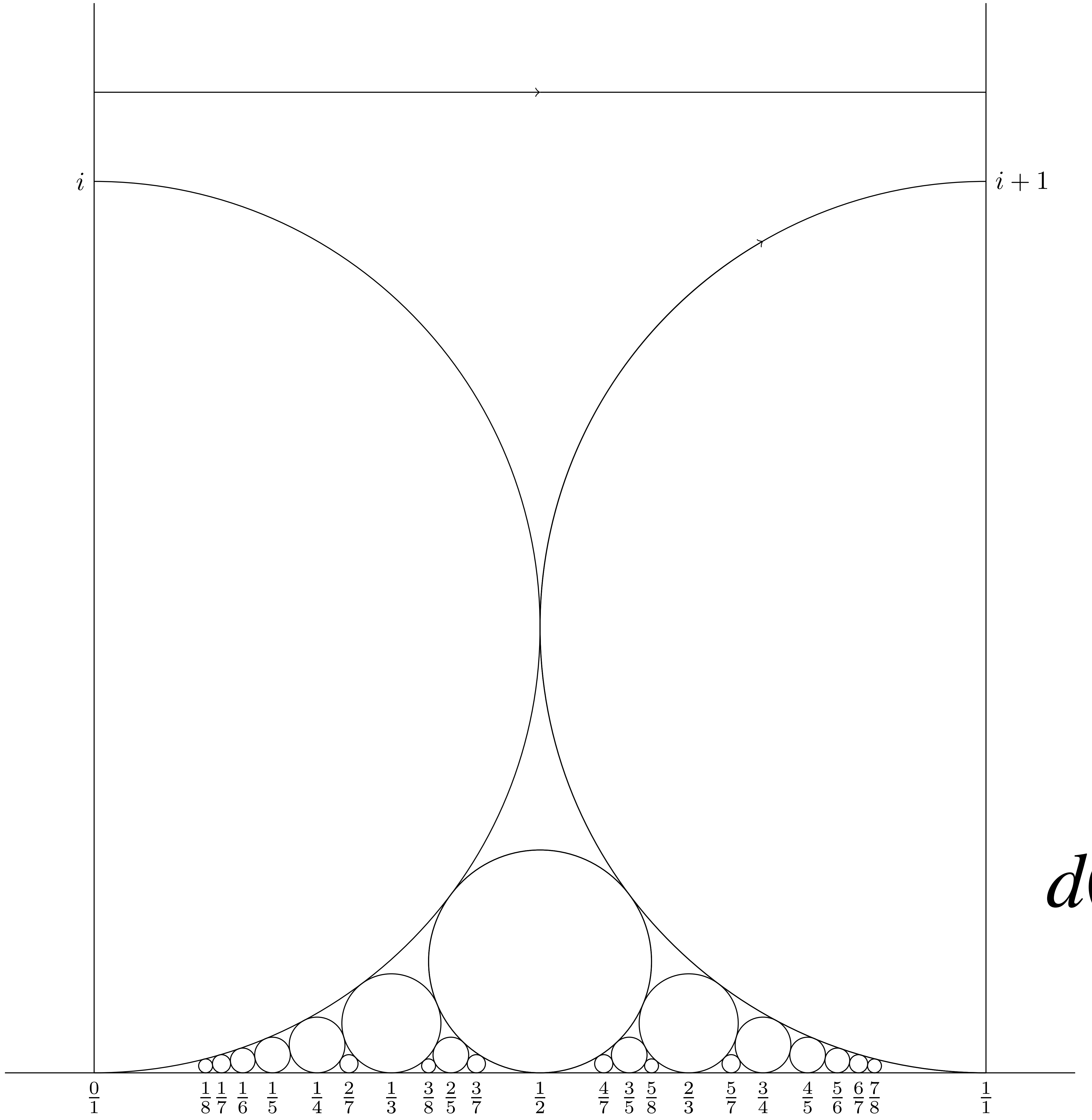
Modularity can be restored at the expense of holomorphicity.

Ex: $\psi_0(\sigma, v) \sim \frac{E_2(\sigma)}{\eta^{24}(\sigma)} + \frac{1}{\eta^{24}(\sigma)} \sum_{s \in \mathbb{Z}} \frac{q^s y}{(1 - q^s y)^2}$

In $\mathscr{R}$, for $0 \leq \ell < 2m$,

$$d(m, n, \ell) = (-1)^{\ell+1} c_m^F(n, \ell)$$

Rademacher expansion



$$\frac{1}{\eta^{24}(\rho)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n \rho}$$

$$d(n) = \int_{z}^{z+1} d\rho\, e^{2\pi i n \rho} \frac{1}{\eta^{24}(\rho)}$$

$$d(n) = \frac{2\pi}{n^{\frac{13}{2}}} \sum_{c>0} \frac{K(-1,n,c)}{c} I_{13}\left(\frac{4\pi\sqrt{n}}{c}\right)$$

Generalized Rademacher expansion

$$
\boxed{c^{\mathrm{F}}_m(n,\ell)} = 2\pi \sum_{k=1}^{\infty} \sum_{\substack{\widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ 4m\widetilde{n}-\widetilde{\ell}^2<0}} \boxed{c^{\mathrm{F}}_m(\widetilde{n},\widetilde{\ell})} \frac{Kl\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right)
$$

$4mn-\ell^2>0$

$$
+ \sqrt{2m} \sum_{k=1}^{\infty} \frac{Kl\big(\frac{\Delta}{4m},-1;k,\psi\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \qquad \text{(A.12)}
$$

$$
- \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g\in\mathbb{Z}/2mk\mathbb{Z} \\ g\equiv j (\mathrm{mod}\ 2m)}} \frac{Kl\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m};k,\psi\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times
$$

$$
\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,\mathrm{d}u\,,
$$

[Ferrari, Reys, '17]

computes the coefficients $c^F_m(n,\ell)$ with $\Delta>0$ in terms of $c^F_m(n',\ell')$ with $\Delta<0$.

Macroscopic entropy

Sen's Quantum Entropy Function

[Sen '08]

$$d_{Micro}(\vec{q}) = \left\langle \exp\left[-iq_i \oint d\theta\, A_\theta^{(i)}\right] \right\rangle_{AdS_2}^{finite} = d_{Macro}(\vec{q})$$

[Dabholkar, Gomes, Murthy '10 '11 '14]

[Gupta, Murthy '12]

[Gupta, Ito, Jeon '15]

[Murthy, Reys '13 '14]

[de Wit, Murthy, Reys '18]

[Jeon, Murthy '18]

Supersymmetric localization in supergravity to compute QEF

For $1/4-$BPS dyons [Murthy, Reys '15] [Gomes '15]

$$d(m,n,\ell) \sim \sum_{\substack{0 \le \tilde{\ell} \le m \\ \tilde{\Delta} < 0}} (\tilde{\ell} - 2n)\, d(m + \tilde{n} - \tilde{\ell})\, d(\tilde{n}) \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{m}\sqrt{|\tilde{\Delta}|\Delta}\right)$$

$c_m^F(\tilde{n}, \tilde{\ell})$ small mismatch

Extra: $\Delta = 0$

New relevant discrete invariant: $\gcd(m, n, \ell)$

For (m, n, ℓ) with $4mn - \ell^2 = 0$, use $\ell/2m = [0; a_1, \ldots, a_r]$ obtain

$$m_{\gamma_*} = 0, \qquad \ell_{\gamma_*} = 0, \qquad n_{\gamma_*} = \gcd(m, n, \ell) \equiv \tilde{g} \qquad d(0,\tilde{g},0)\,?$$

$$\psi_0^F(\sigma) = 2\,\frac{E_2(\sigma)}{\eta^{24}(\sigma)} = -2 \sum_{n \geq -1} n\, d(n)\, q^n$$

Therefore

$$d(m, n, \ell) = 2\,\tilde{g}\, d(\tilde{g}) - \sum_{\gamma \in W(m,n,\ell)} |\ell_\gamma|\, d(m_\gamma)\, d(n_\gamma)$$

Note For $\Delta = 0$ the immortal degeneracy is only a function of $\tilde{g}$: $d_{immortal}(m, n, \ell)_{\Delta=0} = 2\tilde{g} d(\tilde{g})$

Extra: CHL models $N > 1$

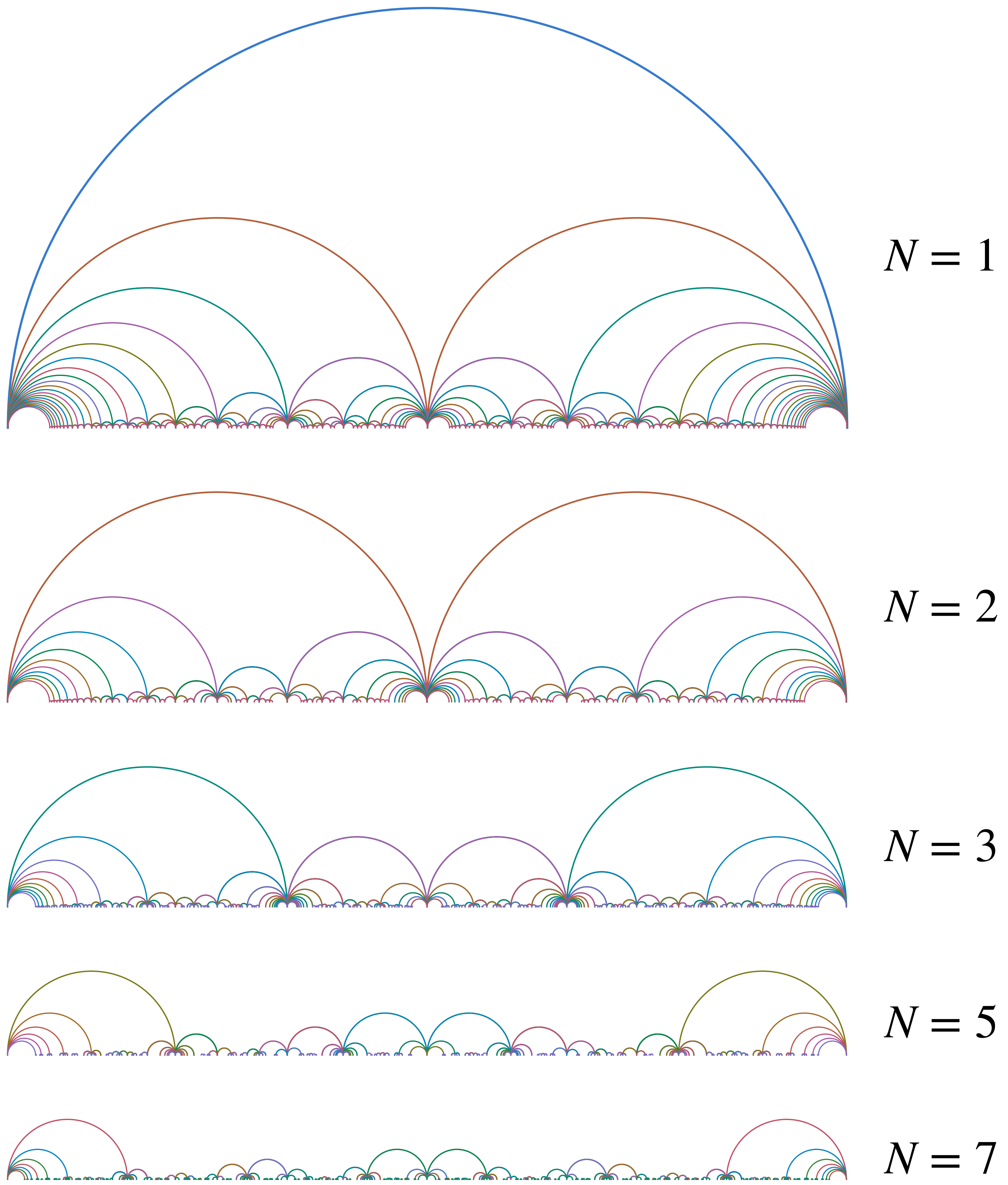
Heterotic string theory on $T^5 \times S^1/\mathbb{Z}_N$ with $N = 2,3,5,7$

Generating functions $\Phi_k(\rho, \sigma, v)^{-1}$. The poles

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma_0(N)$$

The logic is the same, but the details more intricate.

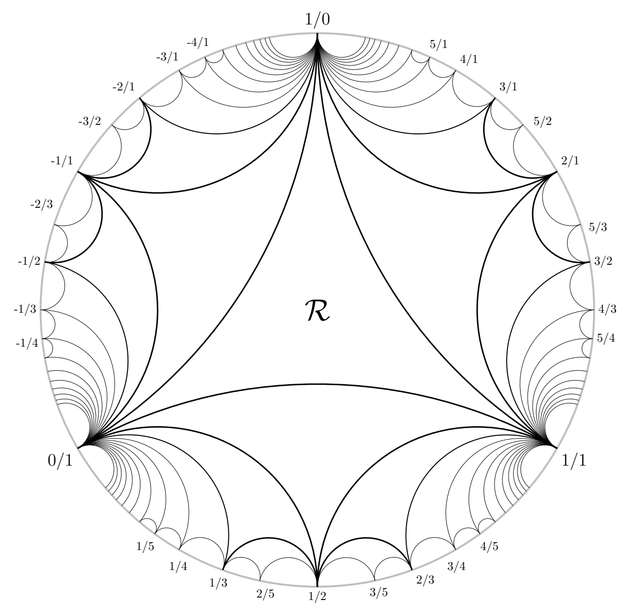
Proceed as earlier, build set $W(m, n, \ell)$ from the continued fraction of $\ell/2m$ but now select the matrices in $\Gamma_0(N)$.



Summary

We use continued fractions to set up an arithmetic of decay walls which we used to explicitly compute all the polar coefficients of

$$\frac{1}{\Phi_k(\rho, \sigma, v)}$$



The appearance of continued fractions is naturally explained by the theory of Binary Quadratic Forms $(m, n, \ell) \leftrightarrow mx^2 - \ell xy + ny^2$.

Consistent with [Moore '98]

[Benjamin, Kachru, Ono, Rolen '18], [Banerjee, Bhand, Dutta, Sen, Singh '20], [Borsten, Duff, Marrani '20] ...

Thank you