Supermoduli of SUSY curves: with NS and RR punctures

Daniel Hernández Ruipérez

Departamento de Matemáticas & IUFFYM, Universidad de Salamanca

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Outline

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   - Superspaces and morphisms
   - Examples. Projective superspaces and super Grassmanians
   - Differentials, cotangent and tangent sheaves
   - Splitness
2. Punctured SUSY curves
   - SUSY curves
   - NS and RR punctures
3. Supermoduli of supercurves with punctures
   - Statement of the problem
   - Bosonic moduli
   - Local supermoduli
   - Global supermoduli

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- Various first approaches (Kostant-Leites, De Witt, Rogers).
  - After Kostant and Manin, the Kostant-Leites model prevailed. Moreover, the definition can be also adapted for holomorphic and algebraic varieties (or schemes).

Diferentiable supermanifolds

- Differentiable supermanifolds have locally graded coordinates $(z_1, \ldots, z_m, \theta_1, \ldots, \theta_n)$, $|z_i| = 0$ (even), $|\theta_j| = 1$ (odd). The algebra of (local) superfunctions is the $\mathbb{Z}_2$-graded algebra

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- How the local models glue together? One takes a differentiable manifold X and a local atlas $\{U_i\}$ with coordinates $(z^i_1, \dots, z^i_m)$ and transition functions ϕ_{ij} and glue $\bigwedge_{\mathcal{C}^i} \langle \theta^i_1, \dots, \theta^i_n \rangle$ and $\bigwedge_{\mathcal{C}^j} \langle \theta^j_1, \dots, \theta^j_n \rangle$ on U_{ij} with $\mathbb{Z}_2$-graded algebra isomorphisms Φ_{ij} such that

$$\begin{array}{ccc}
\bigwedge_{\mathcal{C}^i} \langle \theta^i_1, \dots, \theta^i_n \rangle_{|U_{ij}} & \xrightarrow{\Phi_{ij}} & \bigwedge_{\mathcal{C}^j} \langle \theta^j_1, \dots, \theta^j_n \rangle_{|U_{ij}} \\
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\end{array}$$

  commutes.

Spaces

To simplify the exposition we use the following notation and terminology:

- Scheme = Complex algebraic variety X (may have singularities and nilpotent functions). Technically they are noetherian and locally of finite type over $\mathbb{C}$. $\mathcal{O}_X$ denote the sheaf of algebraic functions.

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- Differentiable supermanifold. $\mathcal{O}_X$ = sheaf of (real or complex, depending on the context) differentiable functions on X.

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   2. $G_{\mathcal{J}}\mathcal{O}_{\mathcal{X}} := \mathcal{O}_X \oplus \mathcal{J}/\mathcal{J}^2 \oplus \mathcal{J}^2/\mathcal{J}^3 \oplus \dots$ *is a coherent* $\mathcal{O}_X$*-module and locally* $\mathcal{O}_{\mathcal{X}} \simeq G_{\mathcal{J}}\mathcal{O}_{\mathcal{X}}$

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Then, all types of superschemes, super analytic spaces, differentiable supermanifolds are graded-commutative locally ringed spaces.

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When $\mathcal{X}$ is locally split, we define $\dim \mathcal{X} = m|n$, where $m = \dim X$ and $n = \operatorname{rk} \mathcal{E}$.

- Any locally split superscheme of dimension $m|1$ is split. In this case, $\mathcal{J} = \mathcal{E}$, and then $0 \to \mathcal{E} \to \mathcal{O}_\mathcal{X} \to i_*\mathcal{O}_X \to 0$ gives $\mathcal{E} = \mathcal{O}_{\mathcal{X},1}$, $\mathcal{O}_{\mathcal{X},0} \simeq \mathcal{O}_X$.

Examples

1. If $X = \mathbb{A}^m$ and $\mathcal{E} = \mathcal{O}_X^{\oplus n}$, then $\mathbb{A}^{m|n} := (\mathbb{A}^m, \bigwedge_{\mathcal{O}_{\mathbb{A}^m}} \mathcal{E})$ is the superaffine space of dimension $m|n$.

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3. Write $m = a + b$ and $n = c + d$. Mimicking the construction of the Grassmanian by glueing 'big cells', one defines the supergrassmanian

$$\mathbb{G}\mathrm{r}(a|c; k^{m,n}) = (Gr(a; k^m) \times Gr(c; k^n), \mathcal{O}_{\mathbb{G}\mathrm{r}})$$

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- It is locally split of dimension $ac + bd|ad + bc$.
- $\mathbb{G}\mathrm{r}(1|0; k^{m,n}) \xrightarrow{\sim} \mathbb{P}^{m|n}$.

Differentials

$f\colon \mathcal{X} \to \mathcal{S}$, $g\colon \mathcal{Z} \to \mathcal{S}$, morphisms of superspaces.
There exists the fibre product $f \times g\colon \mathcal{X} \times_{\mathcal{S}} \mathcal{Z} \to \mathcal{S}$ together with two projections $p_1\colon \mathcal{X} \times_{\mathcal{S}} \mathcal{Z} \to \mathcal{X}$, $p_2\colon \mathcal{X} \times_{\mathcal{S}} \mathcal{Z} \to \mathcal{Z}$ and the diagonal morphism $\mathcal{X} \hookrightarrow \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$.

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Then

$$\Omega_X \xrightarrow{\sim} \Omega_+\mathcal{X} := (\Omega_{\mathcal{X}|X})_0\,, \quad \mathcal{E} \xrightarrow{\sim} \Omega_-\mathcal{X} := (\Omega_{\mathcal{X}|X})_1\,.$$

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Taking duals,

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Obstructions to splitness

There are classes

$$\omega_i \in H^1(X, \Theta_{(-1)^i}\mathcal{X} \otimes \bigwedge^i \mathcal{E})$$

depending on several choices, that control the splitness of $\mathcal{X}$.

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6. A locally split superscheme of dimension $m|2$ is projected if and only if it is split.

Examples of non projected superschemes

There exists a notion of very ample locally free sheaf of rank $1|0$ on a superscheme, similar to the ordinary one, so that very ample line bundles give immersions into projective superspaces.
Let $\mathcal{X}$ be a superscheme.

1. An invertible sheaf $\mathcal{L}$ is very ample on $\mathcal{X}$ $\iff$ the restriction $\mathcal{L}_{|X}$ is very ample on X (Le Brun-Poon-Wells).

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4. $a(m-a)b(n-b) \neq 0, \implies \mathbb{G}\mathrm{r}(a|c; k^{m,n})$ is not superprojective (Penkov) $\implies \mathbb{G}\mathrm{r}(a|c; k^{m,n})$ is not projected.

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- For the bosonic string, these are computed by integrating the Polyakov measure on a compactification of the moduli spaces of algebraic curves (or Riemann surfaces).
- The compactification introduces poles in the measure, fermions were introduce to compensate them.
- Since then, the moduli of SUSY curves (with and without punctures) has attracted a lot of attention.

Definition of SUSY curve

- A SUSY curve over a superscheme $\mathcal{S}$ of genus g is a relative (smooth) supercurve $\pi\colon \mathcal{X} \to \mathcal{S}$ of genus g endowed with a *superconformal structure*, that is, a locally free subsheaf of rank $0|1$ of the relative tangent sheaf, $\mathcal{D} \hookrightarrow \Theta_{\mathcal{X}/\mathcal{S}}$, such that the composition

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- That is, $\mathcal{D}$ is totally non-integrable.
- Locally, there exist superconformal relative graded coordinates (z, θ) such that

$$\mathcal{D} = \langle D \rangle \,, \qquad D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \,, \quad D \otimes D \mapsto 2\overline{\frac{\partial}{\partial z}} \,.$$

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  - A NS N-puncture on a SUSY curve $(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$ is a unordered family $(x_1, \dots, x_N)$ of ($\mathcal{S}$-valued) points of $\pi\colon \mathcal{X} \to \mathcal{S}$ (i.e. sections $x_i\colon \mathcal{S} \hookrightarrow \mathcal{X}$ of π).

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- Ramond-Ramond (RR) punctures. These correspond to divisors where the superconformal structure degenerates and are related to the insertion of fermionic operators.

Ramond-Ramond punctures

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- We also say that $(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$ is a RR-SUSY curve and that $\mathcal{D}$ is a *Ramond-Ramond conformal structure* for $(\mathcal{X} \to \mathcal{S}, \mathcal{Z})$.

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- The irreducible components of $\mathcal{Z}$ are called RR-punctures.

Local equations

The local expression of a RR-superconformal structure is similar to the one for SUSY curves, but with a difference in the relative case.
Let $(\mathcal{X}, \mathcal{Z}, \mathcal{D}) \to \mathcal{S}$ be a RR-SUSY curve.

- There exists an étale covering $\mathcal{T} \to \mathcal{S}$ for which, on the base-change RR-SUSY curve $(\mathcal{X}_\mathcal{T}, \mathcal{Z}_\mathcal{T}, \mathcal{D}_\mathcal{T}) \to \mathcal{T}$, there exist locally relative graded coordinates (z, θ) (superconformal coordinates) such that

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  2. $$\mathcal{D}_\mathcal{T} = \langle D \rangle \,, \qquad D = \frac{\partial}{\partial \theta} + z\theta \frac{\partial}{\partial z} \,.$$
- For a single RR-SUSY curve (that is, $\mathcal{S} = \operatorname{Spec} k$ is one point), no étale covering is required (or better, $\mathcal{T} \to \mathcal{S}$ is the identity)

RR-Spin structures

When the base superscheme is an ordinary scheme S, RR-SUSY curves $(\pi\colon \mathcal{X} \to S, \mathcal{Z}, \mathcal{D})$ are RR-Spin curves:

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$$\mathcal{L} \otimes \mathcal{L} \simeq \kappa_{X/S} \otimes \mathcal{O}_X(Z) = \kappa_{X/S}(Z)\,, \quad \mathcal{L} \simeq \kappa_{X/S}(Z)^{1/2}\,.$$

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That is, for a scheme S:

$$\left\{ \begin{matrix} \text{RR-SUSY curves} \\ (\mathcal{X} \to S, \mathcal{Z}, \mathcal{D}) \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{Relative RR-spin curves} \\ (X \to S, Z, \mathcal{L}) \end{matrix} \right\}$$

Morphisms of RR-SUSY curves

$\pi\colon(\mathcal{X},\mathcal{Z},\mathcal{D})\to\mathcal{S}$, $\pi'\colon(\mathcal{X}',\mathcal{Z}',\mathcal{D}')\to\mathcal{S}$ RR-SUSY curves of degree *n* over $\mathcal{S}$.
A morphism of RR-SUSY curves over $\mathcal{S}$ is a morphism $\phi\colon\mathcal{X}\to\mathcal{X}'$ of $\mathcal{S}$ superschemes that preserves the divisor and the superconformal structure, i.e. such that $\phi(\mathcal{Z})\subseteq\mathcal{Z}'$ and $\phi_*\mathcal{D}\subseteq\mathcal{D}'$.

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Then, a RR-SUSY curve always has a non-trivial automorphism.

Moduli functor of RR-SUSY curves on superschemes

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Moduli problem: To find a superscheme $\mathcal{S}\mathcal{M}^{RR}_{gn}$ "representing SC^{RR}_{gn}". This means that for every superscheme $\mathcal{S}$, one has:

$$\mathrm{Hom}(\mathcal{S}, \mathcal{S}\mathcal{M}^{RR}_{gn}) \xrightarrow{\sim} SC^{RR}_{gn}(\mathcal{S})\,.$$

That is, every relative RR-SUSY curve over $\mathcal{S}$ has to be obtained as the pull-back by a unique morphism $\mathcal{S} \to \mathcal{S}\mathcal{M}^{RR}_{gn}$ of a certain "universal RR-SUSY curve" over $\mathcal{S}\mathcal{M}^{RR}_{gn}$.

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$$\text{Hom}(\mathcal{S}, \mathcal{SM}^{RR}_{gn}) \xrightarrow{\sim} SC^{RR}_{gn}(\mathcal{S})\,.$$

That is, every relative RR-SUSY curve over $\mathcal{S}$ has to be obtained as the pull-back by a unique morphism $\mathcal{S} \to \mathcal{SM}^{RR}_{gn}$ of a certain "universal RR-SUSY curve" over $\mathcal{SM}^{RR}_{gn}$.
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Moduli functor of RR-SUSY curves on superschemes

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However, we can slightly modify the definitions so that there will exist a supermoduli for RR-SUSY-curves, which is no longer a superscheme but a more general kind of object.

Steps in the construction of the supermoduli

The supermoduli for RR-SUSY curves is constructed in the same way as the supermoduli for SUSY curves.

- We assume first that curves have genus $g \geq 2$ and an n-level structure ($n \geq 3$) so that they have no automorphisms but the identity.

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1. Construction of the bosonic supermoduli M_{gn}^{RR}.
2. Construction of the "local supermoduli superscheme".
3. Construction of the (global) supermoduli.

The bosonic moduli of RR-SUSY curves

The bosonic moduli M_{gn}^{RR} is constructed as follows:

- Consider $X_g \to M_g$ universal curve of genus g. There is an open M_{gn} of the n-symmetric power $X_g^{[n]} \to M_g$ that parametrizes families of non-ramified positive divisors of degree n. The pull-back $X_{gn} \to M_{gn}$ of $X_g \to M_g$ has a "universal" relative positive divisor $Z_n \hookrightarrow X_{gn}$ of relative degree n over M_{gn}.

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- For every d, one has the relative Jacobian (or Picard scheme) $\rho_d \colon J^d = J^d(X_{gn}/M_{gn}) \to M_{gn}$ endowed with a universal "degree d line bundle class" Υ_d.

The bosonic moduli of SUSY curves, II

- One has a cartesian diagram that defines the bosonic moduli M_{gm}^{RR} RR-SUSY of curves of genus g along a positive divisor of degree n:

$$\begin{array}{ccc} J^{g-1+n/2} & \xrightarrow{\mu_2} & J^{2g-2+n} \\ \uparrow & & \uparrow \iota \\ M_{gn}^{RR} & \overset{\rho}{\dashrightarrow} & M_{gn} \end{array}$$

$$\mu_2(\mathcal{N}) = \mathcal{N}^{\otimes 2}$$

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- $\rho\colon M_{gn}^{RR} \to M_{gn}$ is an étale covering of degree 2^{2g}, $\implies M_{gn}^{RR}$ is a quasi-projective scheme of dimension $3g-3+n$.

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- $\rho\colon M^{RR}_{gn} \to M_{gn}$ is an étale covering of degree 2^{2g}, $\implies$ M^{RR}_{gn} is a quasi-projective scheme of dimension $3g-3+n$.
- There exists a "universal class" $\Upsilon \in \operatorname{Pic}(X_{gn}/M^{RR}_{gn})$ such that

$$\Upsilon^2 = [\kappa(\mathcal{Z}_n)]\,, \quad \kappa = \kappa_{X_{gn}/M_{gn}}\,.$$

Local universal RR-SUSY curve

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Now,

$$\pi_U \colon \mathcal{X}_{gnU} = (X_{gnU}, \mathcal{O}_{X_{gnU}} \oplus \Pi\mathcal{L}_U) \to U\,,$$

is a ‘local universal RR-SUSY curve over the bosonic moduli with RR-punctures along Z_{nU}.

Fermionic structure of the supermoduli

The fermionic structure os the supermoduli is determined by the odd deformations of the locally universal' RR-SUSY curve $\pi_U \colon \mathcal{X}_{gnU} \to U$:

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$$(\mathcal{SM}_{gn}^{RR})_{|V} = (V, \bigwedge_{\mathcal{O}_V} \mathcal{E}_V)\,.$$

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- The sections of $\Theta_-(\mathcal{SM}_{gn}^{RR})$ in U (the odd vector fields) are the odd infinitesimal deformations of $\pi_U\colon \mathcal{X}_{gnU} \to U$.
- They are given by $[R^1\pi_{U*}\mathcal{G}_{\pi_U}]_1$, where

$$\mathcal{G}(U) = \{D' \in \mathcal{D}er(\mathcal{O}_\mathcal{X}) \,|\, [D', D] \in \mathcal{D}(U), \text{for every } D \in \mathcal{D}(U)\}$$
$$\mathcal{G}_\pi = \mathcal{G} \cap \Theta_{\mathcal{X}/\mathcal{S}} \,.$$

The local supermoduli superscheme of RR-SUSY curves

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Then, the candidate to "local supermoduli supescheme" is

$$\mathcal{U} = (U, \bigwedge \pi_{U*}(\kappa \otimes \kappa(Z_{nU})^{1/2}))\,.$$

One has $\dim \mathcal{U} = (3g-3+n, 2g-2+n/2)$.

Global construction of the supermoduli

Generalizing results of LeBrun and Rothstein one proves that:

- The "local universal RR-SUSY curve over the bosonic moduli", $\pi_U \colon \mathcal{X}_{gnU} \to U$, can be extended to a "local universal supercurve":

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$\implies$ the restriction to the étale covering $U \to M^{RR}_{gn}$ of $\mathcal{SC}^{RR}_{gn}$, is representable by the superscheme $\mathcal{U}$.

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For SUSY curves without punctures the corresponding statement was proved by Domínguez Pérez-HR-Sancho de Salas (97).

Supermoduli of NS-RR-SUSY curves

The case of NS punctures is simpler (Bruzzo-HR):

- The sheaf of relative SUSY curves of genus g with N NS-punctures and n RR-punctures is representable by the N-symmetric power

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There are recent results on that direction:

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There exists a smooth and proper DM-stack over $\mathbb{C}$ representing the functor of families of stable supercurves of genus g with N NS punctures and n RR punctures.

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The boundary of this compactification has been also described, as well a "Mumford formula" in this situation (earlier considered by Rosly-Schwarz-Voronov)

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  - Consequence for pertubative string theory: Cannot integrate on the supermoduli by first integrating over the fibres of a (non-existing) porjection to the ordinary moduli.

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