Seiberg-Witten theory, string theory and WKB analysis

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Introduction

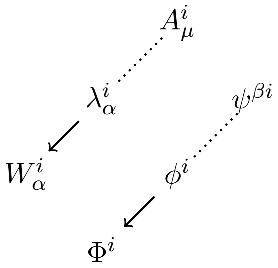
- Goal of this talk: discuss the role of WKB in supersymmetric gauge theories
- Seiberg-Witten theory: find the effective action of $\mathcal{N} = 2$ supersymmetric theories
- Appears in string theory in describing CY manifolds for compactification
- Seiberg-Witten theory can be discussed in terms of WKB approximation
- Present: studying via WKB a deformation of classical quantum mechanics

Outline

Seiberg-Witten theory

The setting

$SU(2)$ pure gauge theory



$SU(2)$ breaking ground state: $\phi = a\sigma_3$

Gauge invariant parameter: $u(a) = \mathrm{tr}\,\phi^2 = 2a^2$

Seiberg-Witten theory

Moduli space

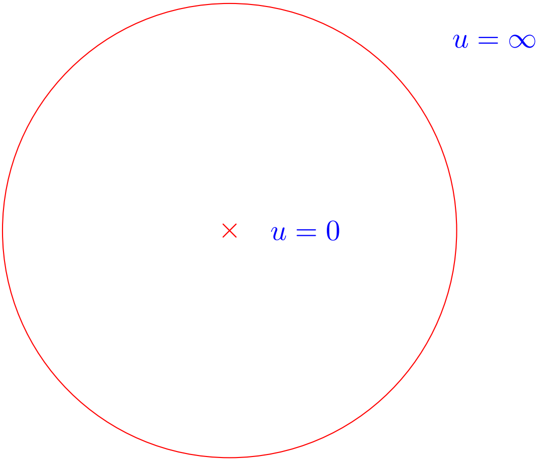
u coordinate on the moduli space of inequivalent vacua

Not all u points are equal!

- $u \neq 0 \rightarrow SU(2)$ is broken into $U(1)$: one massless boson, two massive ones
- $u = 0 \rightarrow SU(2)$ is realized: three massless bosons
- $u = \infty \rightarrow$ weak coupling region

Seiberg-Witten theory

The manifold of vacua



Seiberg-Witten theory

Effective Lagrangian

Quantum theory is determined by computing an effective Lagrangian

Most generic Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \operatorname{Im} \left(\int \mathrm{d}^4\theta K(a, \bar{a}) + \int \mathrm{d}^2\theta \frac{1}{2} \tau(a) W^\alpha W_\alpha \right)$$

The Kähler potential $K(a, \bar{a})$ and the effective coupling $\tau(a)$ are expressed as

$$K(a, \bar{a}) = \frac{\partial \mathcal{F}(a)}{\partial a}, \quad \tau(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial^2 a} = \frac{\theta(a)}{\pi} + \frac{8\pi \mathrm{i}}{g^2(a)}$$

$\mathcal{F}(a)$: prepotential

Seiberg-Witten theory

The prepotential

Classical: the prepotential must reproduce the classical $SU(2)$ Lagrangian

$$\mathcal{F}_{\mathrm{cl}}(a) = \frac{1}{2}\tau_0 a^2$$

Quantum version: instanton corrections!

$$\mathcal{F}(a) = \frac{1}{2}\tau_0 a^2 + \frac{\mathrm{i}}{\pi} a^2 \log\left(\frac{a^2}{\Lambda^2}\right) + \frac{1}{2\pi\mathrm{i}} a^2 \sum_{l=1}^{\infty} c_l \left(\frac{\Lambda}{a}\right)^{4l}$$

Goal of SW: compute c_l

Seiberg-Witten theory

Non trivial monodromies

A consequence: $\tau(a)$ is multi valued

As an example: around ∞

$$\tau = \text{const.} + \frac{2\mathrm{i}}{\pi} \log\left(\frac{u}{\Lambda^2}\right) + \text{single-valued}$$

Under rotation of $2\pi\mathrm{i}$, $\tau \to \tau - 4$

This is good: $\mathrm{Im}\{(\tau)\}$ is harmonic, cannot have a minimum if globally defined

Now it can have a minimum $\to$ metric is positive definite

Seiberg-Witten theory

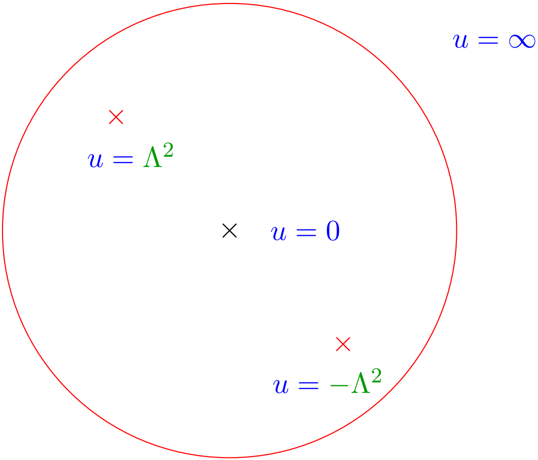
The quantum picture

Quantum theory moduli space:

- The singularity at ∞ stays there
- $u = 0$ is no more a singular point
- Two new singularities at a scale Λ: $u = \pm\Lambda^2$

Seiberg-Witten theory

Quantum moduli space



Seiberg-Witten theory

Quantum singularities

Some motivation:

- $u = 0$ not a singularity: if there was, there would be an $R-$symmetry relating the massive bosons becoming massless
- $u = \pm\Lambda^2$ singularities: BPS solitons become massless! New massless particles enter the spectrum
- Total number of singularities dictated by Witten index for $SU(2)$, that is 2

Seiberg-Witten theory

BPS states

A good way to understand masses: BPS states

For central charge Z, mass of solitons is bound from below: $m^2 \geq |Z|^2$

In $SU(2)$: central charge

$$Z = qa + ga_D$$

a_D: "Higgs magnetic field" dual to the "Higgs electric field" a

$$a_D = \frac{\partial \mathcal{F}}{\partial a}$$

Seiberg-Witten theory

BPS singularities

(q, g) - “electric” and “magnetic” charges

BPS states saturate the bound: $m^2 = |Z|^2$

Around the particular points:

- $u = 0$: not a singular point, $u = 2a^2$ cannot hold
- $u = \Lambda^2$: if $a \neq 0$ and $a_D = 0$, a BPS state with purely magnetic charge can be a vacuum configuration of zero mass
- $u = -\Lambda^2$: similar situation (not necessarily $a = 0$ and $a_D \neq 0$)

To describe prepotential: three patches (one around each singularity) are needed. No global coordinate!

Seiberg-Witten theory

Patching the local data

(a_D, a) forms a vector bundle on the manifold of minima

Around singularities: non trivial monodromy

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_P \begin{pmatrix} a_D \\ a \end{pmatrix}$$

Monodromy matrices detrmined by asymptotic behaviour and charge of BPS singularities

$$M^{(q,g)} = \begin{pmatrix} 1+qg & q^2 \\ -g^2 & 1-gq \end{pmatrix}$$

Seiberg-Witten theory

Determining monodromies

At infinity: knowing semiclassical solution and its monodromy

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}$$

Global consistency condition

$$M_\infty = M_{\Lambda^2} M_{-\Lambda^2}$$

Determines solution uniquely, up to trivial conjugations: $(1, 0)$ for Λ^2, $(1, -2)$ for $-\Lambda^2$

The problem is now mathematical: find $a(u)$ and $a_D(u)$ with those explicit monodromies (Riemann-Hilbert problem, uniquely defined)

WKB and SW

Motivation

The natural setting for this problem: WKB analysis

Moduli space:

$$\mathcal{M}_u \sim \mathbb{H}^+/\Gamma_0(4)$$

$\Gamma_0(4)$ subgroup of $SL(2,\mathbb{Z})$ generated by monodromy matrices

Realize $\mathcal{M}_q$ as a Riemann surface with $\mathcal{M}_q$ as moduli space!

In this case, the surface is the subset of $\mathbb{C}^2$ is given by (x, y) with

$$y^2 = (x^2 - u)^2 - \Lambda^4$$

Analogy:

$$-\hbar^2 {\partial_x}^2 \psi(x, \hbar) = 2(E - V(x))\psi(x, \hbar)$$

WKB and SW

All orders WKB

- Define the quantum momentum as

$$\psi(x,\hbar) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{x_0}^{x} P(t,\hbar)\mathrm{d}t\right)$$

- The momentum solves

$$\mathrm{i}\hbar\partial_x P(x,\hbar) - [P(x,\hbar)]^2 = 2(E - V(x)) = [p(x)]^2$$

- Series expansion of the momentum in $\hbar$:

$$P(x,\hbar) = \sum_{n=0}^{\infty} \hbar^n P_n(x)$$

- Determines recursion relations for the functions $P_n(x)$

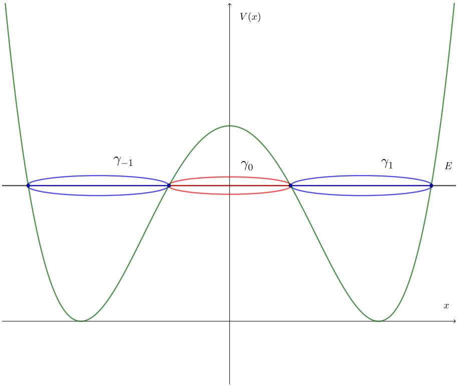
WKB and SW

Integration cycles

- Relevant quantities for non perturbative effects: integrations over cycles

$$S_i(\hbar) = \oint_{\gamma_i} P(t, \hbar)$$

- If γ_i is in a forbidden region, S_i is related to tunneling amplitudes
- Number of γ_i: properties of the spectral curve $y^2 = 2(E - V(x))$



WKB and SW

The bridge

Bridge between WKB and SW:

$$a(u) = \oint_\alpha \lambda_{SW}, \quad a_D(u) = \oint_\beta \lambda_{SW}, \quad \lambda_{SW} = \frac{1}{\sqrt{2}\pi} x^2 \frac{\mathrm{d}x}{y(x,u)}$$

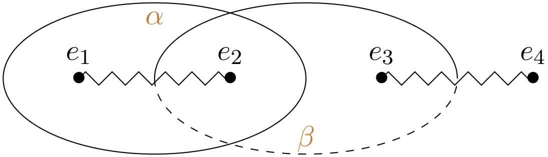
(α, β) standard cycle basis of the torus: cycles between zeroes e_1 and e_2 of y, and e_2 and e_3

Behaviour of cycles at varying u: singularities when two zeroes degenerate

Proposed $a(u)$ and $a_D(u)$ have the correct monodromies

WKB and SW

The torus



WKB and SW

The solution

Solving the WKB system is done through Picard-Fuchs equations

$$\left((\Lambda^4 - u^2)\partial_u{}^2\partial_u{}^2 - 2u\partial_u - \frac{1}{4}\right)\partial_u a(u) = 0,$$

and analogous equation for $a_D(u)$

The cycles are hypergeometric functions!

Asymptotic form $\rightarrow$ determination of quantum corrections to prepotential

WKB and SW

Explicit solution

Explicit solution:

$$a(u) = \frac{\mathrm{i}}{4}\Lambda(u-1)\,{}_2F_1\left(\frac{3}{4},\frac{3}{4},2;1-u\right)$$
$$a_D(u) = \frac{\mathrm{i}}{4}\Lambda u^{\frac{1}{4}}\,{}_2F_1\left(-\frac{1}{4},\frac{1}{4},1;u\right)$$

Prepotentials are obtained by integration

WKB and SW

Conclusion

- SW theory: BPS solitons in $SU(2)$ pure gauge theory
- Through SW theory, find quantum corrections of prepotential
- SW problem can be obtained as a Riemann-Hilbert problem
- WKB gives a very natural setting for solving this problem
- At the end: all non perturbative information determined!

WB and SW

Generalizations

A note: for other gauge groups, the same procedure has been generalized

What can change is the moduli space $\mathcal{M}_u$ and the spectral curve

For $SU(N)$

$$y^2 = (P_N(x, u_i))^2 - \Lambda^{2N}$$

$P_N(x, u_i)$ degree N polynomial with coefficients determined by u_i

General WKB problems are interesting!

Finite difference WKB

Why?

Present work - understand a deformation of classical quantum mechanics

$$H = 2\cosh\left(-\mathrm{i}\hbar\partial_x\right) + V(x)$$

Spectral curves:

$$2\cosh y = E - V(x)$$

Finite difference WKB

The expansion

Same expansion as ordinary WKB:

$$\psi(x,\hbar) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{x_0}^{x} P(t,\hbar)\mathrm{d}t\right)$$

$P(x,\hbar)$ solves an integral equation

Basics of WKB are still working: instantons described by

$$S_i(\hbar) = \oint_{\gamma_i} P(t,\hbar)$$

Finite difference WKB

Motivations from string theory

The spectral curves arise in Type IIA string theory and in the dual heterotic description

Equation of the spectral curve for $SU(N_c)$ symmetry

$$y + \frac{1}{y} = 2P_{N_c}(x)^2$$

Finite differences also appear in Toda lattice theory

The model is integrable

Finite difference WKB

Our goals

- Reproduce known results in a WKB setting
- Give a semiclassical interpretation to the phenomena arising in this deformed model
- Exact quantization conditions including instanton corrections
- Find spectra of important operators

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