UNIVERSITY OF ICELAND

# Entanglement in semi-classical black hole evolution

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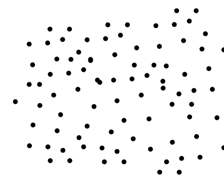
# Outline

- Introduction
- Average entropy of a subsystem — Page curve
- Generalized entropy
- 1+1-dimensional dilaton gravity (CGHS/RST model)
- Black hole solutions
- Hawking effect and semi-classical back-reaction
- Page curves for RST black holes
  - evaporating black hole
  - two-sided eternal black hole
- Holographic complexity of semi-classical bh's (if time permits)

# Black hole evolution

Hawking (1976)

matter in a pure quantum state

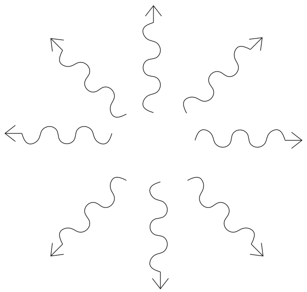


gravitational collapse

black hole

Hawking effect Hawking (1974)

outgoing radiation



# Unitary black hole evolution

D.N.Page (1980)
G. 't Hooft (1990)
L.Susskind, LT, J.Uglum (1993)
K.Schoutens, E.Verlinde, H.Verlinde (1993)
⋮

Assumptions: Susskind, LT, Uglum (1993)

(1) A black hole is a quantum system with discrete energy levels and finite density of states

(2) The dimension of the subspace of states that describe a black hole of mass M is $\exp S_{\rm BH}(M)$

(3) If the initial state of collapsing matter is a pure quantum state then the system as a whole remains in a pure state at all times

(4) After BH forms, the full system can be divided into subsystems

A - (distant) outgoing Hawking radiation

B - everything else (including BH)

# Average entropy of a subsystem

D.N. Page (1993), S. Sen (1996)

Consider a quantum system with Hilbert space of dimension $m \times n$ in a random pure state.

A subsystem of dimension $m < n$ has average entanglement entropy

$$S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}$$

For $n \geq m \gg 1$

$$S_{m,n} \approx \log m - \frac{m}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

Information contained in subsystem:

$$I_{m,n} = \log m - S_{m,n}$$

For $n \geq m \gg 1$ we have $I_{m,n} \leq \dfrac{1}{2}$

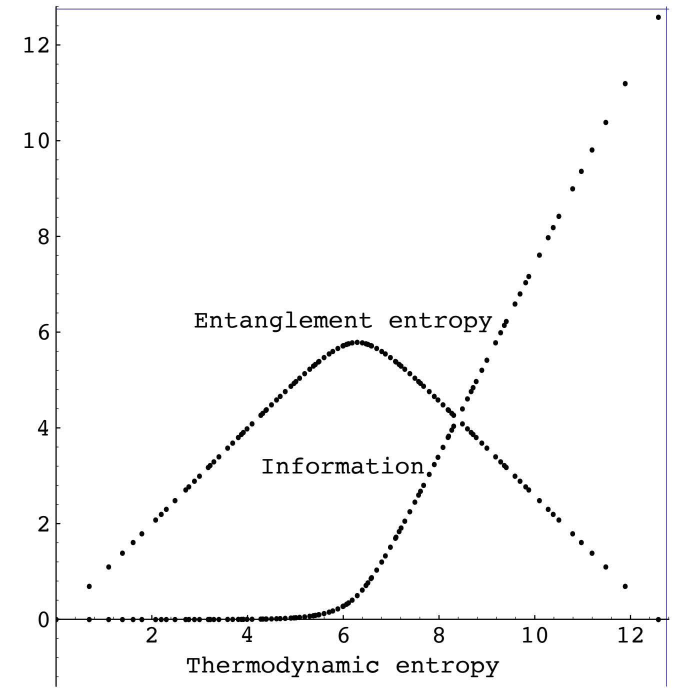
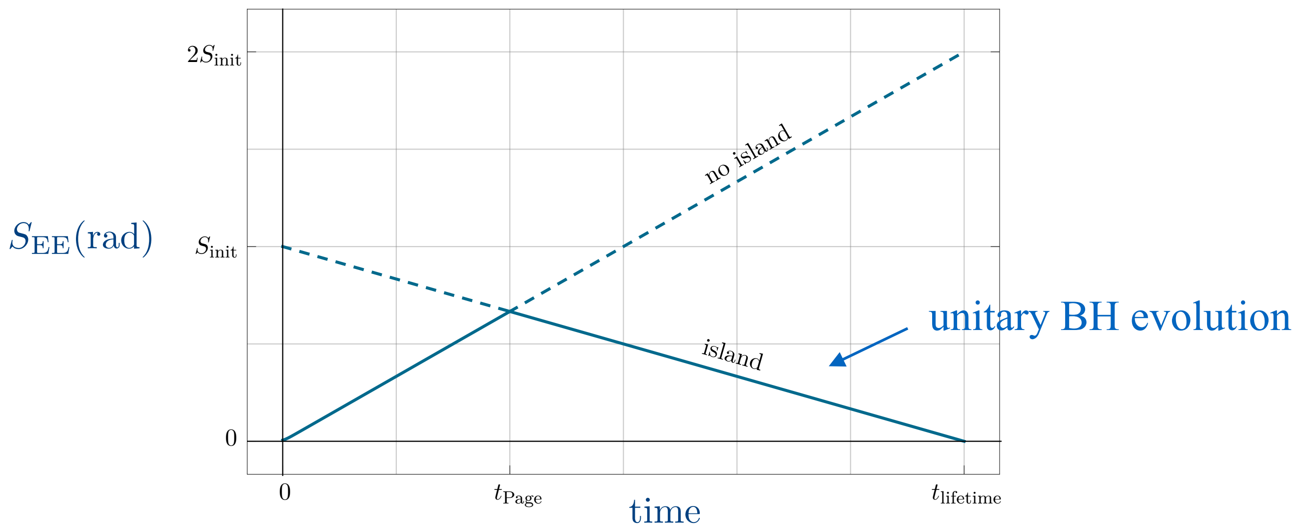



figure from Page (1993)

# Page curve for unitary BH evolution

Entanglement between outgoing Hawking radiation and remaining black hole




Can semi-classical results for $S_{\text{EE}}(\text{rad})$ be reconciled with unitary BH evolution?

Answer is "yes" if a correction motivated by holographic duality is included

Penington (2019)
Almheiri, Engelhardt, Marolf, Maxfield (2019)

Generalised entropy: $$S_{\text{gen}} = \frac{\text{Area}(I)}{4G_N} + S_{\text{Bulk}}[\mathcal{S}_{AI}]$$

Area term dominates after Page time and gives a result consistent with unitarity

The semi-classical theory is surprisingly effective!

# Semi-classical Page curves

## I: Black holes in AdS coupled to external CFT

Penington (2019)
Almheiri, Engelhardt, Marolf, Maxfield (2019)

Almheiri, Mahajan, Maldacena (2019)
Almheiri, Mahajan, Santos (2019)
⋮

Extract Hawking radiation via coupling to external CFT Rocha (2008)

Subsystems: rad — external CFT containing Hawking radiation

bh — CFT dual of AdS containing evaporating BH

Full system is in a pure state: $S_{\rm EE}(\mathrm{rad}) = S_{\rm EE}(\mathrm{bh})$

Use (quantum corrected) holographic entanglement entropy to evaluate $S_{\rm EE}(\mathrm{bh})$

## II: Black holes in asymptotically flat spacetime

Gautason, Schneiderbauer, Sybesma, LT (2020)
Anegawa, Iizuka (2020)
Hashimoto, Iizuka, Matsuo (2020)
Hartman, Shagoulian, Strominger (2020)
⋮

Gautason *et al.*

Adapt semi-classical entropy prescription to two-dimensional dilaton gravity model

Explicit analytic results for semi-classical RST black holes

# Generalized entropy

Ryu, Takayanagi (2006)
Hubeny, Rangamani, Takayanagi (2007)
Faulkner, Lewkowycz, Maldacena (2013)
Engelhardt, Wall (2014)

Quantum corrected holographic entanglement entropy of boundary region $A$

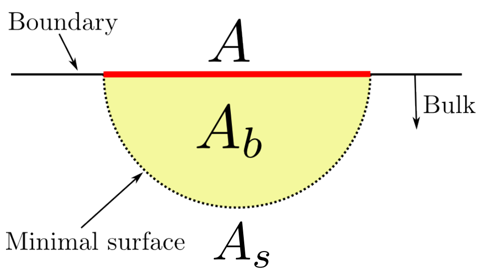



figure from Faulkner *et al.* (2013)

$$S_{\text{gen}}(A) = \min_{A_s}\left\{\text{ext}_{A_s}\left[\frac{\text{Area}(A_s)}{4G_N} + S_{\text{bulk}}(A_b)\right]\right\}$$

For a black hole in AdS coupled to external CFT: Penington (2019)

— $A$ is the entire spatial boundary (where dual CFT is defined)

— $A_s$ is a co-dimension two surface *inside* the bulk geometry

For a black hole in asymptotically flat spacetime: Gautason et al. (2019)

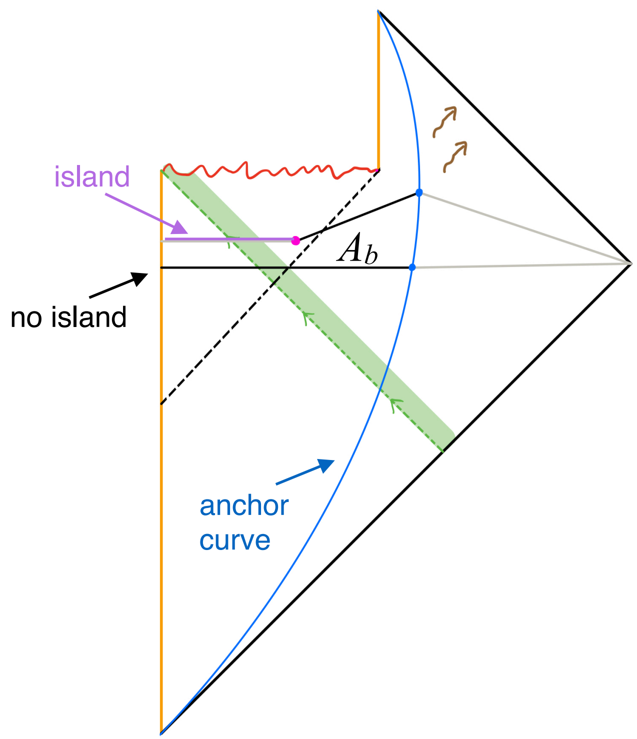
— $A$ is a spatial boundary outside black hole (in asymptotic region)

— $A_s$ is a co-dimension two surface *inside* the bulk geometry

When there is more than one quantum extremal surface $A_s$ we are instructed to choose the one that gives the smallest entropy

# Page curve for evaporating RST black holes

Gautason *et al.* (2020)




i) work with solvable 2d semi-classical model

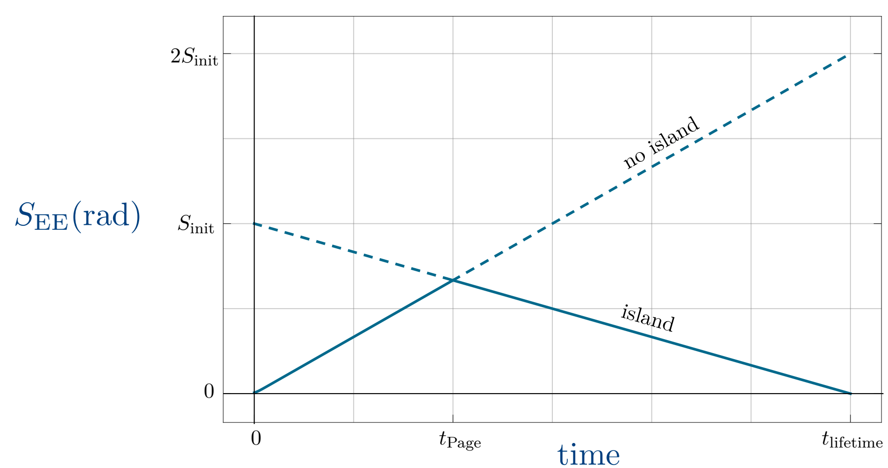
ii) adapt generalised entropy to asymptotically flat background (with linear dilaton)

iii) 2d matter described by strongly coupled CFT

⟶ use AdS_3 holography to calculate $S_{\text{Bulk}}$
cf. Almheiri, Mahajan, Maldacena, Zhao (2019)

iv) coupling to an external bath is unnecessary - 'inside' and 'outside' separated by anchor curve

v) explicit analytic result for generalised entropy



# CGHS model

Callan, Giddings, Harvey, Strominger (1991)

$$S_0 = \frac{1}{2\pi}\int d^2y\sqrt{-g}\Big[e^{-2\phi}(R+4(\nabla\phi)^2+4\lambda^2) - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\Big]$$

Effective action for radial modes of near-extremal magnetically charged black holes in $D = 4$ dilaton gravity.

Length scale $\lambda^{-1}$ set by magnetic charge of $4D$ extremal black hole,

conformal gauge:

$$ds^2 = -\frac{1}{2}e^{2\rho}dy^+dy^-$$

$$\partial_+\partial_- f_i = 0\,,$$

$$\lambda = 1$$

$$\partial_+\partial_-(e^{-2\phi}) = -e^{2(\rho-\phi)}\,,$$

$$\partial_+\partial_-(\rho-\phi) = 0\,.$$

Conformal reparametrisation to Kruskal gauge: $\rho = \phi$

$$-\partial_+\partial_- e^{-2\phi(x^+,x^-)} = 1\,, \qquad -2\,\partial_\pm^2 e^{-2\phi(x^+,x^-)} = \sum_{i=1}^{N}\partial_\pm f_i\,\partial_\pm f_i$$

More general matter sector leads to the same black hole geometries

We'll assume a strongly coupled CFT with large central charge: $c \gg 24$

# Two-sided classical black hole

Linear dilaton vacuum: $e^{-2\phi} = e^{-2\rho} = -x^+ x^-$ ← Kruskal coordinates

Static black hole solution: $e^{-2\phi} = e^{-2\rho} = M - x^+ x^-$

Rescaled coordinates: $x^+ = \sqrt{M} v \,, \quad x^- = \sqrt{M} u$

$$e^{-2\phi} = M(1 - v\,u) \qquad ds^2 = -\frac{dv\,du}{1 - v\,u}$$

← 2d 'cigar' in Lorentzian signature

Mandal, Sengupta, Wadia (1991)
Witten (1991)

Asymptotically flat coordinates:

$$v = e^{t+\sigma}\,, \qquad u = -e^{-t+\sigma}$$

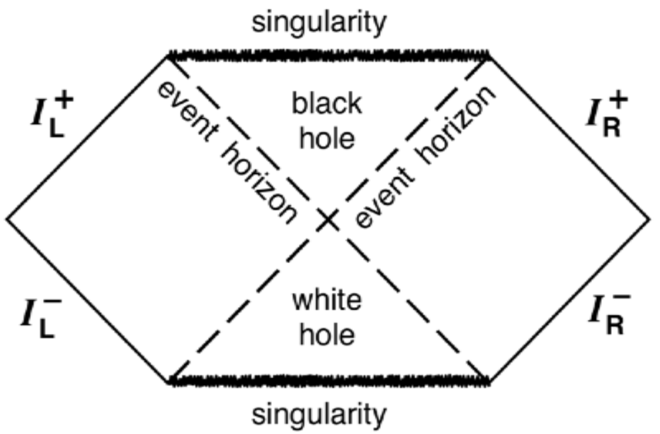
Thermodynamic variables:

$$\mathcal{M} = \frac{\lambda}{\pi} M$$

$$T = \frac{\lambda}{2\pi}$$

$$S = 2\,e^{-2\phi}\Big|_{\text{horizon}} = 2M = \frac{2\pi\mathcal{M}}{\lambda}$$

Penrose diagram



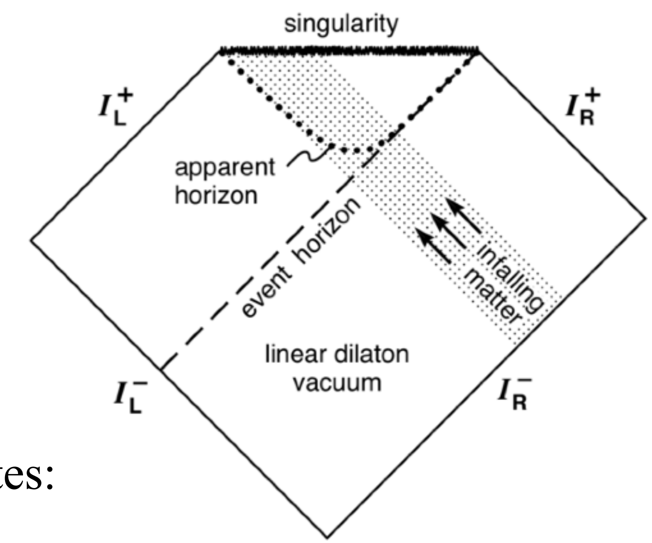
# Dynamical black hole

$$e^{-2\phi} = e^{-2\rho} = M(x^+) - x^+\big(x^- + P_+(x^+)\big)$$

$$M(x^+) = \int_0^{x^+} dy^+ y^+ \, T_{++}(y^+)\,,$$

$$P_+(x^+) = \int_0^{x^+} dy^+ \, T_{++}(y^+)$$

Penrose diagram




Shock wave solution in rescaled Kruskal coordinates:

$$\mathrm{e}^{-2\rho(v,u)} = \frac{1}{M}\,\mathrm{e}^{-2\phi(v,u)} = \begin{cases} -vu & \text{if } v<1 \\ (1-v(u+1)) & \text{if } v>1 \end{cases}$$

Manifestly flat coordinates in linear dilaton region:

$$v = e^{\omega^+}\,, \qquad u = -e^{-\omega^-}$$

Asymptotically flat coordinates:

$$v = e^{\sigma^+}\,, \qquad u = -1 - e^{-\sigma^-}$$

Asymptotic time: $t = \frac{1}{2}(\sigma^+ + \sigma^-)$

# Hawking effect from conformal anomaly

CGHS (1991), Christensen & Fulling (1977)

$$\langle T^\mu{}_\mu \rangle = \frac{c}{12} R \quad \longrightarrow \quad \langle T_{+-} \rangle = -\frac{c}{6} \partial_+ \partial_- \rho$$

$$0 = \partial_- \langle T_{++} \rangle + \partial_+ \langle T_{-+} \rangle - 2\partial_+ \rho \, \langle T_{-+} \rangle$$

$$0 = \partial_+ \langle T_{--} \rangle + \partial_- \langle T_{+-} \rangle - 2\partial_- \rho \, \langle T_{+-} \rangle$$

$$\longrightarrow \quad \langle T_{\pm\pm} \rangle = \frac{c}{12} \left( 2\partial_\pm^2 \rho - 2(\partial_\pm \rho)^2 - t_\pm \right)$$

Conformal reparametrization: $x^\pm \to y^\pm(x^\pm) \quad \longrightarrow \quad \rho(y^+, y^-) = \rho(x^+, x^-) - \frac{1}{2} \log \frac{\mathrm{d}y^+}{\mathrm{d}x^+} \frac{\mathrm{d}y^-}{\mathrm{d}x^-}$

$$\left( \frac{dy^\pm}{dx^\pm} \right)^2 T_{\pm\pm}(y^\pm) = T_{\pm\pm}(x^\pm) - \frac{c}{12} \{y^\pm, x^\pm\} \, , \qquad \{y, x\} = \frac{y'''}{y'} - \frac{3}{2} \frac{(y'')^2}{(y')^2}$$

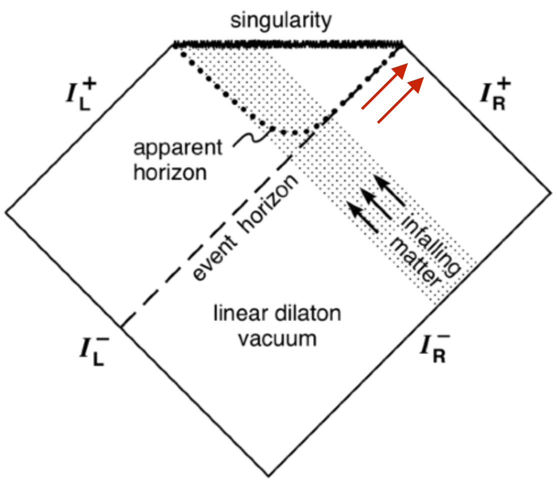
$$\left( \frac{dy^\pm}{dx^\pm} \right)^2 t_\pm(y^\pm) = t_\pm(x^\pm) + \{y^\pm, x^\pm\}$$

No outgoing radiation in linear dilaton vacuum:

$$\rho(\omega^+, \omega^-) = 0 \, , \qquad t_-(\omega^-) = 0$$

Outgoing energy flux at $\sigma^+ \to +\infty$ :

$$T_{--}(\sigma^-) = -\frac{c}{12} t_-(\sigma^-) = \frac{c}{24} \left( 1 - \frac{1}{(1 + \mathrm{e}^{\sigma^-})^2} \right)$$



# Semi-classical back reaction

Add Polyakov-Liouville term to action and take the limit $c \gg 24$ Callan et al.(1991)

$$S_{\mathrm{Q}} = -\frac{c}{96\pi}\int d^2y\sqrt{-g(y)}\int d^2y'\sqrt{-g(y')}\,R(y)\,G(y,y')\,R(y')$$
$$= -\frac{c}{12\pi}\int d^2y\,\partial_+\rho\partial_-\rho$$

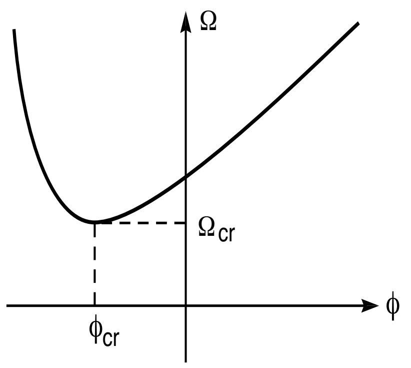
Add RST term to make semiclassical model solvable Russo, Susskind, LT (1992)

$$S_{\mathrm{RST}} = -\frac{c}{96\pi}\int d^2y\sqrt{-g}\,R\,\phi = -\frac{c}{12\pi}\int d^2y\,\phi\,\partial_+\partial_-\rho$$

Introduce new field variable: $\Omega = \mathrm{e}^{-2\phi} + \frac{c}{24}\phi$

Semi-classical field equations in Kruskal coordinates: $\partial_+\partial_-\Omega + 1 = 0\,, \quad -\partial_\pm^2\Omega = \frac{c}{24}t_\pm$

Linear dilaton vacuum: $\Omega = -x^+x^- - \frac{c}{48}\log(-x^+x^-)$




Field redefinition is not one-to-one —> boundary at critical value

$$\Omega_{cr} = \frac{c}{48}\left(1-\log\left(\frac{c}{48}\right)\right)\,, \qquad \phi_{cr} = -\frac{1}{2}\log\left(\frac{c}{48}\right)$$

Curvature singularity: $R = \frac{4}{1-\frac{c}{48}e^{2\phi}}\left(1-(\nabla\phi)^2\right)$

RST boundary conditions: $\partial_+\Omega\big|_{\Omega=\Omega_{\mathrm{cr}}} = 0 = \partial_-\Omega\big|_{\Omega=\Omega_{\mathrm{cr}}}$

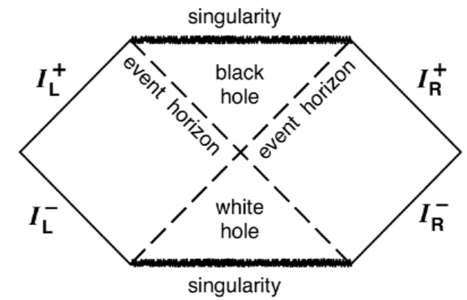
⟶ finite curvature on critical curve where it is timelike

# Semi-classical black holes

(i) Two-sided eternal black hole solution:

$$\Omega = M(1 - vu) + \Omega_{\text{crit}}$$

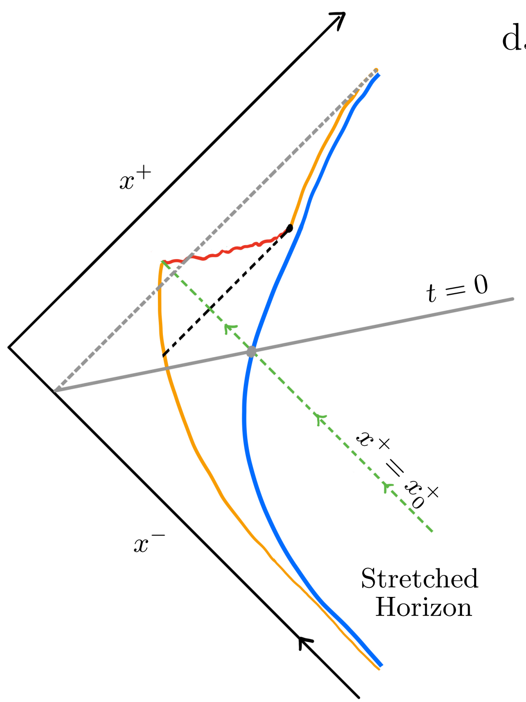
Equilibrium with thermal bath at $T = T_{\text{H}}$

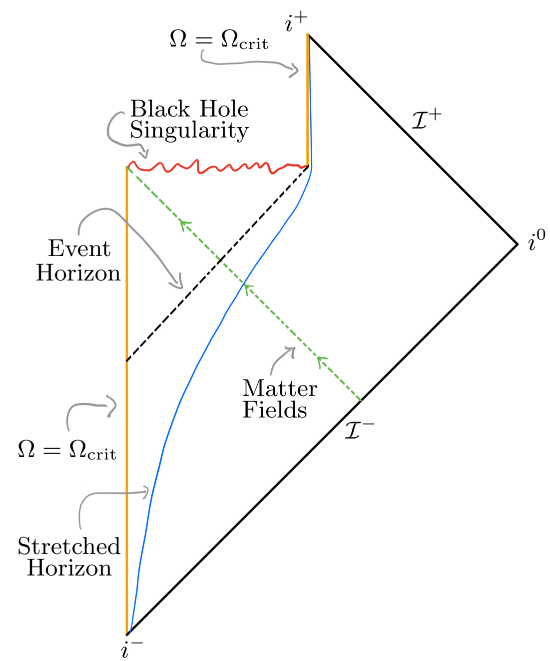



(ii) Dynamical black hole from shock wave collapse:

$$\Omega = M\Big(1 - v(u+1) - \epsilon \log(-Mvu)\Big)$$

$$\mathrm{d}s^2 = -Me^{2\phi}\,\mathrm{d}v\mathrm{d}u$$

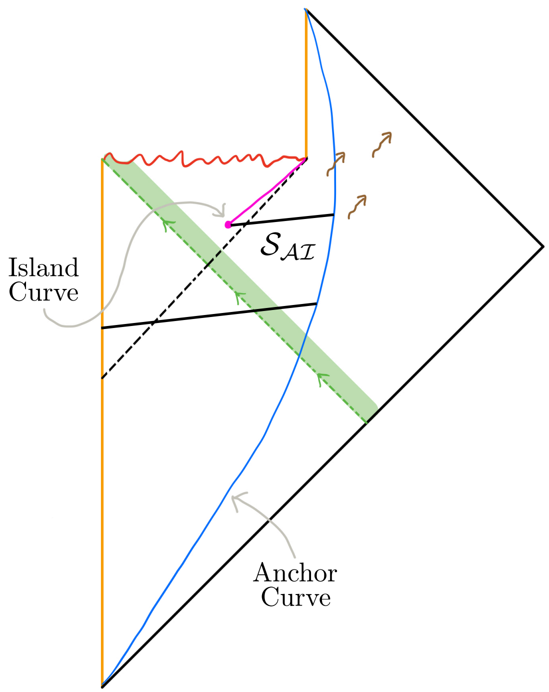



Assume initial black hole mass $M$ is large compared to scale set by central charge:

$$\epsilon \equiv \frac{c}{48M} \ll 1$$

# Generalised entropy in 1+1 dimensions




$$S_{\text{gen}} = \frac{\text{Area}(I)}{4G_N} + S_{\text{Bulk}}[\mathcal{S}_{AI}]$$

Adapt prescription to 1+1 dimensional theory:

AdS boundary $\longrightarrow$ anchor curve

$\mathcal{A} \longrightarrow$ point on anchor curve

$\mathcal{I} \longrightarrow$ point 'inside' anchor curve

$\mathcal{S}_{\mathcal{A}\mathcal{I}} \longrightarrow$ spacelike curve connecting $\mathcal{A}$ and $\mathcal{I}$

Area term: Recall higher-dimensional origin of 1+1 dimensional theory

area of transverse 2-sphere $\longleftrightarrow \; 8\, e^{-2\phi(I)}$

Semi-classical theory: $\quad \dfrac{\text{Area}(I)}{4G_N} = 2\big(\Omega(I) - \Omega_{\text{crit}}\big)$

Bulk term: Use $AdS_3/CFT_2$ and standard RT prescription

cf. Almheiri, Mahajan, Maldacena, Zhao (2019)

$$S_{\text{Bulk}}[\mathcal{S}_{AI}] \simeq \frac{\text{Length}}{4G_{(3)}}$$

← length of 3d geodesic connecting endpoints

AdS3 scale determined by 2d central charge: $\quad c = \dfrac{3L_3}{2G_{(3)}}$

Brown, Henneaux (1986)

# Embedding into AdS3

(1) Look for light-cone coordinates $\mathrm{d}s^2 = -\mathrm{e}^{2\rho}\mathrm{d}y^+\mathrm{d}y^-$ such that $t_\pm(y^\pm) = 0$

(2) Do a Weyl rescaling to $d\tilde{s}^2 = -dy^+\,dy^-$ so that $\langle \widetilde{T}_{\pm\pm}\rangle = 0$

Then the matter CFT is in vacuum state and 3d dual geometry is pure AdS3

$$\mathrm{d}s_3^2 = \frac{L_3^2}{z^2}\left(\mathrm{d}z^2 - \mathrm{d}y^+\mathrm{d}y^-\right)$$

The Weyl transformation in (2) can be implemented as a coordinate transformation in 3d

UV cutoff

$\longrightarrow$ regulated holographic boundary at $z = \delta\,\mathrm{e}^{-\rho(y^+,y^-)}$

Length of AdS3 geodesic gives:

$$S_{\mathrm{Bulk}}[\mathcal{S}_{AI}] = \frac{c}{6}\log\left[d(A,I)^2\mathrm{e}^{\rho(A)}\mathrm{e}^{\rho(I)}\right]\Big|_{t_\pm=0}$$

distance in flat 2d metric

# Page curve for evaporating black hole

Consider a dynamical solution of RST model with incoming matter shock wave into vacuum

$\longrightarrow$ $t_-(\omega^-) = 0$ in manifestly flat coordinates in initial linear dilaton vacuum

but energy of incoming shock gives $t_+(\omega^+) \neq 0$

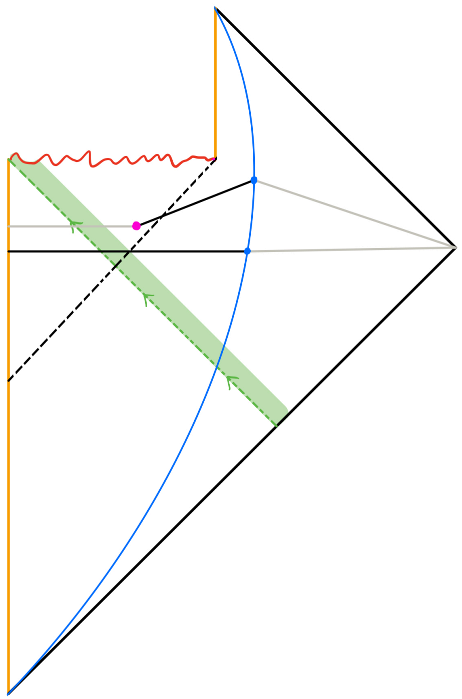
Take infalling matter to be in a coherent state built on the $(\omega^+, \omega^-)$ vacuum

Fiola, Preskill, Strominger, Trivedi (1994)

Then $S_{\text{Bulk}}[\mathcal{S}_{AI}]$ is the same as in the vacuum

$\longrightarrow$ standard Ryu-Takayanagi prescription for $AdS_3/CFT_2$ using $(\omega^+, \omega^-)$ coordinates

# Island configuration



area term

bulk term

$$S_{\text{gen}}^{\text{island}} = 2M\Big(1 - v_I(1+u_I) - \epsilon \log\left(-M v_I u_I\right)\Big)$$
$$+\frac{c}{12}\log\left[\left(\log\frac{v_A}{v_I}\log\frac{u_A}{u_I}\right)^2 \frac{v_A u_A}{(1-v_A(1+u_A))}\frac{v_I u_I}{(1-v_I(1+u_I))}\right]$$

Extremise over location of inside point: $(v_I, u_I)$

$$0 = -2M(1+u_I) + \frac{c}{12 v_I}\frac{1+u_I}{(1-v_I(1+u_I))} + \frac{c}{24 v_I} - \frac{c}{6 v_I \log\left(\frac{v_A}{v_I}\right)}$$
$$0 = -2M v_I + \frac{c}{12 u_I}\frac{v_I}{(1-v_I(1+u_I))} + \frac{c}{24 u_I} - \frac{c}{6 u_I \log\left(\frac{u_A}{u_I}\right)}\,.$$

Solution for $M \gg c \gg 1$

retarded time on anchor curve

$$S_{\text{gen}}^{\text{island}} = 2M - \frac{c}{24}(t_A - \sigma_A) + \dots$$

# No-island configuration

$(v_I, u_I) \longrightarrow (v_0, u_0)$ on boundary curve $\Omega = \Omega_{\text{crit}}$

$$S_{\text{gen}}^{\text{no-island}} = \frac{c}{12}\log\left[\left(\log\frac{v_A}{v_0}\log\frac{u_A}{u_0}\right)^2 \frac{v_A u_A}{(1-v_A(1+u_A))}\right]$$
$$= \frac{c}{12}(t_A - \sigma_A) + \dots\,,$$

# Page curve for evaporating black hole

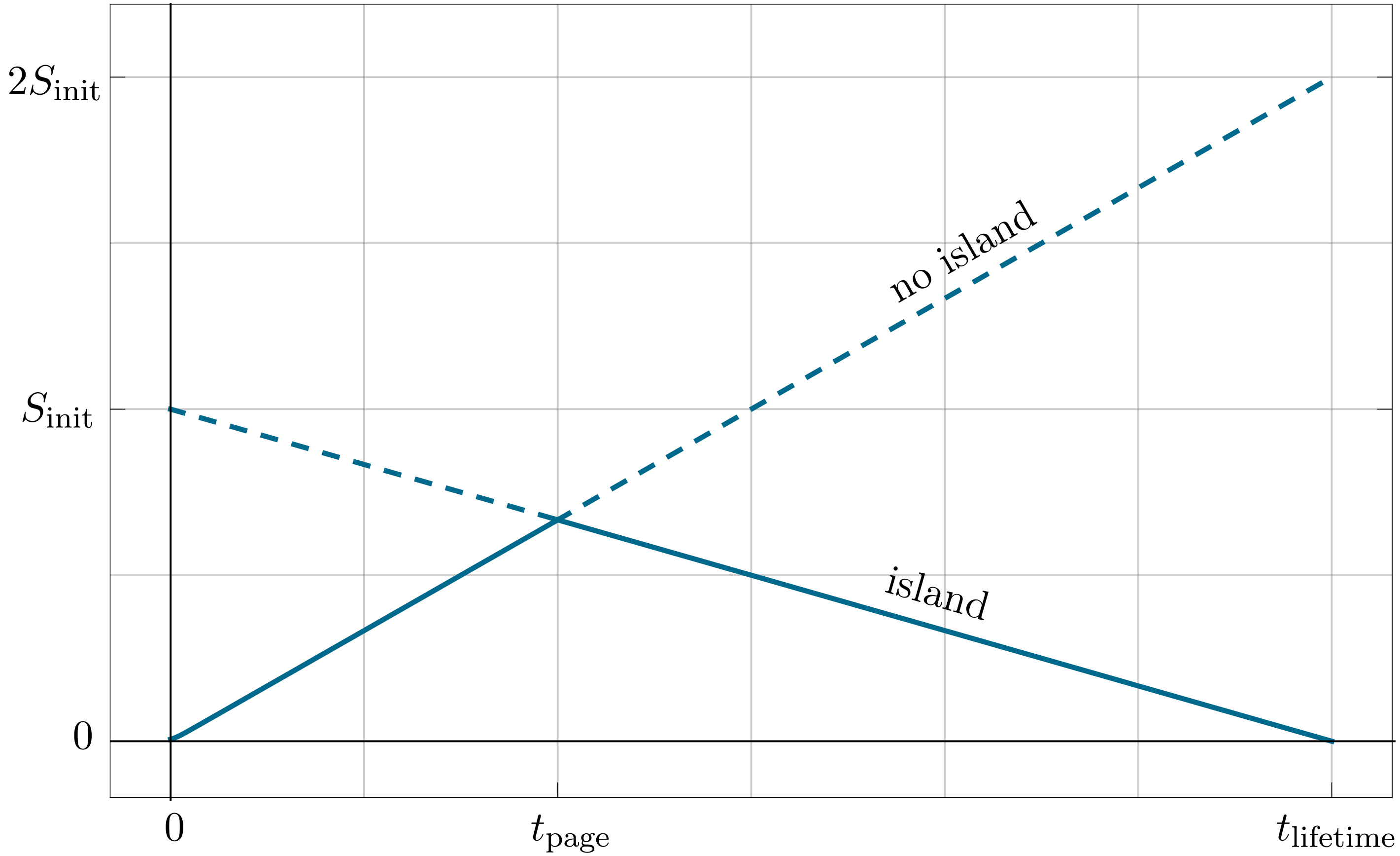
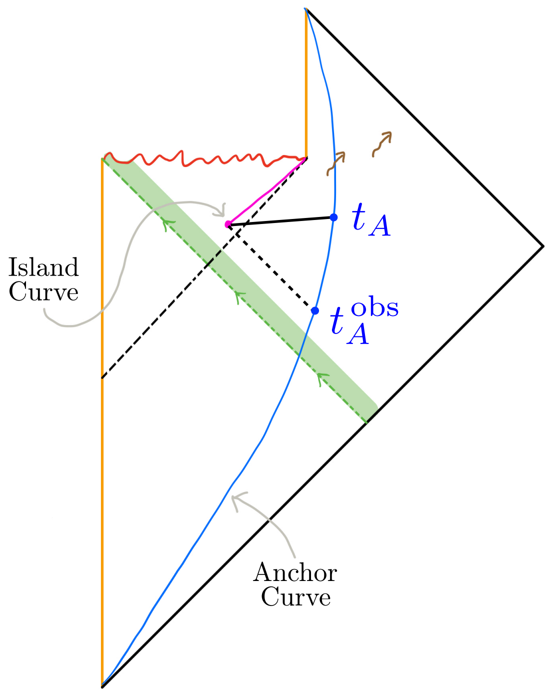



**Figure 5**. The Page curve for the dynamical RST black hole.

# Location of island




Extremisation problem for time $t_A$ on anchor curve is solved by

$$v_I(t_A) > 1\,, \quad u_I(t_A) = -1 + \frac{\epsilon}{v_I(t_A)} + O(\epsilon^2)$$

*i.e.* the island QES is inside black hole (very close to event horizon)

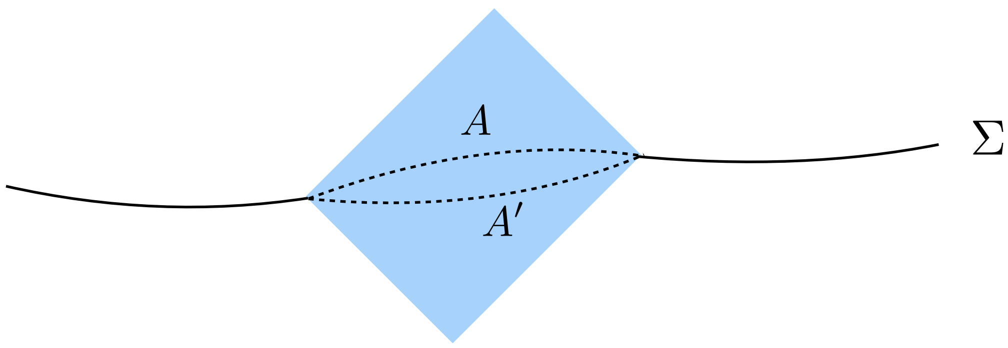
Light signal emitted from anchor curve at time $t_A^{\rm obs}$ is received at QES

if $t_A - t_A^{\rm obs} = 2\sigma_a + t_s$ where $t_s \equiv \log\left(\frac{1}{4\epsilon}\right)$ scrambling time

Valid island solution exists for all $t_A - \sigma_A \gtrsim t_s$

*i.e.* within a scrambling time after the first Hawking particle being emitted

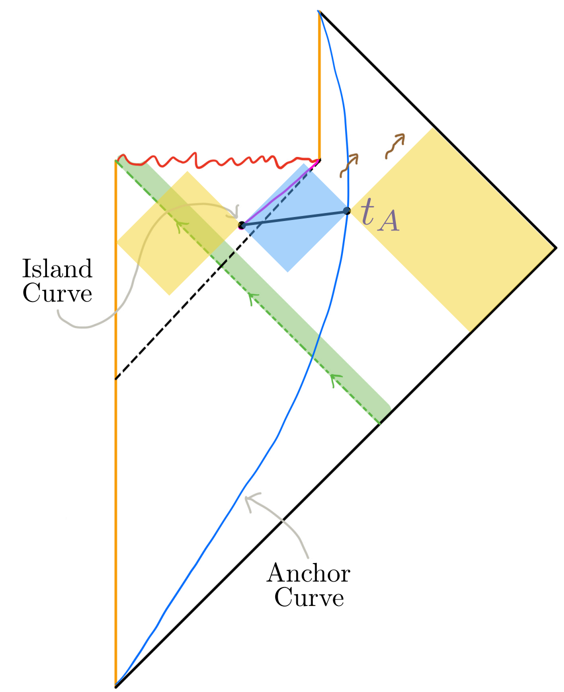
# Entanglement wedges




Initial data on $A$ is sufficient to determine bulk fields inside the *causal diamond* of $A$.

$A$' has the same causal diamond as $A$ and the same entanglement entropy.

Black hole *entanglement wedge* at time $t_A$ is the causal diamond of a spatial region between point $t_A$ on anchor curve and its QES.




Radiation *entanglement wedge* at time $t_A$ is formed from the causal diamonds of the complement of the spatial region.

# Bulk fields in black hole interior

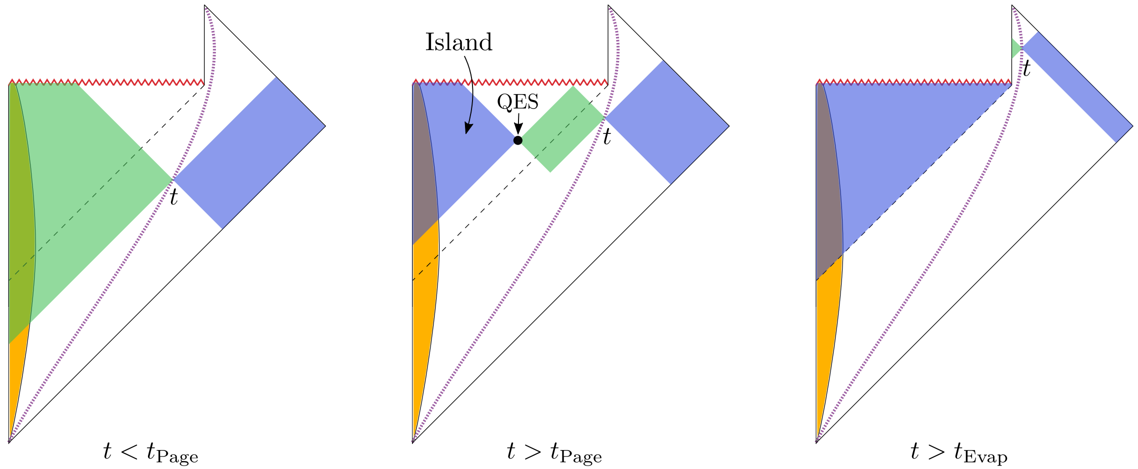



figure from Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini (2020)

Old black hole: ($t_A > t_{Page}$)

Bulk fields on island no longer contribute to generalised entropy of black hole

DOF's on black hole horizon can describe only small part of black hole interior

Do we have enough DOF's to encode observations made by an infalling observer?

— regulate transplanckian modes using infalling lattice Corley, Jacobson (1997)

— infalling observers only encounter modes that originate from a scrambling time or less before they enter Lowe, LT (2015)

— only need to reconstruct bulk fields in the black hole entanglement wedge.

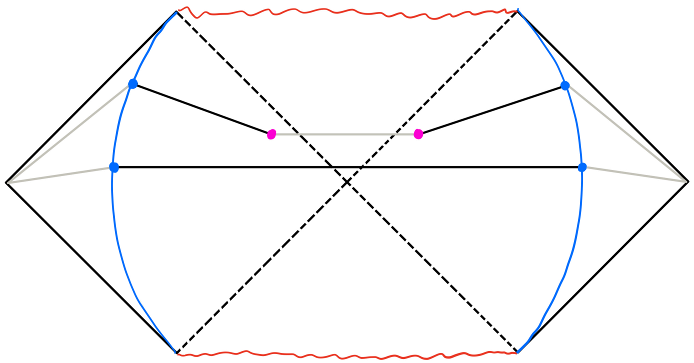
Lowe, LT (2016, 2017)

# Page curve for eternal RST black hole

Two-sided eternal black hole solution: $\Omega = M(1 - vu) + \Omega_{\text{crit}}$

Equilibrium with thermal bath at $T = T_{\text{H}}$

$t_\pm(x^\pm) = 0$ in Kruskal coordinates in eternal black hole background



Symmetric anchor points: $(v^m, u^m) = (u, v)$ located well outside BH horizon

No-island configuration: $S_{\text{bulk}} = \frac{c}{12} \log \left[ (v_A - v_{A^m})^2 (u_A - u_{A^m})^2 \, e^{2\rho(v_A, u_A)} e^{2\rho(v_{A^m}, u_{A^m})} \right]$

$$\approx \frac{c}{12} \log \frac{(v_A - u_A)^4}{(1 - v_A u_A)^2} \approx \frac{c}{3} t_A$$

Island configuration: $S_{\text{gen}}^{\text{island}} = 4M(1 - v_I u_I) + \frac{c}{6} \log \frac{(v_A - v_I)^2 (u_A - u_I)^2}{(1 - v_A u_A)(1 - v_I u_I)}$

Extremising over location of inside point: $\frac{u_I}{v_I} = \frac{u_A}{v_A}, \quad v_I \approx -\frac{4\epsilon}{u_A}$

$$S_{\text{gen}}^{\text{island}} = 4M + \frac{c}{3} \sigma_A + \ldots$$

$$t_{\text{Page}} = \frac{1}{4\epsilon} = \frac{12M}{c}$$

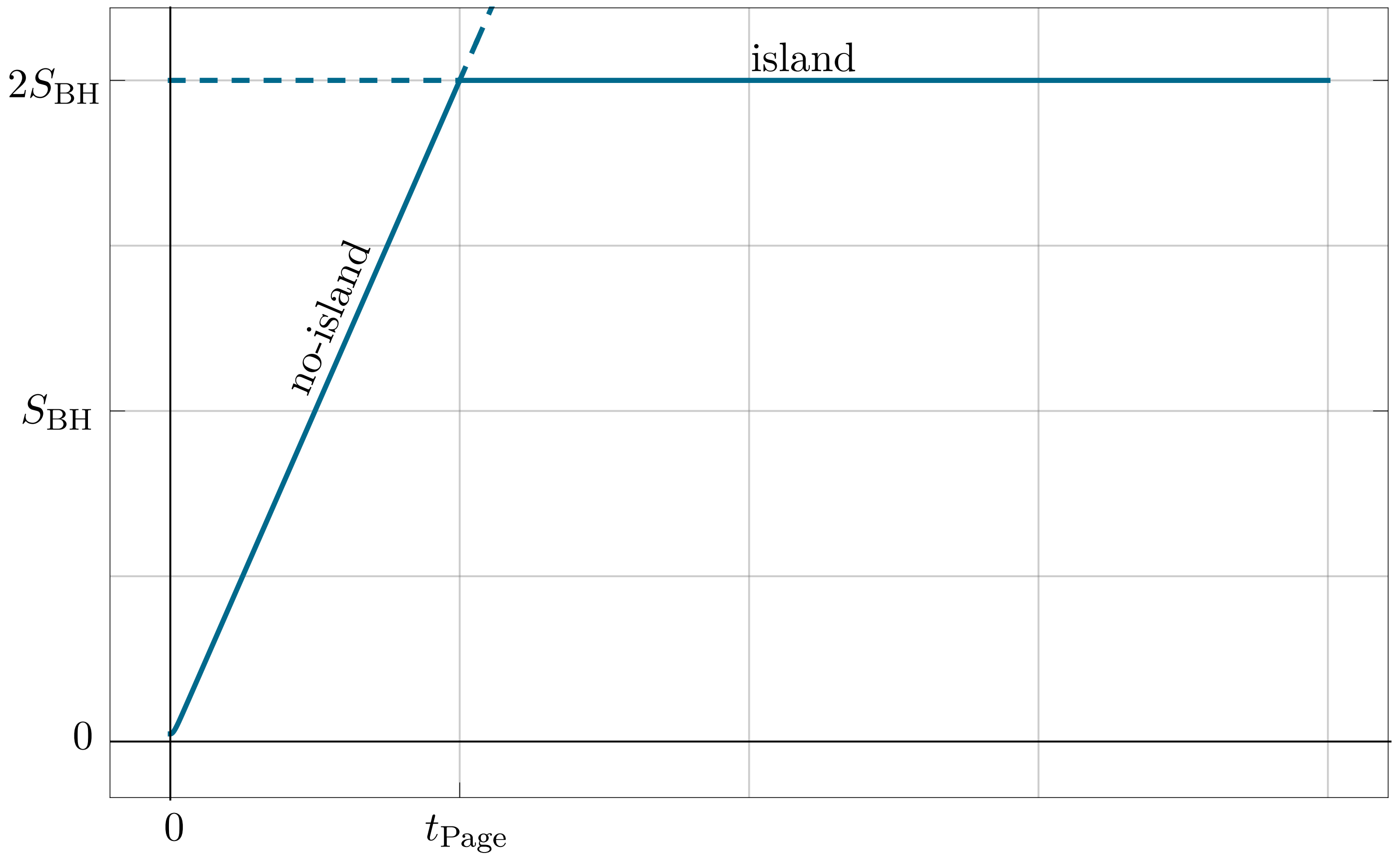



**Figure 4**. Page curve for the eternal RST black hole with $t_{\mathrm{Page}} = 6S_{\mathrm{BH}}/c$. The graph plots $S_{\mathrm{gen}} - \frac{c}{3}\sigma_A$ as a function of retarded time on the anchor curve.

# Summary / extensions

- Semiclassical gravity is suprisingly effective!
- By assuming a QRT formula we explicitly obtained Page curves for semi-classical black holes in asymptotically flat spacetime in a 2d toy model
- This includes both an eternal black hole, supported by an incoming energy flux matching the outgoing Hawking flux, and a black hole formed by gravitational collapse that gradually evaporates
- For the evaporating black hole the Page time is 1/3 of the black hole lifetime
- Result relies on 2d conformal methods and $AdS_3/CFT_2$ and is valid for black hole mass large compared to scale set by the matter central charge.
- Black hole entanglement wedge encompasses only part of black hole interior after Page time — but large enough to encode observations by infalling observer
- Derivation of generalised entropy formula using replica wormholes

G.Penington, S.Shenker, D.Stanford, Z.Yang (2020)
A.Almheiri, T.Hartman, J. Maldacena, E. Shaghoulian, A. Tajdini (2020)
T.Hartman, E. Shaghoulian, A.Strominger (2020)

- Extension to higher dimensions and to cosmological backgrounds ….
- The same toy model provides insight into holographic complexity of black holes

L.Schneiderbauer, W.Sybesma, LT (2020)

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