Generalized Series Expansions in Asymptotically Free Large-N Lattice Field Theories

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Summary:

- Sign problem @ Finite Baryon Density and Diagrammatic Monte-Carlo for lattice QCD
- Strong-coupling vs weak-coupling QCD
- Lattice perturbation theory with Cayley map and IR problems
- Trans-series from lattice perturbation theory
- The fate of Renormalons?
- Monte-Carlo sampling @ weak coupling

Tests of continuity for twisted compactified principal chiral model [work with S. Valgushev]

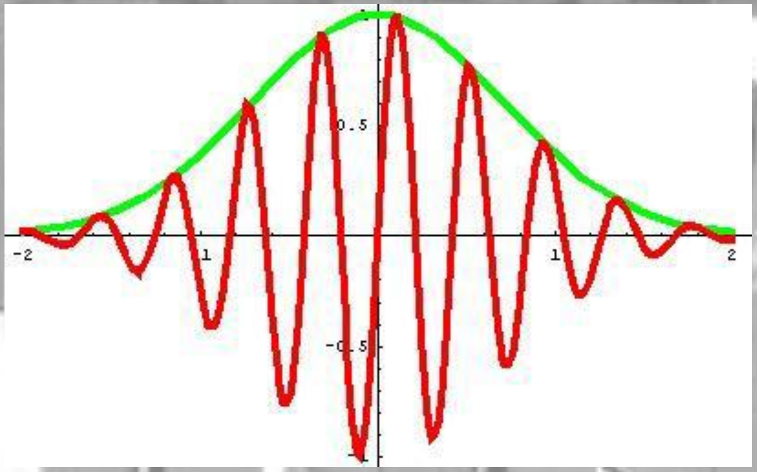
Sign problem in QCD

$$\int dA_\mu \det\left(\mathcal{D}\left[A_\mu\right]\right)^{N_f} e^{-S[A_\mu]},$$

$$\mathcal{D}^\dagger\left[A_\mu\right] \neq -\mathcal{D}\left[A_\mu\right] \Rightarrow \arg\det\left(\mathcal{D}\left[A_\mu\right]\right) \neq 0$$

Lattice QCD
@ finite baryon density:

- **Complex path integral**
- **No positive weight for**
- **Monte-Carlo**




Reweighting
DiagMC
Complex Langevin
Worm algorithm
Lefshetz thimbles
Density of states
Canonical formalism
Subsets

VS

Diagrammatic Monte-Carlo for QCD

So far lattice strong-coupling expansion:

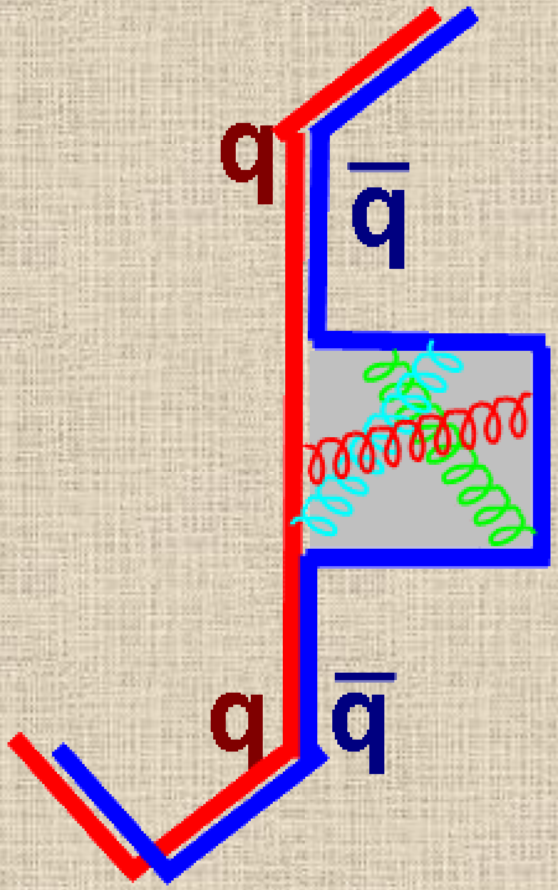
(leading order or few lowest orders)

[de Forcrand,Philipsen,Unger, Gattringer,…]

- **Worldlines of quarks/mesons/baryons**
- **"Worldsheets" of confining strings**

Very good approximation!

Physical degrees of freedom!



Lattice strong-coupling expansion

- **Confinement**
- **Dynamical mass gap generation**

<u>BUT</u>

Continuum physics is at weak-coupling!

… Rigorously relating strong- and weak-coupling might bring you $1.000.000 …

Diagrammatic Monte-Carlo at weak coupling?

… and make it first-principle and automatic

Lattice perturbation theory for QCD:

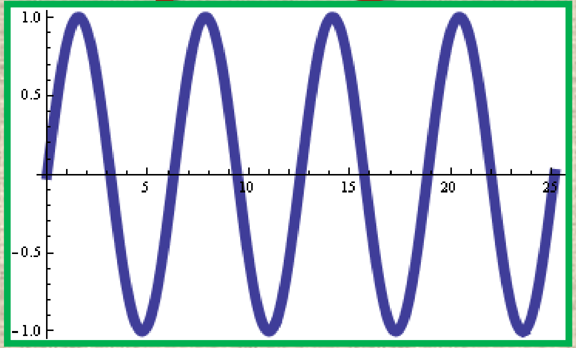
- Small fluctuations of SU(N) fields around vacuum (1)
- Map SU(N) to Hermitian matrices

  Popular choice $g = e^{i\alpha\phi}$
- Maps only a part of all matrices
- Infinitely many vacua
- Exp.small terms due to cut-off

Popular choice

$$g = e^{i\alpha\phi}$$

- **Maps a subset of all Hermitian matrices to the whole U(N)**
- **Infinitely many degenerate vacua**



- **Double-trace terms in the Jacobian**

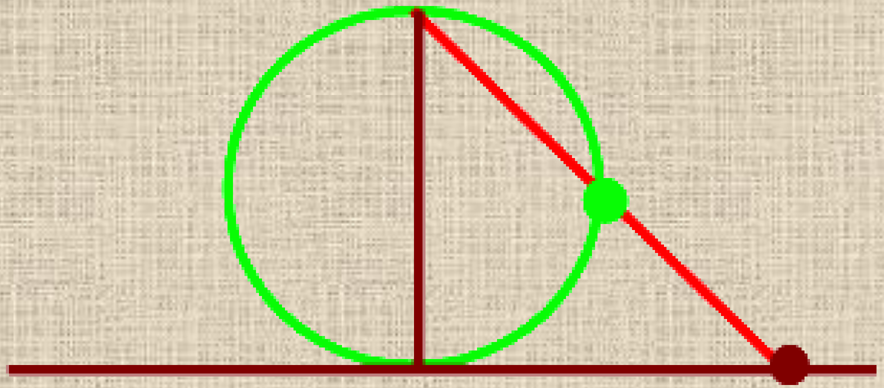
$$\int_{U(N)} dg = \int_{\mathbb{M}} d\phi \exp\left(\sum_{i,j} \log\left(\frac{\sin^2((\lambda_i-\lambda_j)/2)}{(\lambda_i-\lambda_j)^2}\right)\right) =$$

$$= \int_{\mathbb{M}} d\phi \exp\left(-\frac{1}{6}\left(N \mathrm{Tr}\,\phi^2 - \mathrm{Tr}\,\phi\,\mathrm{Tr}\,\phi\right) + O\left(\phi^4\right)\right)$$

Cayley map/Stereographic projection

(Conformal mapping from circle to line)

$$g = \frac{1 + i\alpha\phi}{1 - i\alpha\phi}$$



- **Maps the whole space of Hermitian matrices to the whole U(N)**
- **Perturbative vacuum unique**
- **Jacobian is very simple**

$$\int_{SU(N)} dg \Rightarrow \int_{\mathbb{H}_{N\times N}} d\phi \det\left(1 + \alpha^2\phi^2\right)^{-N} =$$

$$= \int_{\mathbb{H}_{N\times N}} d\phi \exp\left(-N\alpha^2 \mathrm{Tr}\,\phi^2 + O\left(\alpha^4\phi^4\right)\right)$$

Bare mass from the Jacobian?

(Take principal chiral model as example...)

$$\mathcal{Z} = \int_{U(N)} dg_x \exp\left(-\frac{N}{\lambda} \sum_{<x,y>} \mathrm{Tr}\,\left(g_x^\dagger g_y\right)\right)$$

$$g_x = \frac{1+i\alpha\phi_x}{1-i\alpha\phi_x} \qquad \alpha^2 = \frac{\lambda}{8}$$

$$S_0\left[g_x\right] = \lambda^{-1} \sum_{x,y} D_{xy} \mathrm{Tr}\,\left(g_x^\dagger g_y\right) =$$

$$= 4\lambda^{-1} \sum_{\substack{k,l=1 \\ k+l=2n}}^{+\infty} (-1)^{\frac{k-l}{2}} \alpha^{k+l} \sum_{x,y} D_{xy} \mathrm{Tr}\,\left(\phi_x^k \phi_y^l\right)$$

$$\int_{U(N)} dg_x = \int_{\mathbb{H}^{N\times N}} d\phi_x \exp\left(-N \mathrm{Tr}\,\ln\left(1+\alpha^2\phi_x^2\right)\right)$$

Bare mass from the Jacobian? (principal chiral model as example…)

$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{4}\delta_{xy}\right)\mathrm{Tr}\,(\phi_x\phi_y) + \\ + \sum_{n=2}^{+\infty}\left(-\frac{\lambda}{8}\right)^{n-1}\left(\frac{\lambda}{8n}\sum_x \mathrm{Tr}\,\phi_x^{2n} + \\ \frac{1}{2}\sum_{l=1}^{2n-1}(-1)^{l-1}\sum_{x,y} D_{xy}\mathrm{Tr}\,\left(\phi_x^{2n-l}\phi_y^{l}\right)\right)$$

- Massive planar field theory,
- Suitable for Diagrammatic Monte-Carlo
- Bare mass ~ bare coupling,
- Infinitely many interacting vertices

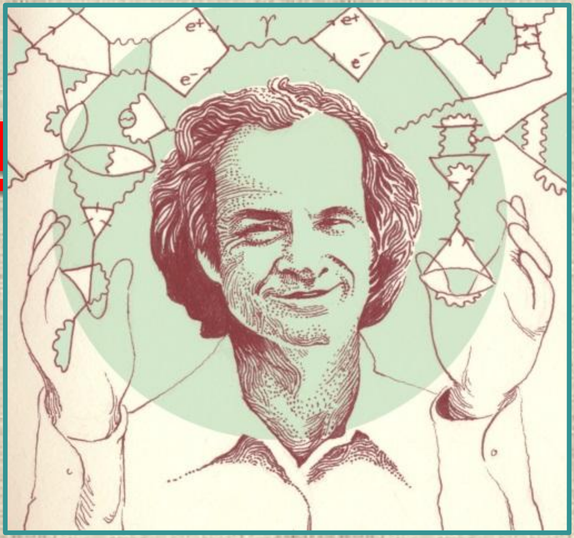
Bare mass from the Jacobian?

Small mass term λ/4 due to Jacobian!

"Conformal anomaly" stemming from integration measure [a-la Fujikawa 79]

Perturbation theory with coupling in vertices AND propagators!!!

Let's try, at least formally, to expand in vertices…



Minimal working example:
O(N) sigma model @ large N

Gauge theory/PCM too hard to start with (one needs DiagMC to sum over all planar diagrams)

Something simpler?

$$\int\limits_{S_N} d\vec{n}_x \exp\left(-\frac{1}{\alpha^2} \sum_{<x,y>} \vec{n}_x \cdot \vec{n}_y\right)$$

$$\langle \vec{n}_x \cdot \vec{n}_y \rangle \sim$$
$$\sim \exp\left(-m^2 |x-y|\right)$$

Exact answer

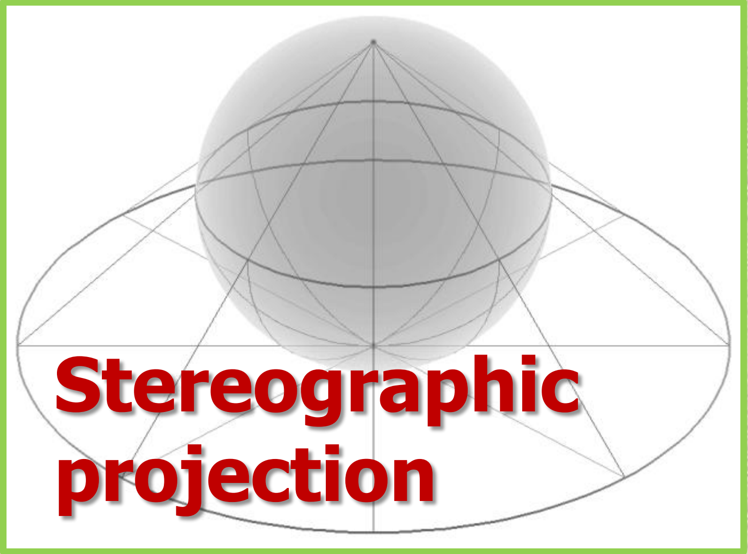
$$m^2 = 32 \exp\left(-\frac{4\pi}{\alpha^2}\right)$$

Non-perturbative mass gap in 2D

O(N) sigma model @ large N

Analogue of Cayley map is

$$S_N \to \mathbb{R}^{N-1}$$



Maps whole sphere to whole hyperplane

$$n_{0\,x} = \frac{1 - \frac{\lambda}{4}\phi_x^2}{1 + \frac{\lambda}{4}\phi_x^2},$$

$$n_{i\,x} = \frac{\sqrt{\lambda}\,\phi_{i\,x}}{1 + \frac{\lambda}{4}\phi_x^2},$$

$$\mathcal{D}n_x = \mathcal{D}\phi_x \left(1 + \frac{\lambda}{4}\phi_x^2\right)^{-N}$$

Again, bare mass term from the Jacobian… **[PB, 1510.06568]**

O(N) sigma model @ large N

Full action in new coordinates

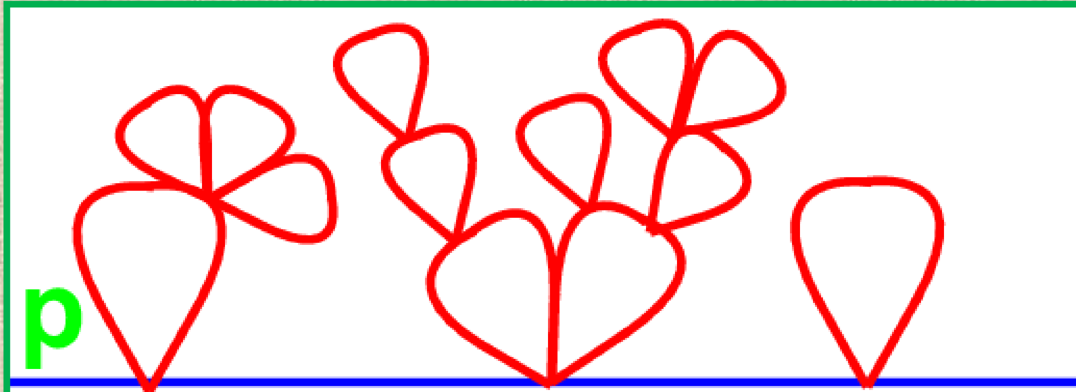
$$S\left[\phi_x\right] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{2}\delta_{xy}\right)\phi_x\cdot\phi_y +$$

$$+ \sum_{k=2}^{+\infty}\frac{(-1)^{k-1}\lambda^k}{4^k\,k}\sum_x\left(\phi_x^2\right)^k +$$

$$+ \sum_{\substack{k,l=0\\ k+l\neq 0}}^{+\infty}\frac{(-1)^{k+l}\lambda^{k+l}}{2\cdot 4^{k+l}}\sum_{x,y} D_{xy}\left(\phi_x^2\right)^k\left(\phi_y^2\right)^l\left(\phi_x\cdot\phi_y\right)$$

We blindly do perturbation theory …

Only cactus diagrams @ large N



O(N) sigma model @ large N

From our perturbative expansion we get

$$m^2 = \sum_{p,q=0}^{+\infty} c_{p,q}\, \lambda^p\, (\log\lambda)^q$$

(automatic analytic recursion)

We get trans-series, only without „saddle point" terms!!!

Can capture non-perturbative effects:

$$\exp\left(-\frac{1}{\lambda}\right) = \exp\left(-e^{-\log(\lambda)}\right) = \sum_k c_k\, (\log\lambda)^k$$

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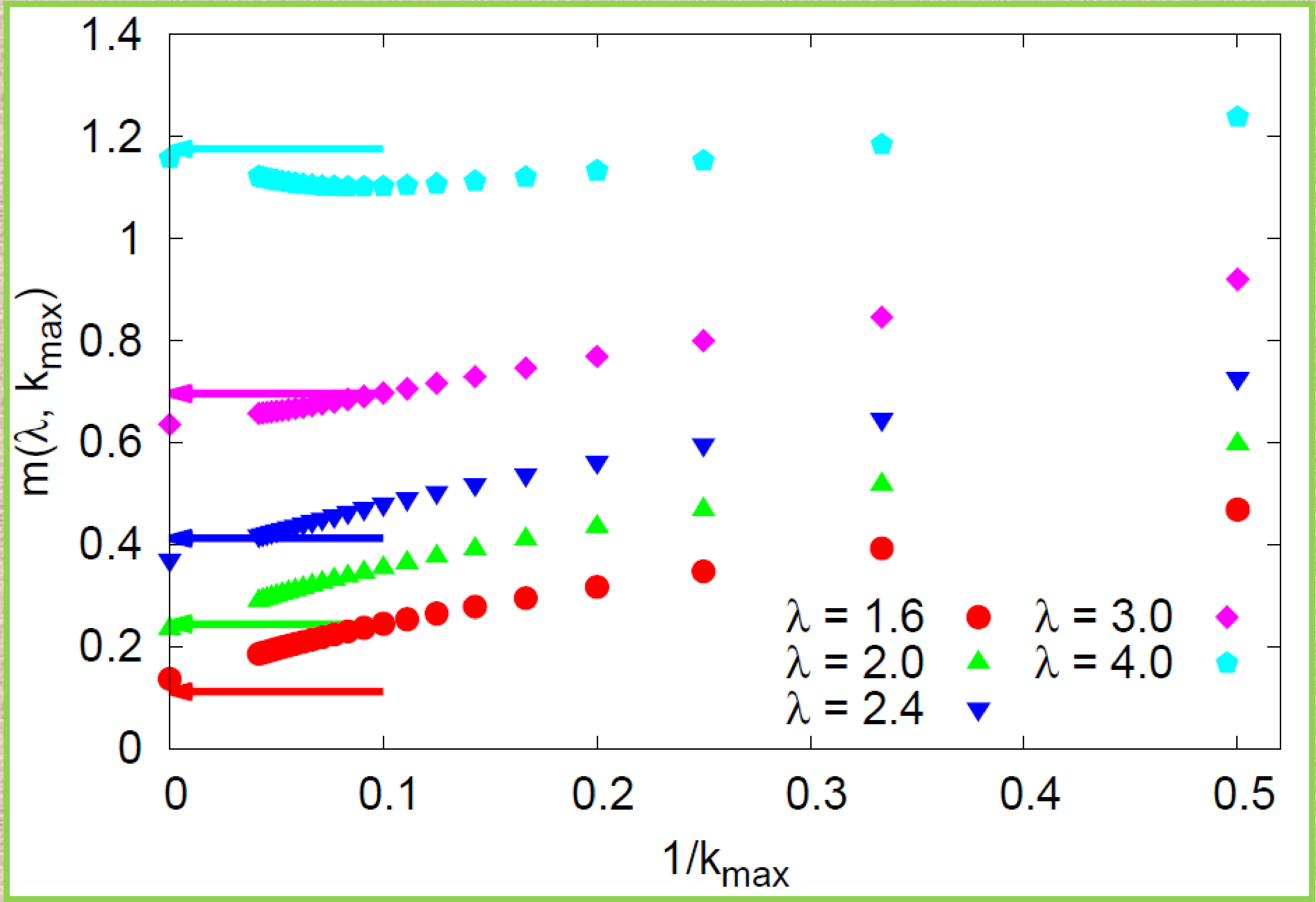
O(N) sigma model @ large N



Good convergence in practice
(But no proof of convergence!!!)

Principal Chiral Model

Next step towards gauge theory...

$$S\left[\phi_x\right] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{4}\delta_{xy}\right)\mathrm{Tr}\left(\phi_x\phi_y\right) +$$

$$+\sum_{n=2}^{+\infty}\left(-\frac{\lambda}{8}\right)^{n-1}\left(\frac{\lambda}{8n}\sum_x \mathrm{Tr}\,\phi_x^{2n} + \right.$$

$$\left.\frac{1}{2}\sum_{l=1}^{2n-1}(-1)^{l-1}\sum_{x,y} D_{xy}\mathrm{Tr}\left(\phi_x^{2n-l}\phi_y^{l}\right)\right)$$

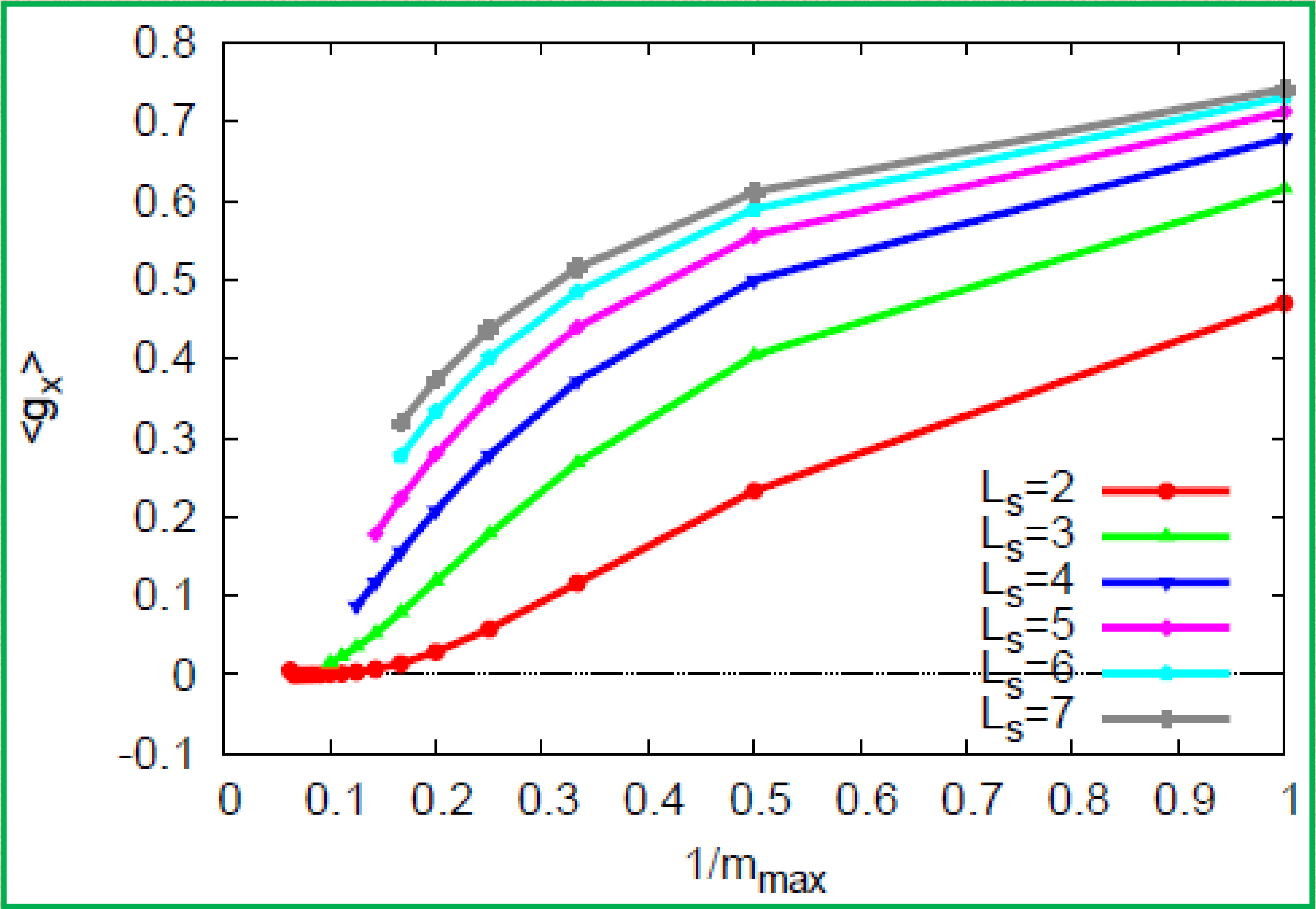
- **Massive planar field theory,**
- **Bare mass ~ bare coupling,**
- **Infinitely many interacting vertices**

Principal Chiral Model@large N

- Explicit recursive calculations for D=1 and L ~ O(10) [Lattice size] [Work with Ali Davody]
- Exact answers for $L = 2, 3, 4, \infty$ [Vicari, Rossi, hep-lat/9609003]
- All momentum sums are discrete and finite
- Rational approximations for finite L

Let`s check the restoration of SU(N) x SU(N) symmetry – we break it by choosing the vacuum $g_x = 1$

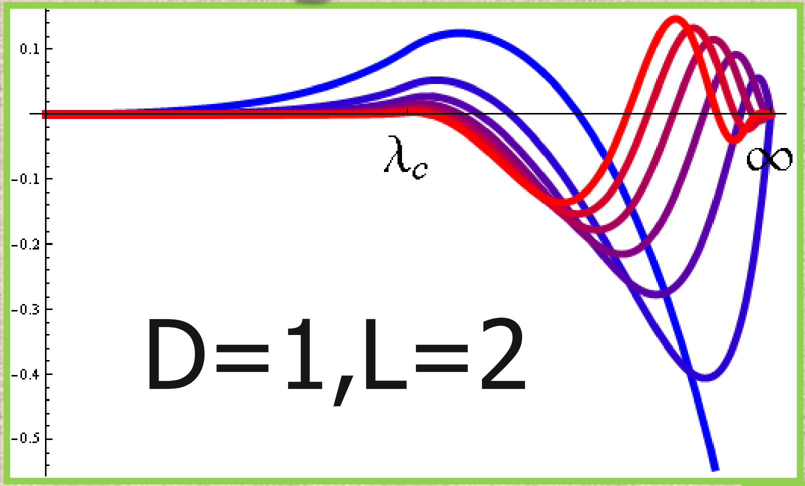
Principal Chiral Model (N=∞,D=1)

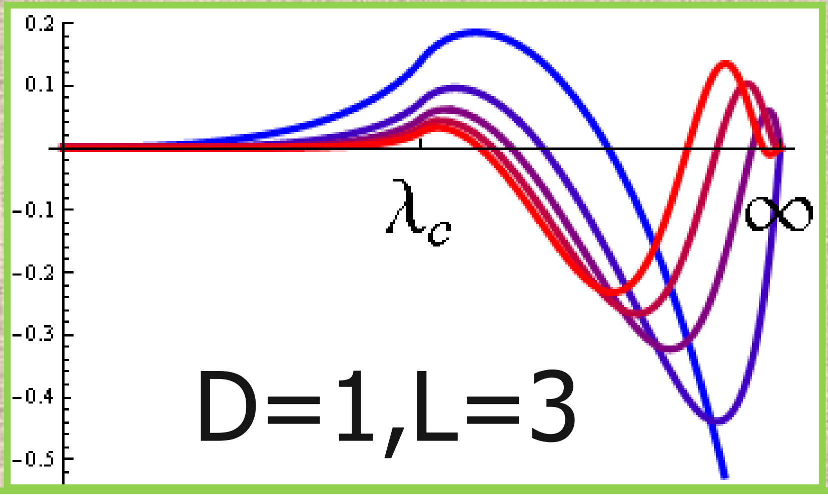


SU(N)xSU(N) restored at large orders

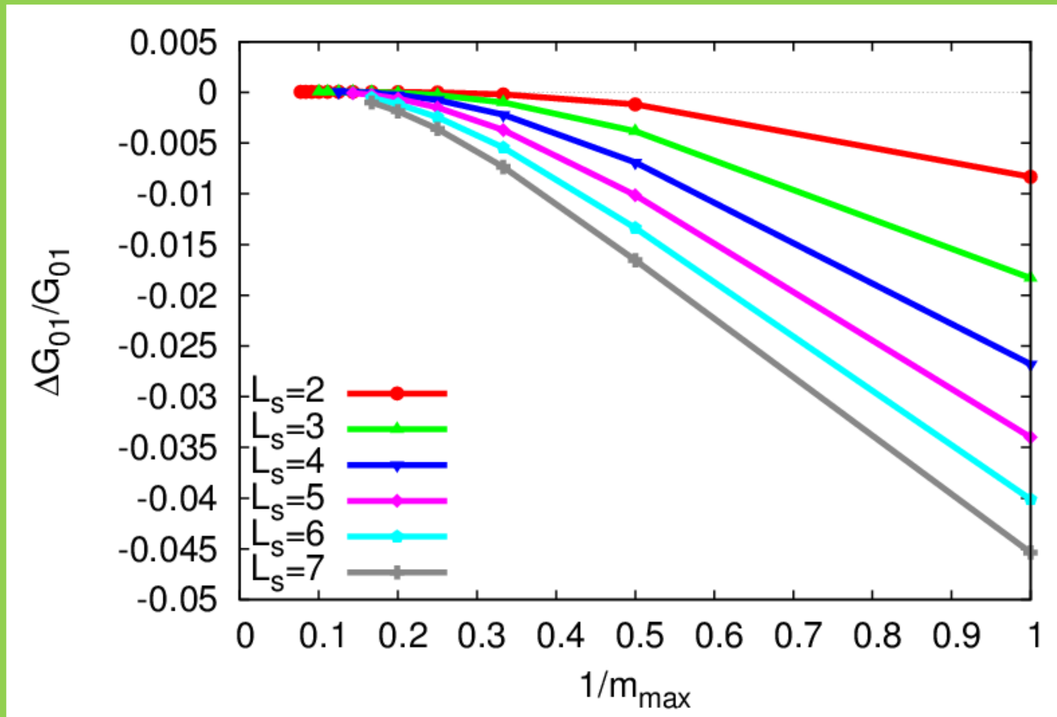
Principal Chiral Model ($N=\infty, D=1$)

Convergence of mean link…

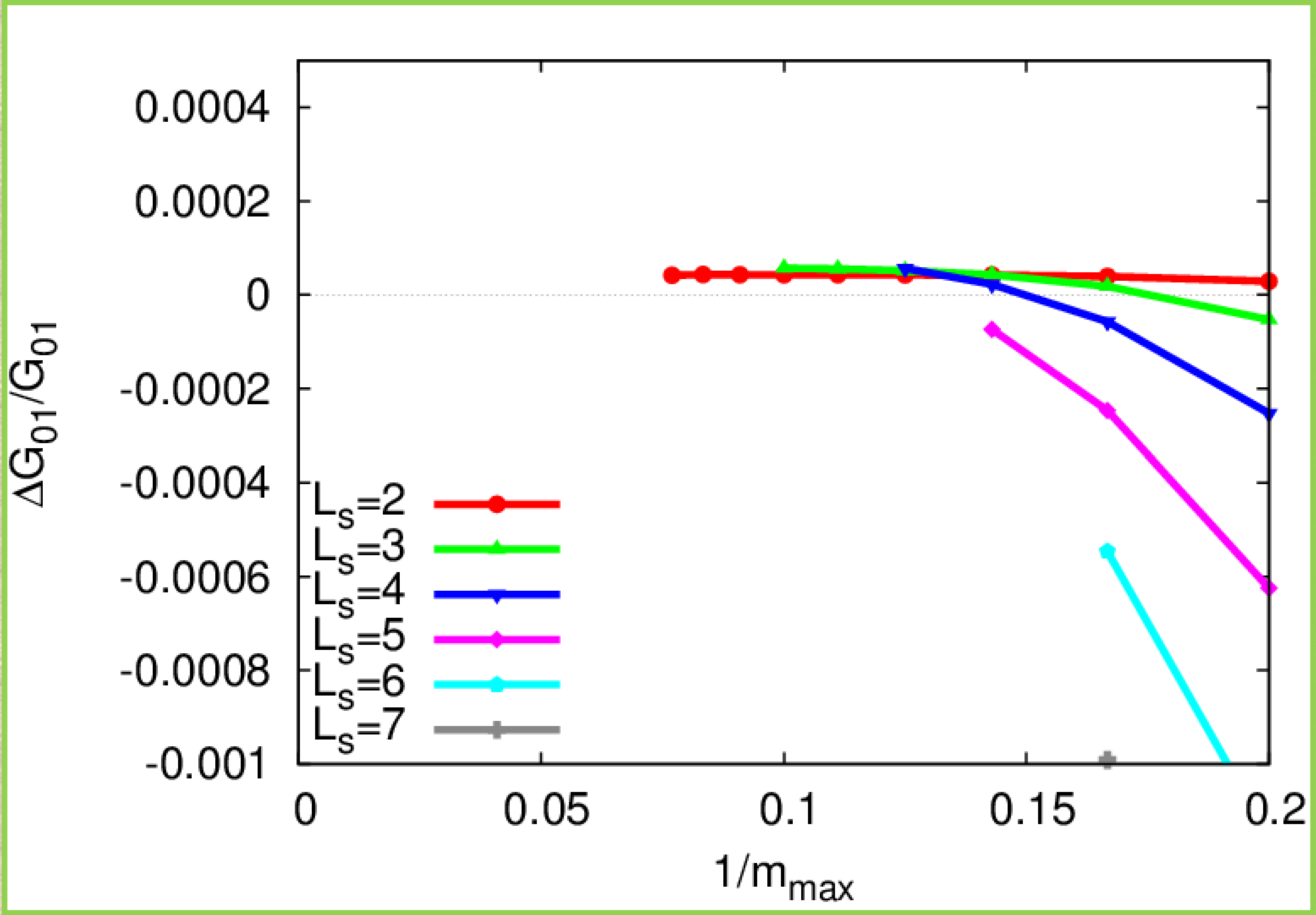




Quite good and fast convergence, BUT …



Principal Chiral Model (N=∞,D=1)



Small systematic error@high orders…

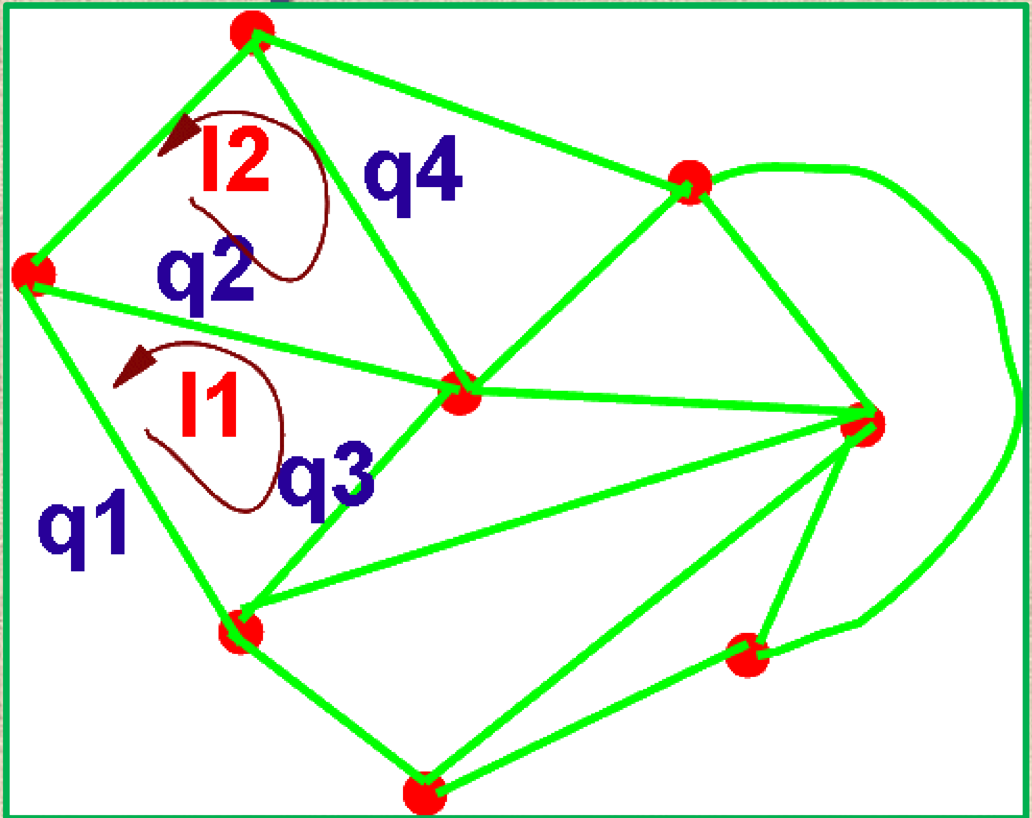
Principal Chiral Model (N=∞,D=2)

* Asymptotically free theory
* Now we need DiagMC, but before…
* Again series of the form

$$m^2 = \sum_{p,q=0}^{+\infty} c_{p,q}\, \lambda^p\, (\log \lambda)^q$$

* No negative powers of λ can appear
  For a planar graph with V vertices,
  L lines and E edges , V – E + F = 2

Principal Chiral Model (N=∞,D=2)



For planar diagrams

V – E + F = 2

⇒Only logs of coupling appear in 2D

$$\lambda^V \int d^2 l_1 \ldots d^2 l_F \, \frac{1}{q_1^2 + m^2} \cdots \frac{1}{q_E^2 + m^2} \sim$$

$$\sim \lambda^V m^{2(F-E)} \sim \lambda^V \lambda^{F-E} \sim \lambda^2$$

Trans-series VS renormalons

In large-N limit:

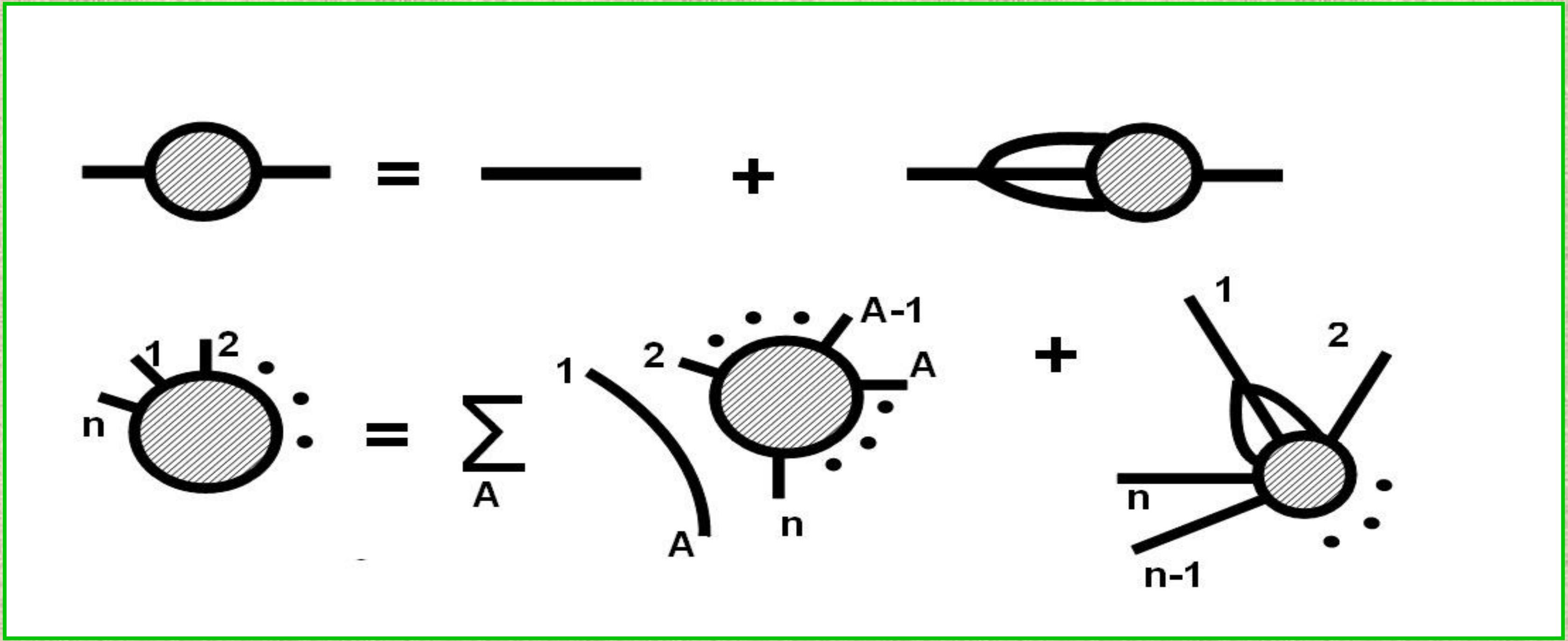
- **# of diagrams grows exponentially**
- **All contributions are finite**
- **Suitable for DiagMC**
- **No IR singularities**
- **No IR renormalons**
- **Simple poles and cuts at most**

BUT: trans-series with powers and logs

- **No „instanton terms"**
- **Jacobian removes classical solutions?**

Diagrammatic Monte-Carlo

Start with Schwinger-Dyson equations



(here for simplicity φ^4) (independent of mass)

Recursive structure for diagrams:

V vertices, L legs -> V-v vertices, L+l legs

Diagrammatic Monte-Carlo

Schwinger-Dyson equations are (infinitely many) linear equations of the form

$$\phi(X) = \sum_{Y} A(X|Y)\,\phi(Y) + b(X)$$

which involve all (disconnected) correlators

$$\phi(X) = \langle \phi(x_1) \ldots \phi(x_n) \rangle, \quad X = \{x_1, \ldots, x_n\}$$

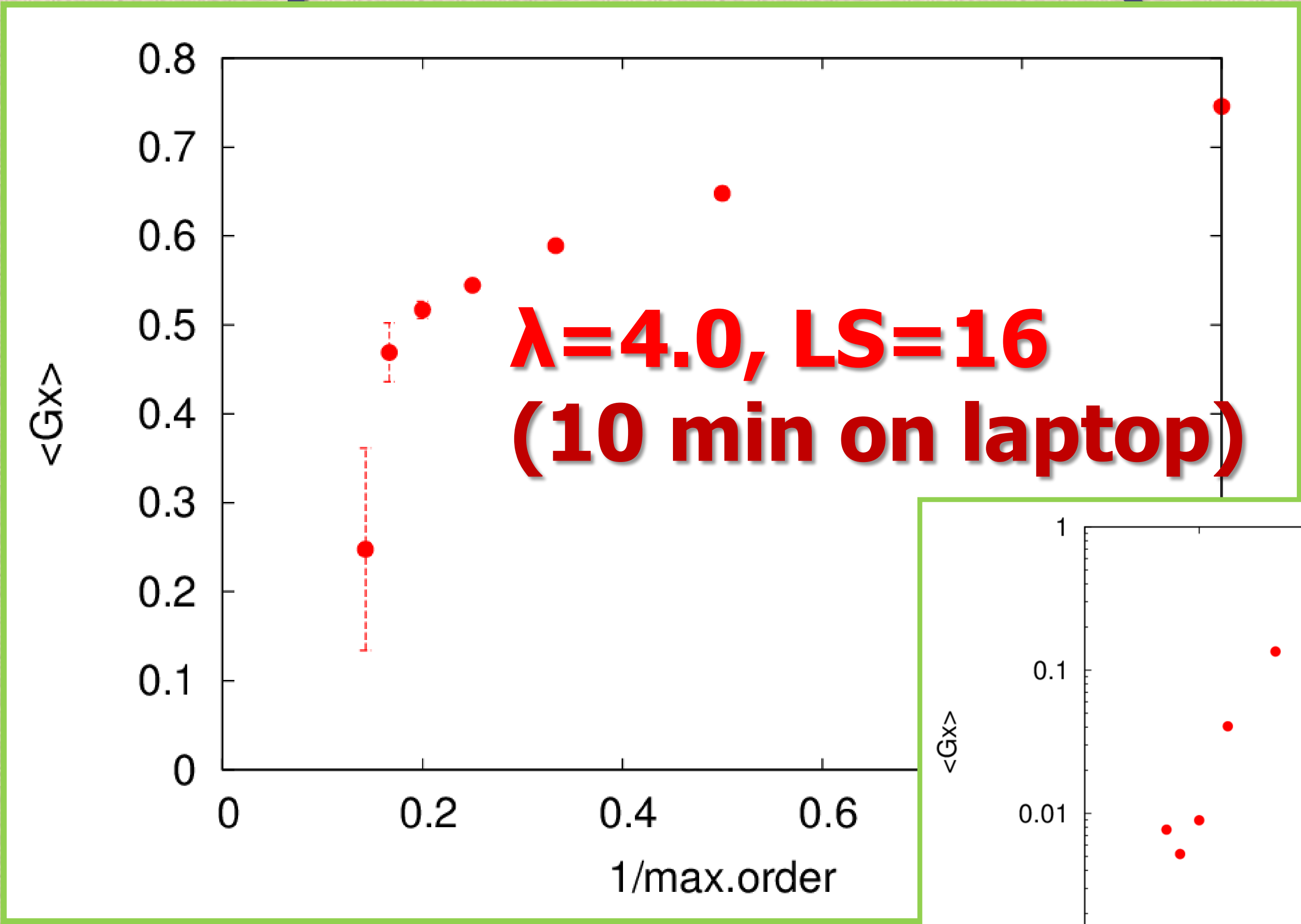
… And no large-N factorization assumed yet!

Solution can be written as geometric series

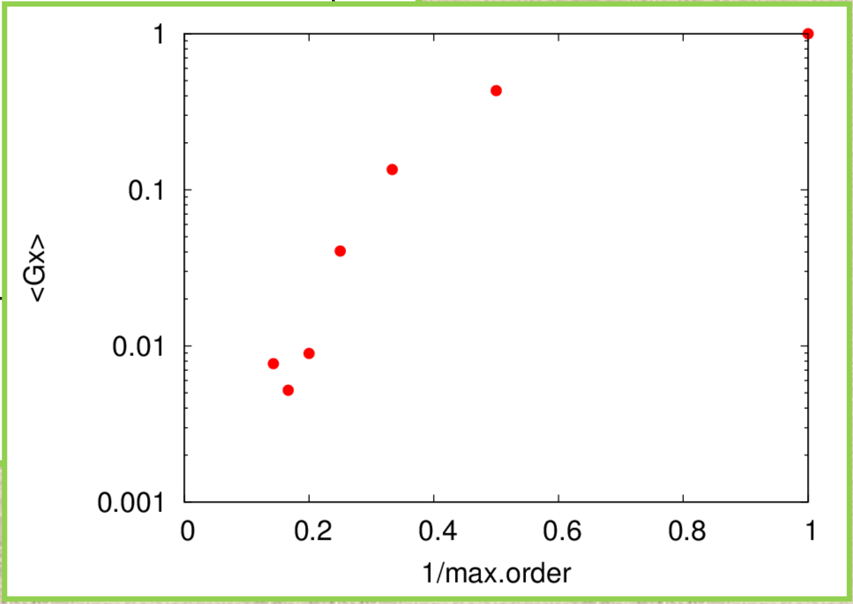
$$\phi(X) = \sum_{n=0}^{+\infty} \sum_{X_0} \ldots \sum_{X_n} \delta(X, X_n)\, A(X_n|X_{n-1}) \ldots A(X_1|X_0)\, b(X_0)$$

The terms in these series are MC sampled

Principal Chiral Model ($N=\infty$, $D=2$)

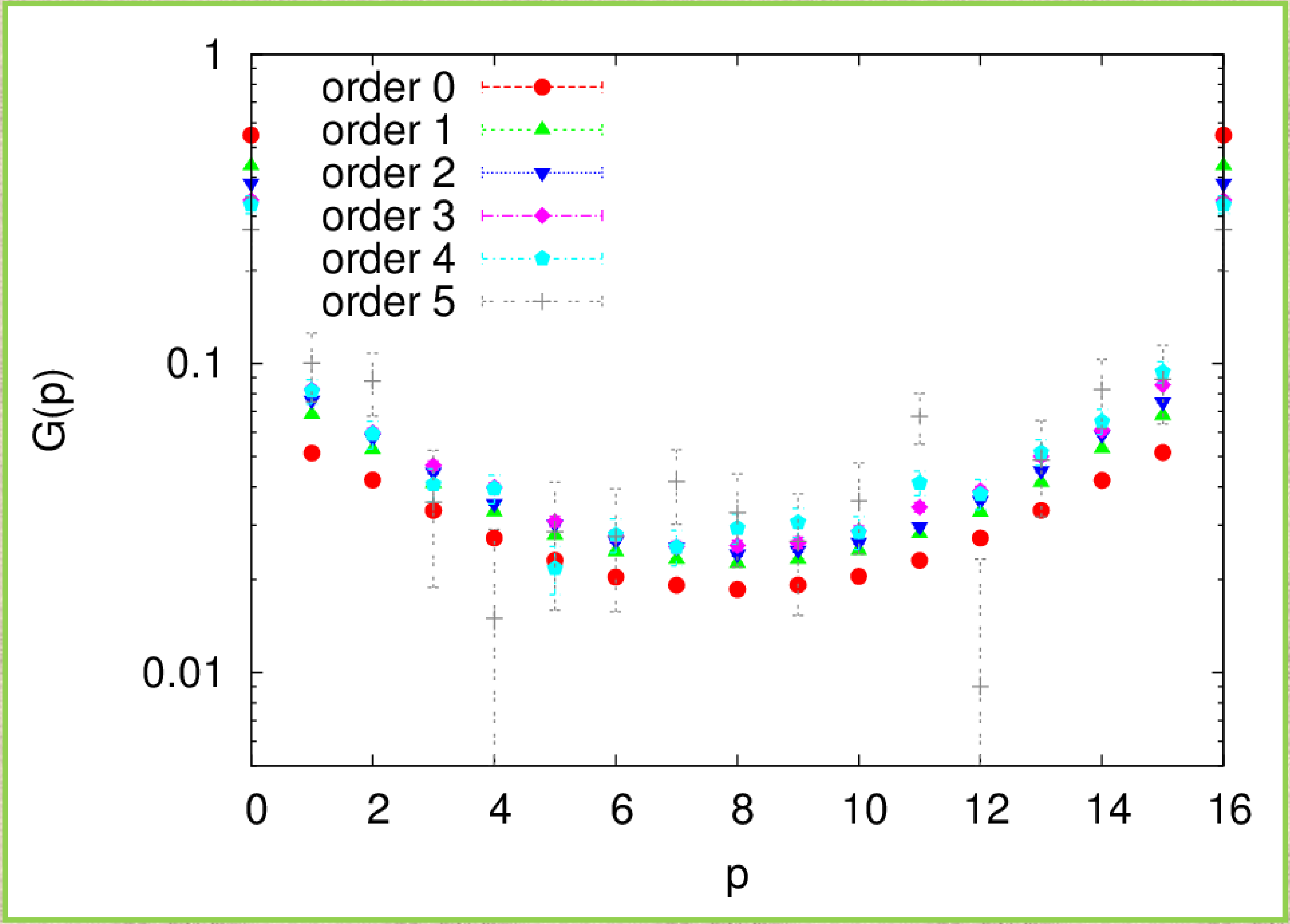


λ=4.0, LS=16
(10 min on laptop)



Restoration of SU(N) x SU(N) symmetry

Principal Chiral Model (N=∞,D=2)



U(N) field correlator in momentum space

Alternative forms of SD equations

SD equations in terms of $g_x \in SU(N)$

- * **Significantly simpler (only 4-vertices)**
- * **Advantageous for fermions**

In momentum space:

$$\mathcal{G}(p_1, q_1) = \frac{1}{V^2} \sum_{x,y} e^{-ipx - iqy} \langle \frac{1}{N} \mathrm{Tr}\, (g_x^\dagger g_y) \rangle$$

$$\mathcal{G}(p_1, q_1) = \lambda \frac{\mathcal{G}_0(p_1)\delta(p_1+q_1)}{V} + \mathcal{G}_0(p_1) \sum_{\tilde{p}_1, \tilde{q}_1, \tilde{p}_2} \delta(p_1, \tilde{p}_1 + \tilde{q}_1 + \tilde{p}_2)\, D(\tilde{q}_1)\, \mathcal{G}(\tilde{p}_1, \tilde{q}_1, \tilde{p}_2, q_1)$$

Bare mass ~ coupling

$$\mathcal{G}_0(p) = (\lambda + D(p))^{-1}$$

Seems to be very general!

Summary: Transseries from DiagMC

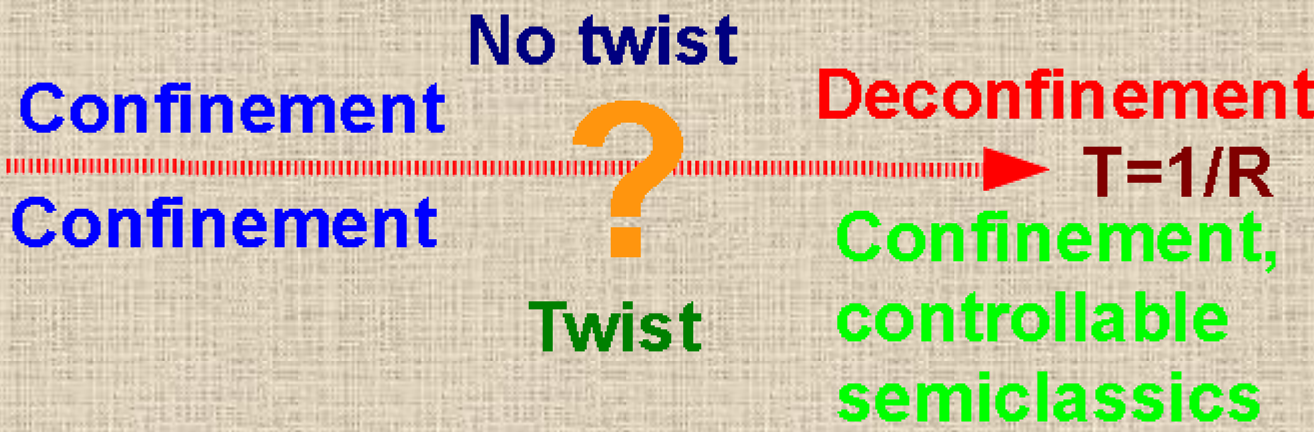
* "Effective mass" in the bare quantum action: general feature of compact field theories
* In 2D, leads to trans-series with powers and logs
* Series expansion suitable for DiagMC
* No factorial divergences!

Disclaimer: Bare Lattice Perturbation Theory

* Running coupling etc. hidden in the structure of the series in a complex way

Lattice tests of continuity

* Twisted compactifications are important tools in exploring the resurgent structure of QFT
* O(N), CP^{N-1}, SU(N) sigma-models



* Continuity is a conjecture (AFAIK)
* Analyticity at intermediate T not rigorously proven

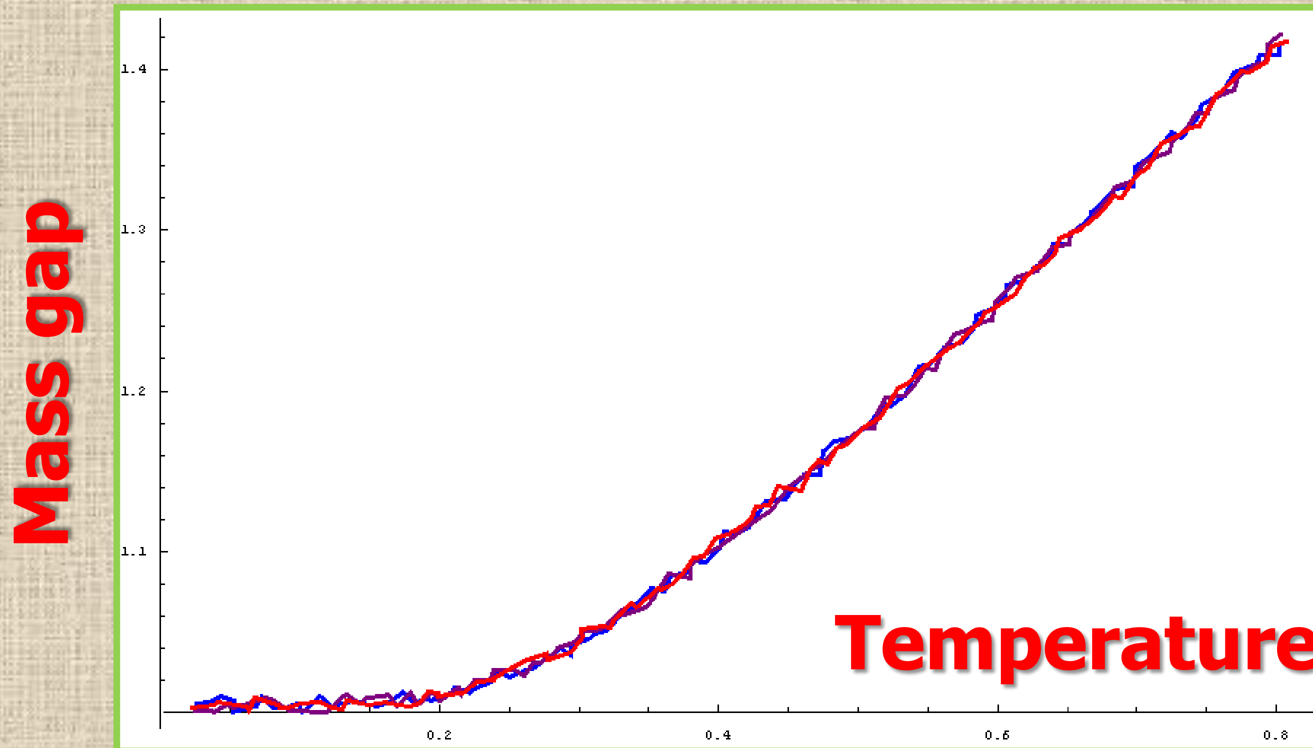
Lattice tests of continuity: SU(N) PCM [work with S. Valgushev]

* Monte-Carlo simulations, N=3 and N=6
* Cabibbo-Marinari heat-bath algorithm
* Temporal compactification vs
* Twisted compactification
* Correlation length ~ 10-12
* t'Hooft coupling ~ 3

Thermodynamics not well studied,
Order of transition unknown (to me?)

Order of transition?

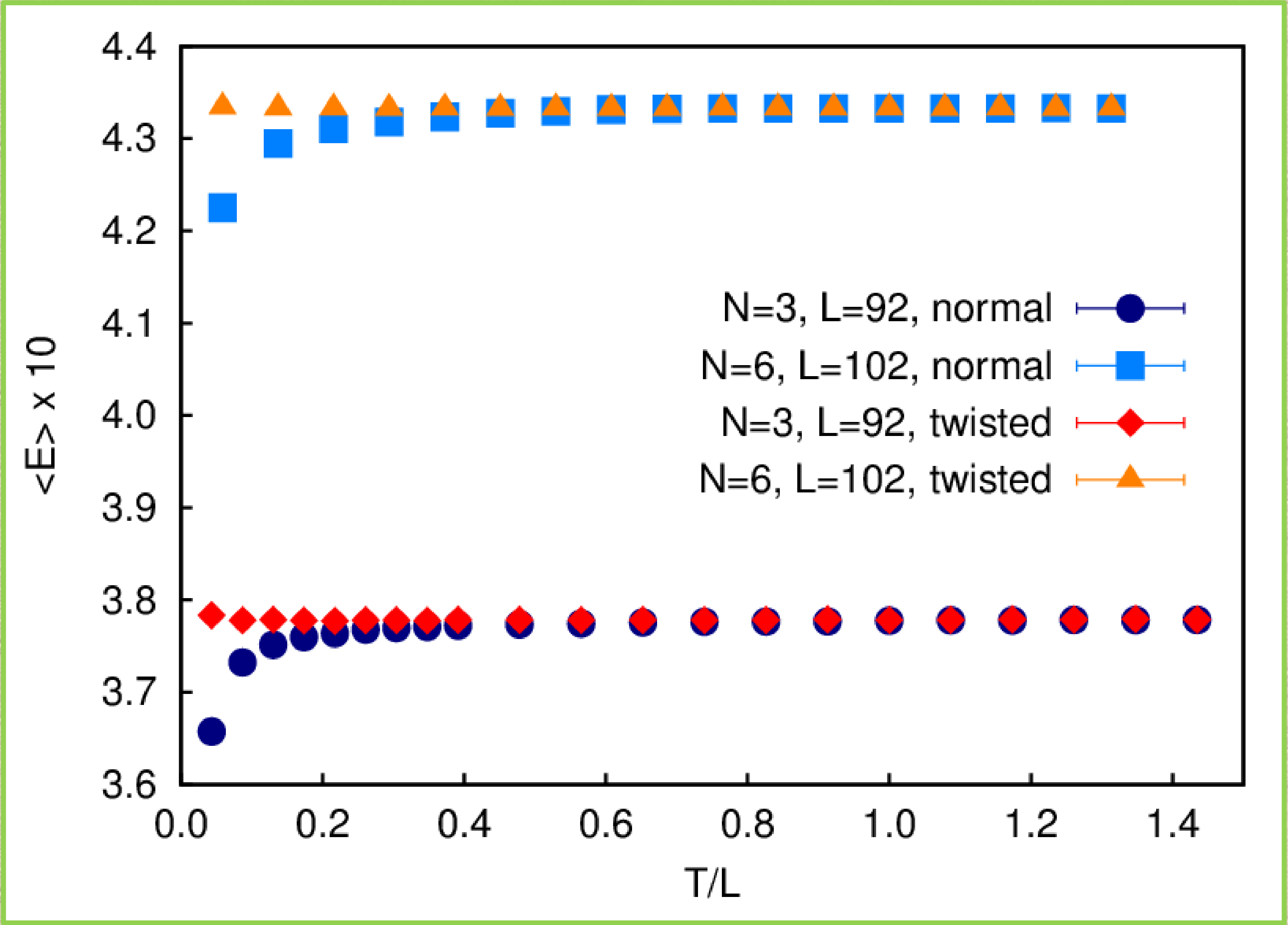
Seems to be a crossover in large-N O(N) sigma model, but of course in PCM things may be very different



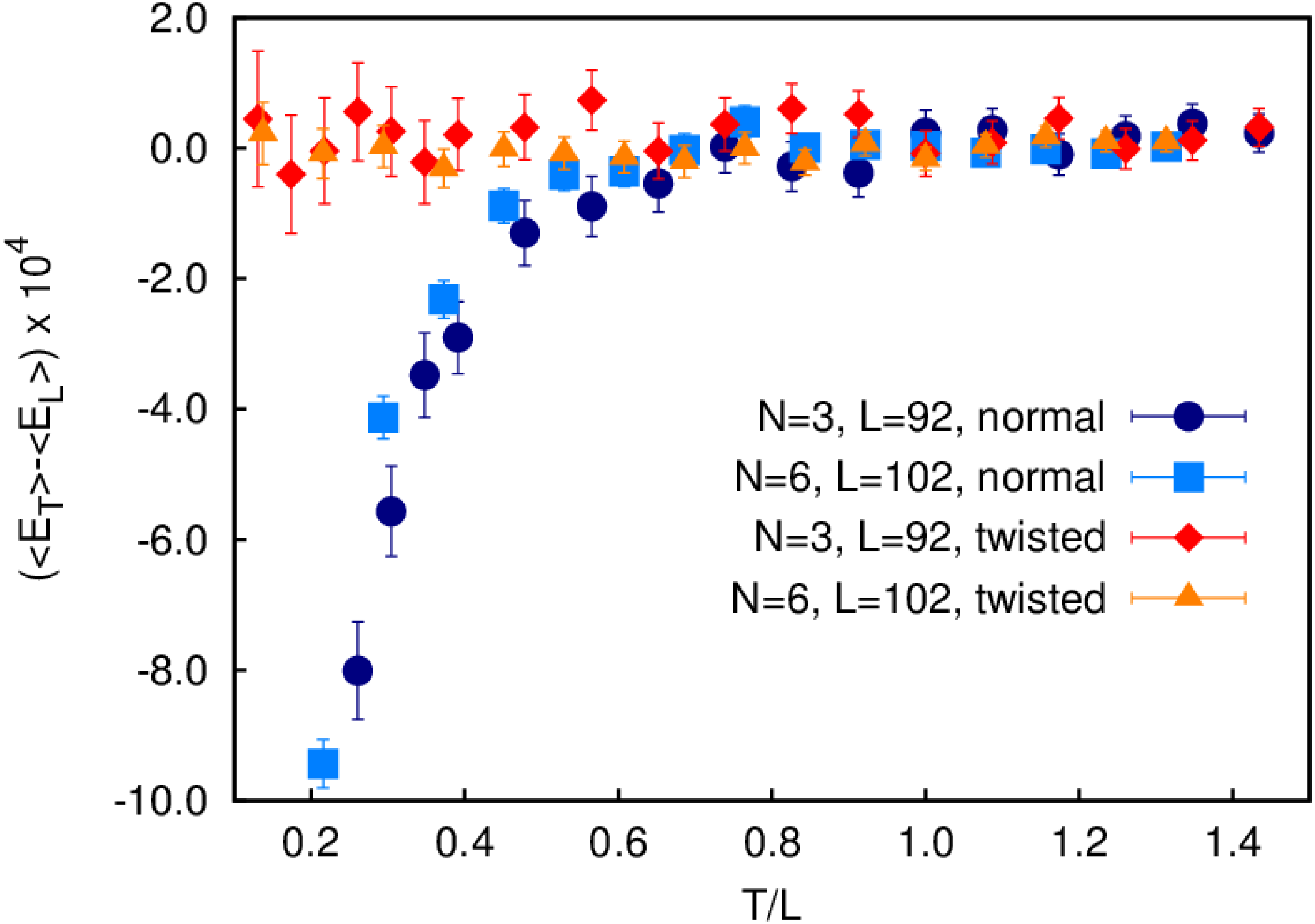
T/M(T=0) ~ 0.2

UV cutoff varies by 2 orders of magnitude

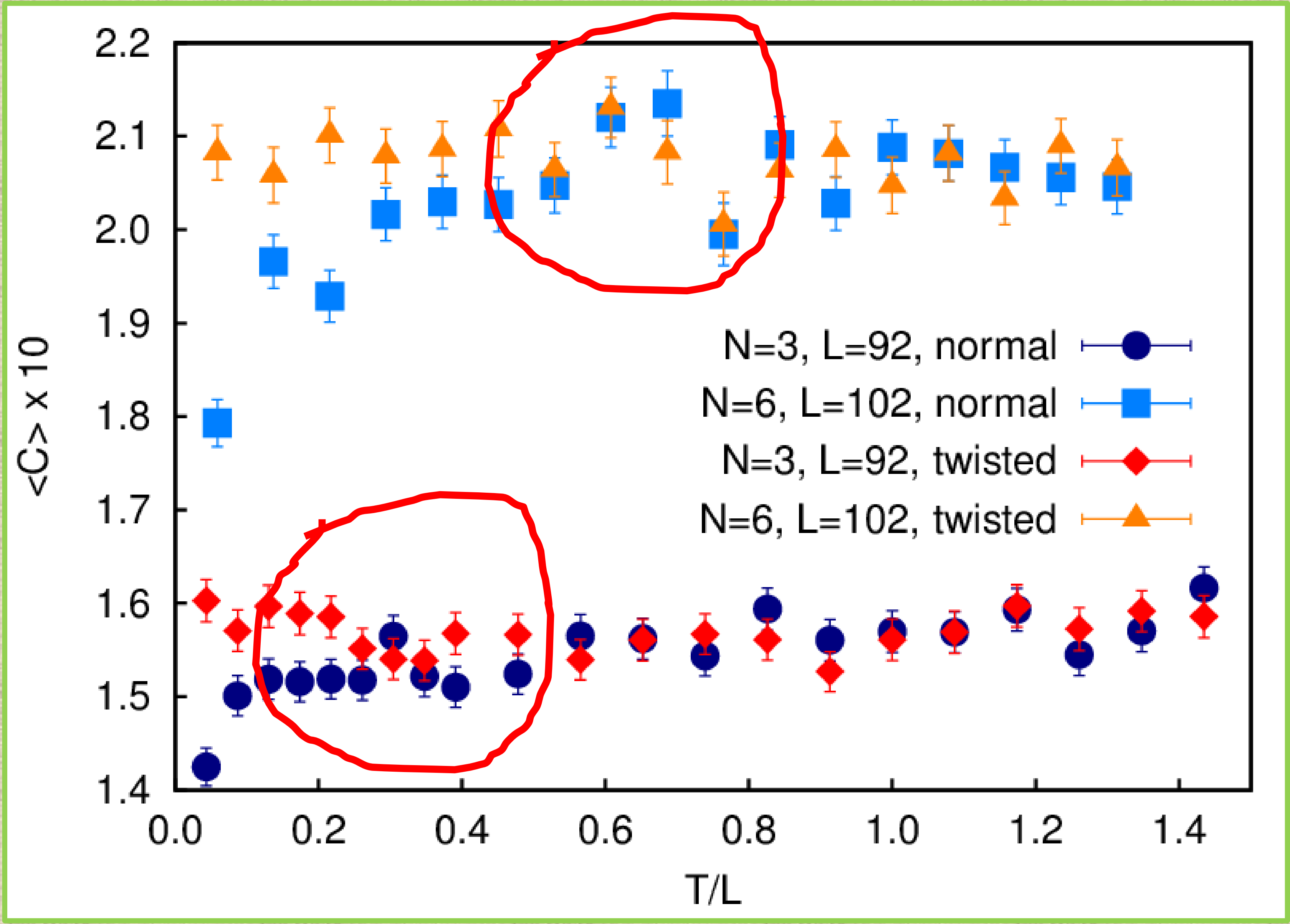
Mean link vs. "Temperature"



Space-time links vs. "Temperature"



Link susceptibility vs. "Temperature"



Thank you for your attention!!!