Resurgent Properties of Minimal String Theory

Maximilian Schwick, Roberto Vega

Instituto Superior Técnico

Iberian Strings 2021,

January 21, 2021

Overview

Bibliography

- I. Aniceto, R. Schiappa, M. Vonk, *The Resurgence of Instantons in String Theory*, Commun. Number Theor. Phys. **6** (2012) 339, `arXiv:`*1106.5922* `[hep-th]`.
- S. Baldino, R. Schiappa, M. S., R. V.
  arxiv: to appear..

Revisit Roberto´s talk: Painlevé Solution

Painlevé I:

$$u(z)^2 - \frac{1}{6}\,u''(z) = z$$

$\Rightarrow$ resurgent transseries sectorial solutions written in:

$$x = z^{-\frac{5}{4}}$$

$$u(x,\,\sigma_1,\sigma_2) = \sum_{n,\,m=0}^{\infty} \sigma_1^n\,\sigma_2^m\,\mathrm{e}^{-(n-m)\,\frac{A}{x}} \sum_{k=0}^{k_{n\,m}} \left(\frac{\log(x)}{2}\right)^k \underbrace{\sum_{g=0}^{\infty} u_{2g}^{(n|m)[k]}\,x^{g+\beta_{n\,m}^{[k]}}}_{:=\,\Phi_{(n|m)}^{[k]}}$$

Going global:

- numerical Stokes Data
- *Can we go analytical*?

Why is Stokes Data important? The Airy Example

The Rainbow:
Almost 200 years ago George Biddell Airy tried to describe a rainbow. He came up with the so called Airy function to describe the light distribution:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_0^\infty \cos\left(\frac{t^3}{3} + z\,t\right) \mathrm{d}t$$

- Hard to compute values
- George Stokes tried to approximate

Why is Stokes Data Important?

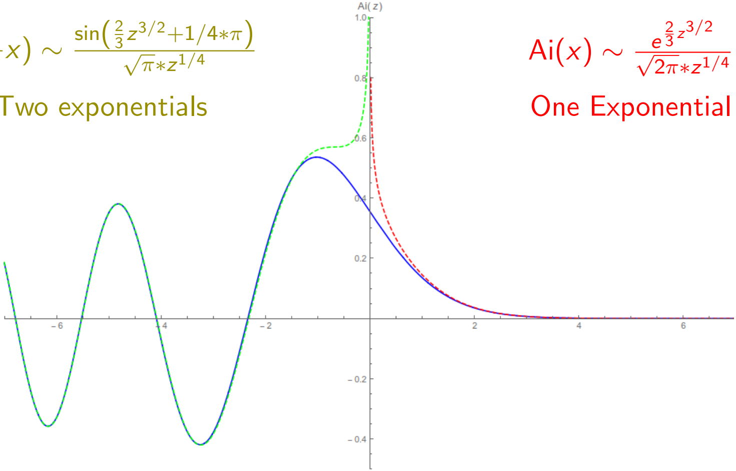
Example: Airy Function:

$$\mathrm{Ai}(-x) \sim \frac{\sin\left(\frac{2}{3}z^{3/2}+1/4*\pi\right)}{\sqrt{\pi}*z^{1/4}}$$

Two exponentials

$$\mathrm{Ai}(x) \sim \frac{e^{\frac{2}{3}z^{3/2}}}{\sqrt{2\pi}*z^{1/4}}$$

One Exponential



⇒ Second exponential? ⇒ Crossed a Stokes line ⇒ Stokes Data controls behavior in different regions of the complex plane

Overview

How does Stokes Data do its Job?

Stokes Automorphism

$$\underline{\mathfrak{S}}_{\text{Stokes line}}\, u(z,\sigma_1,\,\sigma_2) \;=\; \exp\left(\sum_{s\in \substack{\text{singularities along}\\ \text{Stokes line}}} \dot{\Delta}_s \right) u(z,\sigma_1,\,\sigma_2)$$

- Bridge equation (note the vector notation):

$$\Delta_{\boldsymbol{s}\cdot\boldsymbol{A}}\, u_{\boldsymbol{n}} = \sum_{\boldsymbol{p}\in\ker\mathfrak{P}} \boldsymbol{S}_{\boldsymbol{s}+\boldsymbol{p}}\cdot(\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{s})\, u_{\boldsymbol{n}+\boldsymbol{p}+\boldsymbol{s}}$$

- Borel Residues:

$$\underline{\mathfrak{S}}_{\text{Stokes line}}\, u_{\boldsymbol{n}} \;=\; \sum_{\boldsymbol{k}\,\in \substack{\text{Resulting sectors}\\ \text{allowed by Stokes Line}}} \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{k}}\, u_{\boldsymbol{k}}$$

A minimal set of Borel Residues

Relation Borel Residues ⇔ Stokes Constants (forward):

$$\mathsf{S}_{(n,m)\to(n+s-p,m-p)} = -\boldsymbol{S}_{(s-p,-p)} \cdot \begin{bmatrix} n+s-p \\ m-p \end{bmatrix} + P^{(n,m)}_{(n+s-p,\, m-p)},$$

$P \Rightarrow$ combination of forward Stokes Vectors of lower step size

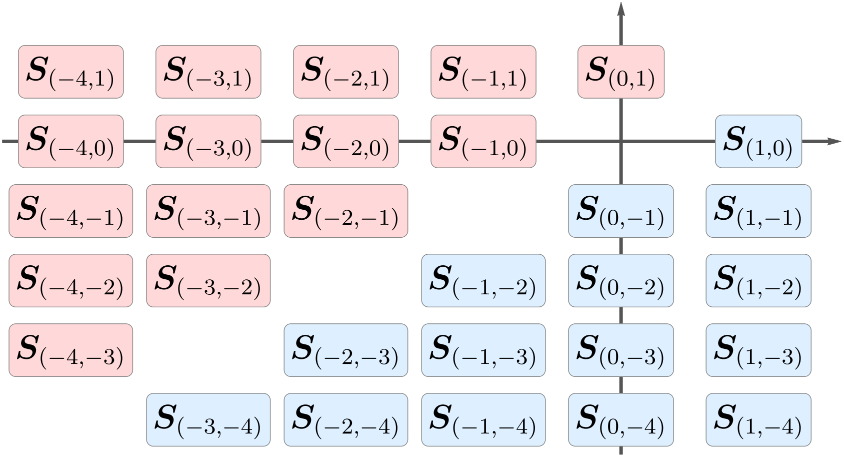
A minimal Set:

- Borel Residues depend on n, m, s, p ⇔ Stokes Constant depend on s, p.
- $p \leq \min(n+s, m)$.

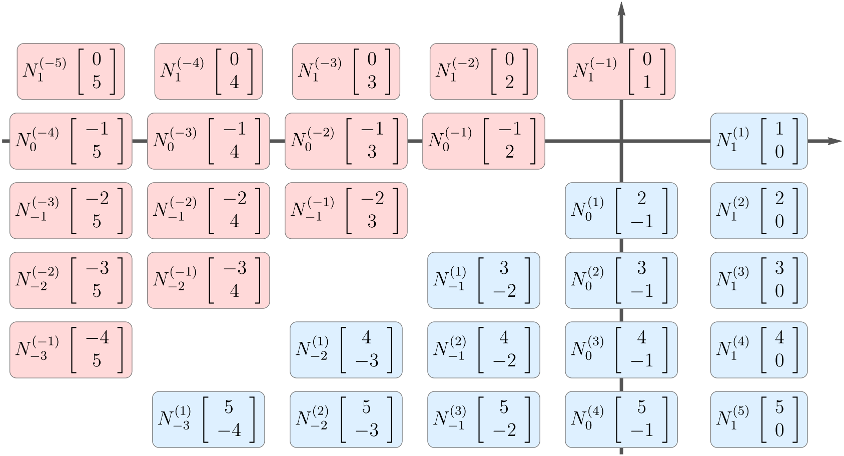
$\Rightarrow$ choice in n, m $\Rightarrow$ set $n = m$

Conclusion: We can always start at the diagonal sectors

Structure of Stokes Data for Painlevé



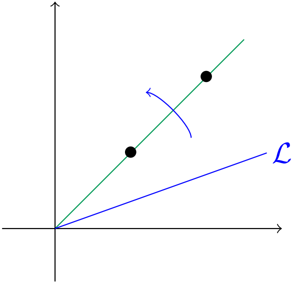
Vector Structure of Stokes Data



Overview

Calculating Stokes Data

Numerical Method:



Smoothness of the resummed (Laplace resummation $\mathcal{L}$) solution across Stokes lines

Asymptotics:

Given: Asymptotic Series

Asymptotic Behavior

$\Downarrow$

Stokes Data

An Introduction to Asymptotic Relations

Toy Example: The First Pole on the Borel Plane:

Known Quantities:

$$\mathcal{B}\left[\Phi_{\boldsymbol{n}}\right](s)\Big|_{s\approx A} \sim \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(1,0,\dots)} \frac{Z_{\boldsymbol{n}+(1,0,\dots)}}{2\pi\mathrm{i}\,(s-A)} + \text{hol.}$$

As a Series around 0:

$$\mathcal{B}\left[\Phi_{\boldsymbol{n}}\right](s) \approx \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(1,0,\dots)} \frac{Z_{\boldsymbol{n}+(1,0,\dots)}}{-2\pi\mathrm{i}\,A} \sum_{k=0}^{\infty} \left(\frac{s}{A}\right)^k + \sum_{j=0}^{\infty} h_j\, s^j$$

Devide by A Supressed Growth

Take coefficients:

$$\frac{1}{k!}\left(\Phi_{\boldsymbol{n}}\right)_{k+1} \approx \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(1,0,\dots)} \frac{Z_{\boldsymbol{n}+(1,0,\dots)}}{-2\pi\mathrm{i}\,A} \left(\frac{1}{A}\right)^k + h_k$$

An Introduction to Asymptotic Relations:

Large Order:

The h_k are supressed for large k:

$$\frac{1}{k!}\left(\Phi_{\boldsymbol{n}}\right)_{k+1} \simeq \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(1,0,\dots)} \frac{Z_{\boldsymbol{n}+(1,0,\dots)}}{-2\pi\mathrm{i}\,A} \left(\frac{1}{A}\right)^k, \quad k \text{ large}$$

This is called an asymptotic relation.

General Singularities

The above idea also works for general singularities:

$$\mathcal{B}\left[\Phi_{\boldsymbol{n}}\right](s)\Big|_{s\approx\ell A} \sim \mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(\ell,0,\dots)} \left(\frac{Z_{\boldsymbol{n}+(\ell,0,\dots)}}{2\pi\mathrm{i}(s-\ell A)} + \mathcal{B}\left[\Phi_{\boldsymbol{n}+(\ell,0,\dots)}\right](s-\ell A)\,\frac{\log(s-\ell A)}{2\pi\mathrm{i}}\right) + \text{hol.}$$

Calculating Stokes Data

Sequence Converging towards Borel Residues

Simplify the asymptotic relation above:

$$\mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(1,0,\dots)} \simeq \frac{-2\pi\mathrm{i}}{Z_{\boldsymbol{n}+(1,0,\dots)}} \frac{A^k}{\Gamma(k)} (\Phi_{\boldsymbol{n}})_k$$

- Borel Residue
- asymptotic sectors
- asymptotically growing terms

Asymptotic relation ⇒ limiting procedure

$$\mathsf{S}_{\boldsymbol{n}\to\boldsymbol{n}+(1,0,\dots)} = \lim_{k\to\infty} \frac{-2\pi\mathrm{i}}{Z_{\boldsymbol{n}+(1,0,\dots)}} \frac{A^k}{\Gamma(k)} (\Phi_{\boldsymbol{n}})_k$$

- Numerical approximation of Borel Residues
- This is nicely described in [1]

Asymptotic Relations for Painlevé

Minimal set of Stokes data ⇒ consider only diagonal sectors

Painlevé analogue to large order relation:

For the large order parameter g we have:

Painleve Sectors | Borel Residue

$$u_{4g}^{(n,n)[0]} \simeq -\frac{1}{\pi \mathrm{i}} \sum_{s=1}^{+\infty} \sum_{h=0}^{+\infty} \sum_{p=0}^{n} \sum_{k=0}^{p} \frac{1}{k!} \left(\frac{\alpha}{2} s\right)^k u_{2h}^{(p+s-k,p-k)[0]} \, \mathsf{S}_{(n,n)\to(p+s,p)} \, \tilde{H}_k\left(2g+n-h-\beta_{(p+s,p)}^{(k)}, sA\right)$$

Contains factorial growth, action, Logarithms

$$\tilde{H}_k(g,s) := \frac{\partial^k}{\partial g^k} \frac{\Gamma(g)}{s^g}$$

Overriding Idea

Painlevé analogue to large order relation:

$$u_{4g}^{(n,n)[0]} \simeq -\frac{1}{\pi \mathrm{i}} \sum_{s=1}^{+\infty} \sum_{h=0}^{+\infty} \sum_{p=0}^{n} \sum_{k=0}^{p} \frac{1}{k!} \left(\frac{\alpha}{2} s\right)^k u_{2h}^{(p+s-k,p-k)[0]} \mathsf{S}_{(n,n)\to(p+s,p)} \tilde{H}_k \left(2g+n-h-\beta_{(p+s,p)}^{(k)}, sA\right)$$

- Translate the asymptotic relation into a limit (as in the prior example)
- Simple structure so that the large order behavior is guessable

From asymptotic relations to analytics?

$$u_{4g}^{(n,n)[0]} \simeq -\frac{1}{\pi \mathrm{i}} \sum_{s=1}^{+\infty} \sum_{h=0}^{+\infty} \sum_{p=0}^{n} \sum_{k=0}^{p} \frac{1}{k!} \left(\frac{\alpha}{2} s\right)^k u_{2h}^{(p+s-k,p-k)[0]} \, \mathsf{S}_{(n,n)\to(p+s,p)} \, \tilde{H}_k \left(2g+n-h-\beta_{(p+s,p)}^{(k)}, sA\right)$$

To go to analytics we need to:

- Borel Residues $\Rightarrow$ Stokes Data
- better understand the structure of logarithms.
- understand the large order behavior of the Painlevé coefficients.

Dropping Subleading Terms

- drop subleading terms ⇒ less sums.
- rename conveniently.

$$\sim \frac{\left(u_{4g}^{(n,n)[0]} - \text{subleading}\right)}{\Gamma(2g+n-\frac{s}{2})} (sA)^{2g+n-\frac{s}{2}}$$

Vector Structure

$$D_{g,s}^{(n,n)} \simeq \sum_{p=0}^{n-s+1} \frac{1}{p!} \left(\frac{\alpha}{2} s\right)^p N_{s-n+p}^{(s)} \times$$

$$B_p\left(\psi^{(0)}\left(2g+n-\tfrac{s}{2}\right) - \log(s \cdot A), \psi^{(1)}\left(2g+n-\tfrac{s}{2}\right), \dots, \psi^{(p-1)}\left(2g+n-\tfrac{s}{2}\right)\right)$$

Logarithmic Terms

where $\psi^{(n)}(x)$ is the polygamma function.

Logarithmic Terms

Polygamma Functions

$$\psi^{(n)}(x) = \begin{cases} -\gamma_{\mathrm{E}} + \sum_{k=1}^{x-1} \frac{1}{k}, & n = 0 \\ (-1)^{m+1}\, m!\, \zeta(1+n) - (-1)^{m+1}\, m! \sum_{k=1}^{x-1} \frac{1}{k^{n+1}}, & n \geq 1. \end{cases}$$

$\Rightarrow$ Hide away the sums (large order dependence) in $D^{(n,n)}_{g,s}$.

$\Rightarrow$ same as evaluating the polygamma functions at 1.

The Conjecture

$$\sum_{p=0}^{n-s+1} N^{(s)}_{s-n+p} \frac{1}{p!} \left(\frac{\alpha}{2}\, s\right)^p B_p\left(-\frac{\tilde{z}_s}{\left(\frac{\alpha}{2}\, s\right)},\ \frac{1}{s}\psi^{(1)}(1),\ \ldots,\ \frac{1}{s^{p-1}}\psi^{(p-1)}(1)\right) = \lim_{g->\infty} d^{(n,n)}_{s,0}(g),$$

where we still need to determine:

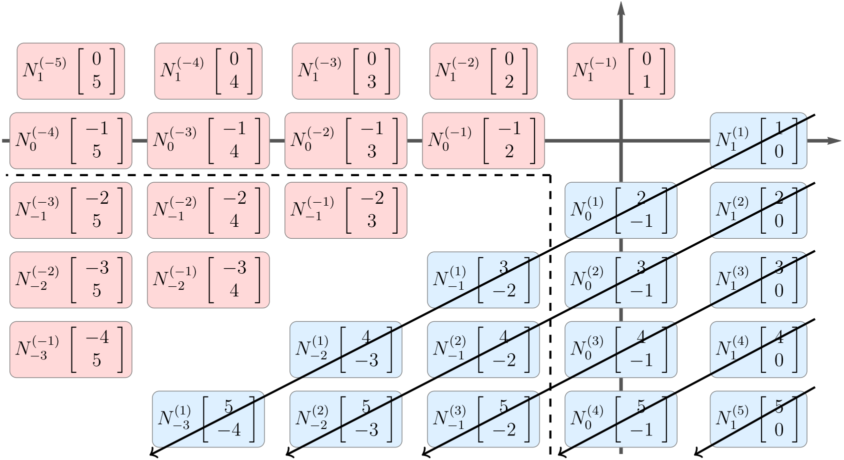
- $d^{(1,1)}_{s,0}(1)$ which can be calculated from $N^{(s)}_1$
- $\tilde{z}_s$ which can be calculated from knowing $N^{(1)}_0$
- $d^{(n,n)}_{s,0}(1)$

Numerically it turns out that

$$d^{(n,n)}_{s,0}(1) = 0, \quad n > 1.$$

Idea: Conjecture this to hold for all $n > 1$.

A graphical Explanation



$s = 1$ $\quad s = 2$ $\quad s = 3$ $\quad s = 4$

We can guess that for Painlevé I $N_0^{(1)} = \frac{2}{3} N_1^{(1)} \left(\gamma_{\mathrm{E}} + \log\left(96\sqrt{3}\right)\right)$

Overview

Solutions to the Painlevé I Equation:

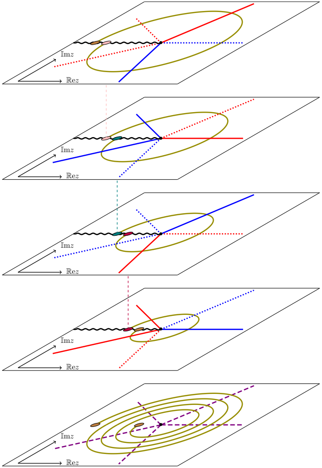
Painlevé I:

$$u(z)^2 - \frac{1}{6}\, u''(z) = z$$

transseries written in:

$$x = z^{-\frac{5}{4}}$$

Stokes Automorphism



- continuous lines: Stokes lines
- dashed lines: Anti-Stokes lines
- blue lines: Forwards
- red line: Backwards

Overview

outlook

- It would be nice to be able to prove the conjecture
- Closed-form expression for full nonperturbative partition function?
- Same analysis for SuperGravity with a non-vanishing RR-flux background, which yields a modified q-PII [Klebanov, Maldacena, Seiberg]

$$u''(z) - \frac{1}{2}u^3(z) + \frac{1}{2}z\,u(z) + \frac{q^2}{u^3(z)} = 0.$$