Sigma models as Gross-Neveu models

Dmitri Bykov
(Steklov Math. Inst., Moscow)

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Based on my papers 20.06... and 20.09...

$$\Sigma \to \mathcal{M}$$

Complex target space
($\mathbb{C}^n$, $\mathbb{CP}^{n-1}$, $Gr(k,n)$, flags ...)

Riemann surface

Although today's talk is not about flag manifold models, the latter provide a fascinating example of the theory and relate to the theory of spin chains, see my upcoming review with I. Affleck & K. Wamer "Flag manifold sigma models: spin chains and integrable theories"

- $\exists$ a wide class of sigma models, equivalent to bosonic Gross-Neveu models
- This formulation makes it easy to construct trigonometric (in principle also elliptic) deformations
- Renormalizability can be proven @ one loop $\mapsto$ Generalized Ricci flow
- Sigma models are models with polynomial interactions!
- A new approach to models with fermions
- A new method of constructing SUSY theories
- Integrable models related to quiver varieties

Example. The $\mathbb{CP}^{n-1}$-model as a GN-model.

/DB 2020

$U, V \in \mathbb{C}^n$, $\Psi := \begin{pmatrix} U \\ V \end{pmatrix}$ 'Dirac boson'

$$\mathcal{L} = \overline{\Psi}_a \not{D} \Psi_a + \varkappa \left(\overline{\Psi}_a \frac{1+\gamma_5}{2} \Psi_a \right) \cdot \left(\overline{\Psi}_b \frac{1-\gamma_5}{2} \Psi_b \right)$$

$U(1)$ ($\mathbb{C}^*$) covariant derivative

Bosonic chiral gauged Gross-Neveu model

Let us write out the Lagrangian in components:

$$\mathcal{L} = V \cdot \bar{D} U + \bar{U} \cdot D\bar{V} + \varkappa (\bar{U} \cdot U)(\bar{V} \cdot V)$$

Eliminating the fields $V, \bar{V}$, we obtain

$$\mathcal{L} = \frac{1}{\varkappa} \frac{D\bar{U} \cdot \bar{D} U}{\bar{U} \cdot U}$$ = $\mathbb{CP}^{n-1}$-model in a GLSM-formulation

($\bar{U} \cdot U = 1$ – standard 'Hopf gauge')

The role of chiral symmetry

With Euclidean signature on Σ,

Chiral symmetry

=

Complexification of original (flavor) symmetry

Zumino '1977
Mehta '1990

In our example $U \to \lambda U$, $V \to \lambda^{-1} V$, $\lambda \in \mathbb{C}^*$

$\Rightarrow$ Therefore $|U|^4$ and $|V|^4$ are prohibited,
but $|U|^2 |V|^2$ is allowed.

Gross–Neveu are classically integrable,
with a simple Lax pair

Fermionic case: Neveu & Papanicolaou '1978

Trigonometric deformation

To construct an integrable deformation
we replace the GN Lagrangian with

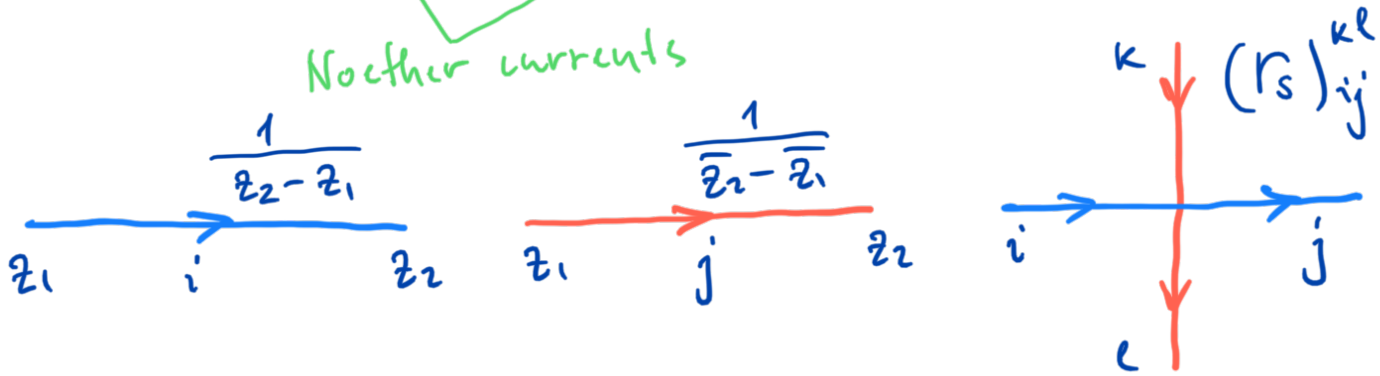
$$\mathcal{L}^{(s)} = \bar{\Psi}_a \not{\partial} \Psi_a + \varkappa (r_s)_{ab}^{cd} \left(\bar{\Psi}_a \frac{1+\gamma_5}{2} \Psi_c\right) \cdot \left(\bar{\Psi}_d \frac{1-\gamma_5}{2} \Psi_b\right)$$

classical r-matrix
('s' is the deformation parameter)

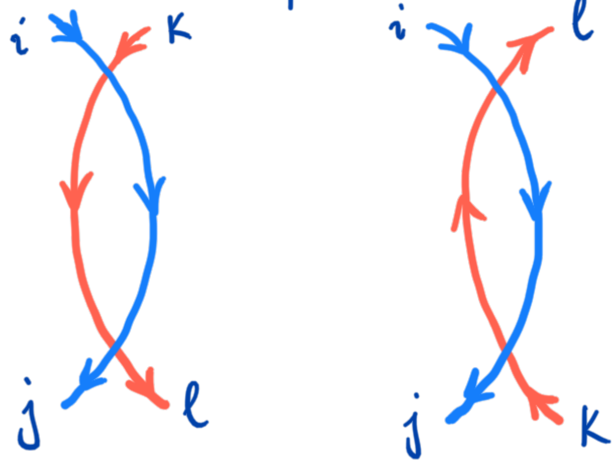
Belavin
Drinfeld
'1980

The family of flat connections $A_u = r_u(J)\,dz - r_{us^{-1}}(\bar{J})\,d\bar{z}$	$\leftrightarrow$	Classical Yang–Baxter equation

Noether currents



To prove renormalizability
we compute the one-loop β-function



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$$\sum^{n} \left((r)^{kq} (r)^{ql} \quad (r)^{ql} (r)^{kq} \right)$$

$$\beta_{ij}^{k\ell} = \sum_{p,q=1} \left((T_s)_{ip} (T_s)_{pj} - (T_s)_{ip} (T_s)_{pj} \right)$$

The renormalization group flow equations $\dot{r}_{ij}^{k\ell} = \beta_{ij}^{k\ell}$ have a remarkable elementary solution $s = e^{n\tau}$

Example.

$\mathbb{CP}^1$: $n = 2 \Rightarrow s = e^{2\tau}$ 'Sausage'

$$ds^2 = (s^{-1} - s) \frac{|dW|^2}{(s + |W|^2)(s^{-1} + |W|^2)} \qquad 0 < s < 1$$

$$-\frac{d}{d\tau} g_{w\bar{w}} = R_{w\bar{w}} \qquad \text{(Ricci flow)}$$

$$\boxed{\text{Length} \sim |\log(s)| = n|\tau|}$$

Fateev
Onofri
Zamolodchikov
'1994

$\mathbb{CP}^{n-1}$: $(\mathbb{C}^*)^{n-1}$ in the UV $\xleftrightarrow{\ RG\ }$ Homogeneous metric on $\mathbb{CP}^{n-1}$ in the IR
(asymptotic freedom) $s \to 0$ $\qquad$ $s \to 1$

- Equivalent to the solution of (complicated) Ricci flow equations in the geometric formulation of the sigma model:

$$-\dot{g}_{ij} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + 2 \nabla_i \nabla_j \Phi$$

$$-\dot{B}_{ij} = -\frac{1}{2} \nabla^k H_{kij} + \nabla^k \Phi \cdot H_{kij}$$

$$-\dot{\Phi} = \text{const.} - \frac{1}{2} \nabla^k \nabla_k \Phi + \nabla^k \Phi \cdot \nabla_k \Phi + \frac{1}{24} H_{kmn} H^{kmn}$$

Models with fermions

The main reason why one needs to introduce fermions is that purely bosonic theories have chiral anomalies

$$\mathcal{L} = \bar{\Psi}_a \not{D} \Psi_a \longmapsto S_{eff.} \sim n \int d^2 z \; F_{z\bar{z}} \frac{1}{\Delta} F_{z\bar{z}}$$

Schwinger '1962

Not invariant w.r.t. complex gauge transformations

$A \to A + \partial \alpha, \quad \bar{A} \to \bar{A} + \bar{\partial} \bar{\alpha}$

Non-abelian version $\longmapsto$ WZNW model

Polyakov Wiegmann '1983

An elementary way to cancel the anomaly:

$$\tilde{\mathcal{L}} = \bar{\Psi}_a \not{D} \Psi_a + \bar{\Theta}_a \not{D} \Theta_a$$

Bosons The determinants simply cancel! Fermions simply cancel.

A more conceptual way is to pass to a general diff.-geom. description

To this end we recall once again the $\mathbb{CP}^{n-1}$-model:

$$\mathcal{L} = \underbrace{V\cdot\bar{D}U + \bar{U}\cdot DV}_{\text{Poincaré-Liouville one-form}} + \underbrace{\varkappa(\bar{U}\cdot U)\times(\bar{V}\cdot V)}_{\text{"Hamiltonian"}}$$

Poincaré-Liouville one-form for complex sympl. structure

$$\sum_{i=1}^{n} dV_i \wedge dU_i$$

"Hamiltonian"

$\beta\gamma$-systems — Costello Yamazaki '2019 + GLSM representation — DB '2017

$\Rightarrow$ Complex symplectic reduction!

The general setup is as follows:

- Complex symplectic quiver supervariety Φ (phase space)

- Matter fields $U\oplus V$ in representation $W\oplus W^\vee$ of a complex supergroup G_{gauge}

- Complex global symmetry group

G_{global}, corresponding moment map μ

The Lagrangian has the following canonical form:

$$\boxed{\mathcal{L} = V \cdot \bar{D} U + \bar{U} \cdot D \bar{V} + \varkappa \, \mathrm{STr}(\mu \bar{\mu})}$$

coupling constant

- Chiral anomaly cancellation condition:

$$\mathrm{STr}_W (T_a T_b) = 0$$

generators of $\mathfrak{g}_{gauge}$

* Conjecturally related to anomalies in the Yangian

Abdalla et.al.'1981-84

Examples. $\mathbb{CP}^{n-1}$-models with fermions

DB '2020

Suitably choosing the phase space Φ and the global symmetry group G_{global}, one can obtain all known integrable $\mathbb{CP}^{n}$-models

– $\Phi = T^* \underbrace{\mathbb{CP}^{n-1|n}}$ $\quad (G_{gauge} = \mathbb{C}^*)$

= Fermionic bundle $\Pi(\mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(1))$ over $\mathbb{CP}^{n-1}$

G_{global} – arbitrary subgroup of the full symplectomorphism group $PSL(n|n)$

$G_{global} = SL(n) \times \mathbb{1} \subset PSL(n|n)$

"minimally coupled fermions"

Abdalla et.al. '1981-84

$G_{global} = PSL(n|n)$: sigma model with target space $\mathbb{CP}^{n-1|n}$

Read, Saleur '2001
Witten '2003
Schomerus et.al.'2010

$- \quad \Phi = T^* E \, , \qquad E = \Pi T(\mathbb{CP}^{n-1})$

Fermionic tangent bundle

$$G_{gauge} = \left\{ \begin{pmatrix} \lambda & 0 \\ \xi & \lambda \end{pmatrix} \in SL(1|1) \subset SL(n|n) \right\}$$

bosonic $\simeq \mathbb{C}^*$

fermionic $\simeq \mathbb{C}$

Symplectic superreduction!

$G_{global} = SL(n) \Rightarrow$ SUSY $\mathbb{CP}^{n-1}$ model

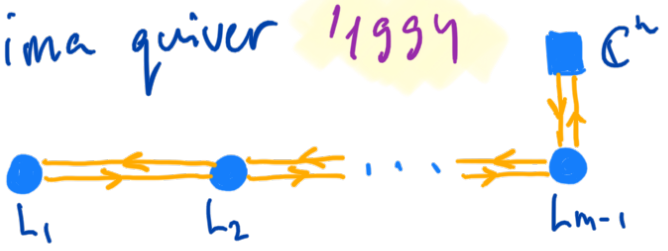
Summary & Outlook

- Sigma models = chiral Gross-Neveu models
- Integrable sigma models related to quiver supervarieties

$\boxed{\text{Classically integrable flag manifold}}$

models are also included DB '2015'

Nakajima quiver '1994



$T^*\mathcal{F}$

$\mathbb{CP}^{n-1}$, $Gr(k,n)$ are special cases

- Polynomial interactions
  (instead of the usual infinite series)
  + ultralocal Lax pairs Delduc et. al. 2019

Polynomial (Dyson-Maleev) variables for spin operators in cond-mat (quantization of complex coadjoint orbits)
(Our review '2021)

4D Ashtekar variables

dim. red. to 2D (Ehlers, Geroch, Belinsky-Zakharov, Maison)

$SL(2,\mathbb{R})/SO(2)$
sigma model
(U, V variables)

Brodbeck
Zagermann
'2000

- It is reasonable to expect that one can construct a quantum theory

- Particularly interesting to study the dependence of the spectrum on the θ-angles (relation to spin chains)

Our review '2021

DB '2011-12

Affleck et. al. '2017+

Seiberg et. al. '2018

Sulejmanpasic et. al. '2018